GENERALIZATION OF HEINZ OPERATOR INEQUALITIES VIA HYPERBOLIC FUNCTIONS

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(Communicated by M. Krnić)

Abstract. In this paper, we use the Taylor series of hyperbolic functions \( \cosh x \) and \( \sinh x \) to get some generalized inequalities for the Heinz operator means.

1. Introduction

Throughout this paper, \( \mathcal{B}^+ \) denotes the set of all positive invertible operators on a Hilbert space \( \mathcal{H} \). For \( A, B \in \mathcal{B}^+ \) and \( \nu \in [0, 1] \), the weighted arithmetic operator mean \( A\nabla_\nu B \) and geometric mean \( A\#_\nu B \), are defined as follows:

\[
A\nabla_\nu B = (1 - \nu)A + \nu B,
\]

\[
A\#_\nu B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}}.
\]

We refer the reader to F. Kubo and T. Ando [5]. When \( \nu = 1/2 \) we write \( A\nabla B \) and \( A\# B \) for brevity, respectively. The Heinz operator mean is defined by

\[
H_\nu(A, B) = \frac{A\#_\nu B + A\#_{1-\nu} B}{2},
\]

where \( A, B \in \mathcal{B}^+ \) and \( \nu \in [0, 1] \). It is easy to see that the Heinz operator mean interpolates the arithmetic-geometric operator mean inequality [4]:

\[
A\# B \leq H_\nu(A, B) \leq A\nabla B.
\]

In this paper, we study some operator inequalities related to Heinz means. Since \( A, B \) are positive and invertible, \( \nu \) can be extended to \( (-\infty, +\infty) \) in the definition of Arithmetic mean, Geometric mean and Heinz mean. For recent results treating the Heinz means, we refer the reader to [2, 3, 6, 7, 8].

Mathematics subject classification (2010): Please write AMS classification!.

Keywords and phrases: Heinz operator inequality, Heinz operator mean, hyperbolic functions.
2. Main results

The main idea is that we can use the Taylor series of hyperbolic functions $\cosh x$ and $\sinh x$ to get some refinements of inequalities of the Heinz means. To be specific, if we let $\alpha = 1 - 2t$ and $x = (\log a - \log b)/2$, then we have

$$\cosh \alpha x = \frac{a^{1-t}b^t + a^t b^{1-t}}{2\sqrt{ab}} = H_t(a,b) \sqrt{ab},$$

and

$$\frac{\sinh \alpha x}{\alpha x} = \frac{a^{1-t}b^t - a^t b^{1-t}}{(1-2t)(\log a - \log b) \sqrt{ab}}.$$

So by improving some inequalities of hyperbolic functions, we can get some refinements of Heinz means inequalities.

**Theorem 2.1.** Let $A, B \in \mathcal{B}^+$, and $r, s, t \in \mathbb{R}$ with $t, r \neq 1/2$. If $|1 - 2r| \leq |1 - 2t|$, then

$$\left(1 - \frac{(1-2s)^2}{(1-2r)^2}\right)A^r B + \frac{(1-2s)^2}{(1-2r)^2}H_r(A,B)$$

$$\leq \left(1 - \frac{(1-2s)^2}{(1-2t)^2}\right)A^t B + \frac{(1-2s)^2}{(1-2t)^2}H_t(A,B).$$  \hspace{1cm} (2.1)

**Proof.** We first show that the following inequality

$$\left(1 - \frac{\beta^2}{\gamma^2}\right) + \frac{\beta^2}{\gamma^2} \cosh \gamma x \leq \left(1 - \frac{\beta^2}{\alpha^2}\right) + \frac{\beta^2}{\alpha^2} \cosh \alpha x \quad (x \in \mathbb{R})$$  \hspace{1cm} (2.2)

holds for real numbers $\alpha, \beta, \gamma$ with $\alpha, \gamma \neq 0$ and $|\gamma| \leq |\alpha|$. By the Taylor series of $\cosh x$ we have

$$\left[ \left(1 - \frac{\beta^2}{\gamma^2}\right) + \frac{\beta^2}{\gamma^2} \cosh \gamma x \right] - \left[ \left(1 - \frac{\beta^2}{\alpha^2}\right) + \frac{\beta^2}{\alpha^2} \cosh \alpha x \right]$$

$$= \left[ 1 - \frac{\beta^2}{\alpha^2} + \frac{\beta^2}{\alpha^2} \left(1 + \frac{\alpha^2 x^2}{2!} + \frac{\alpha^4 x^4}{4!} + \cdots \right) \right]$$

$$- \left[ 1 - \frac{\beta^2}{\gamma^2} + \frac{\beta^2}{\gamma^2} \left(1 + \frac{\gamma^2 x^2}{2!} + \frac{\gamma^4 x^4}{4!} + \cdots \right) \right]$$

$$= \frac{\beta^2}{\alpha^2} \left(\frac{\alpha^2 x^2}{2!} + \frac{\alpha^4 x^4}{4!} + \cdots \right) - \frac{\beta^2}{\gamma^2} \left(\frac{\gamma^2 x^2}{2!} + \frac{\gamma^4 x^4}{4!} + \cdots \right)$$

$$\geq 0.$$
Then (2.2) holds. Now let \( \alpha = 1 - 2t, \beta = 1 - 2s, \gamma = 1 - 2r \) and \( x = (\log a - \log b)/2 \). Then it follows that for \( a, b > 0 \),

\[
\left( 1 - \frac{(1 - 2s)^2}{(1 - 2r)^2} \right) \sqrt{ab} + \frac{(1 - 2s)^2}{(1 - 2r)^2} H_r(a, b) \leq \left( 1 - \frac{(1 - 2s)^2}{(1 - 2r)^2} \right) \sqrt{ab} + \frac{(1 - 2s)^2}{(1 - 2r)^2} H_r(a, b).
\]

Hence, for invertible positive operator \( X \) we have

\[
\left( 1 - \frac{(1 - 2s)^2}{(1 - 2r)^2} \right) X^\frac{1}{2} + \frac{(1 - 2s)^2}{(1 - 2r)^2} H_r(X, 1) \leq \left( 1 - \frac{(1 - 2s)^2}{(1 - 2r)^2} \right) X^\frac{1}{2} + \frac{(1 - 2s)^2}{(1 - 2r)^2} H_r(X, 1).
\]

Substituting \( X \) with \( A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \), and multiplying both sides with \( A^{\frac{1}{2}} \), we have the inequality (2.1).

\( \square \)

**Corollary 2.2.** Under the assumptions of Theorem 2.1, if \( t = 1 \) then for \( |1 - 2r| \leq 1 \),

\[
\left( 1 - \frac{(1 - 2s)^2}{(1 - 2r)^2} \right) \frac{\log a}{\log b} + \frac{(1 - 2s)^2}{(1 - 2r)^2} H_r(A, B)
\]

\[
\leq \left( 1 - \frac{(1 - 2s)^2}{(1 - 2r)^2} \right) \frac{\log a}{\log b} + \frac{(1 - 2s)^2}{(1 - 2r)^2} H_r(A, B). \quad (2.3)
\]

If \( r = 1 \) then for \( 1 \leq |1 - 2t| \),

\[
\left( 1 - \frac{(1 - 2s)^2}{(1 - 2r)^2} \right) \frac{\log a}{\log b} + (1 - 2s)^2 \Delta B
\]

\[
\leq \left( 1 - \frac{(1 - 2s)^2}{(1 - 2r)^2} \right) \frac{\log a}{\log b} + \frac{(1 - 2s)^2}{(1 - 2r)^2} H_r(A, B). \quad (2.4)
\]

If \( s = r \) then for \( |1 - 2s| \leq |1 - 2t| \),

\[
H_s(A, B) \leq \left( 1 - \frac{(1 - 2s)^2}{(1 - 2r)^2} \right) \frac{\log a}{\log b} + (1 - 2s)^2 H_r(A, B). \quad (2.5)
\]

If \( s = t \) then for \( |1 - 2r| \leq |1 - 2s| \),

\[
\left( 1 - \frac{(1 - 2s)^2}{(1 - 2r)^2} \right) \frac{\log a}{\log b} + \frac{(1 - 2s)^2}{(1 - 2r)^2} H_r(A, B) \leq H_s(A, B). \quad (2.6)
\]

Moreover the inequalities (2.3) and (2.4) are equivalent, and (2.5) and (2.6) are equivalent.

Now define a function \( F_\nu : \mathbb{R}_+ \to \mathbb{R}, (\nu \in \mathbb{R}) \) by

\[
F_\nu(x) = \begin{cases} 
\frac{x^\nu - x^{1-\nu}}{\log x}, & x > 0, x \neq 1, \\
2\nu - 1, & x = 1.
\end{cases}
\]

Then we have
THEOREM 2.3. Let \( r,s,t \in \mathbb{R} \) with \( t,r \neq 1/2 \). If \( |1 - 2r| \leq |1 - 2t| \) and \( A,B \in \mathcal{B}^+ \), then

\[
\left( 1 - \frac{(1 - 2s)^2}{(1 - 2r)^2} \right) A^2 B + \frac{(1 - 2s)^2}{(1 - 2r)^2} \frac{1}{2r - 1} A^{\frac{1}{2}} F_r(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}) A^{\frac{1}{2}} \\
\leq \left( 1 - \frac{(1 - 2s)^2}{(1 - 2t)^2} \right) A^2 B + \frac{(1 - 2s)^2}{(1 - 2t)^2} \frac{1}{2t - 1} A^{\frac{1}{2}} F_t(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}) A^{\frac{1}{2}}.
\]

Proof. By a similar argument as in Theorem 2.1, for the case of \( \sinh x/x \) we have

\[
\left( 1 - \frac{\beta^2}{\gamma^2} \right) + \frac{\beta^2 \sinh \gamma x}{\gamma x} \leq \left( 1 - \frac{\beta^2}{\alpha^2} \right) + \frac{\beta^2 \sinh \alpha x}{\alpha x}
\]

holds for real numbers \( \alpha, \beta, \gamma \) with \( \alpha, \gamma \neq 0 \) and \( |\gamma| \leq |\alpha| \). And then for \( a,b > 0 \) and \( r,s,t \in \mathbb{R} \) with \( t,r \neq 1/2 \) and \( |1 - 2r| \leq |1 - 2t| \), we have

\[
\left( 1 - \frac{(1 - 2s)^2}{(1 - 2r)^2} \right) + \frac{(1 - 2s)^2}{(1 - 2r)^2} \frac{a^{1-r} b^r - a^r b^{1-r}}{(1 - 2r)(\log a - \log b)} \frac{1}{\sqrt{ab}} \\
\leq \left( 1 - \frac{(1 - 2s)^2}{(1 - 2t)^2} \right) + \frac{(1 - 2s)^2}{(1 - 2t)^2} \frac{a^{1-t} b^t - a^t b^{1-t}}{(1 - 2t)(\log a - \log b)} \frac{1}{\sqrt{ab}}.
\]

Hence the conclusions follow. \( \Box \)

THEOREM 2.4. Let \( r,s \in \mathbb{R} \) with \( r,s \neq \frac{1}{2} \). If

\[
\frac{(1 - 2s)^2}{(1 - 2r)^2} \geq \frac{5}{3},
\]

then for \( A,B \in \mathcal{B}^+ \),

\[
\left( 1 - \frac{1}{3} \frac{(1 - 2s)^2}{(1 - 2r)^2} \right) A^2 B + \frac{1}{3} \frac{(1 - 2s)^2}{(1 - 2r)^2} H_r(A,B) \leq \frac{1}{2s - 1} A^{\frac{1}{2}} F_r(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}) A^{\frac{1}{2}}.
\]

Proof. For real numbers \( \beta, \gamma \) with \( \beta, \gamma \neq 0 \), and \( \beta^2/\gamma^2 \geq 5/3 \), we have

\[
\frac{\sinh \beta x}{\beta x} = 1 + \frac{\beta^2 x^2}{3!} + \frac{\beta^4 x^4}{5!} + \cdots
\]

\[
= 1 - \frac{1}{3} \frac{\beta^2}{\gamma^2} + \frac{1}{3} \frac{\beta^2}{\gamma^2} + \frac{1}{3} \frac{\beta^2}{\gamma^2} \frac{\gamma^2 x^2}{2!} + \frac{1}{3} \frac{\beta^2}{\gamma^2} \frac{\gamma^2 x^4}{4!} + \frac{1}{3} \frac{\beta^2}{\gamma^2} \frac{\gamma^2 x^6}{6!} + \cdots
\]

\[
\geq 1 - \frac{1}{3} \frac{\beta^2}{\gamma^2} + \frac{1}{3} \frac{\beta^2}{\gamma^2} \left( 1 + \frac{\gamma^2 x^2}{2!} + \frac{\gamma^2 x^4}{4!} + \frac{3}{7} \left( \frac{5}{6} \right) \frac{\gamma^2 x^6}{6!} + \cdots \right)
\]

\[
\geq 1 - \frac{1}{3} \frac{\beta^2}{\gamma^2} + \frac{1}{3} \frac{\beta^2}{\gamma^2} \left( 1 + \frac{\gamma^2 x^2}{2!} + \frac{\gamma^2 x^4}{4!} + \frac{\gamma^2 x^6}{6!} + \cdots \right)
\]

\[
= 1 - \frac{1}{3} \frac{\beta^2}{\gamma^2} + \frac{1}{3} \frac{\beta^2}{\gamma^2} \cosh \gamma x.
\]
Let $\beta = 1 - 2s, \gamma = 1 - 2r$, and $x = (\log a - \log b)/2$. It follows that
\[
\left(1 - \frac{1}{3} \frac{(1 - 2s)^2}{(1 - 2r)^2}\right) + \frac{1}{3} \frac{(1 - 2s)^2}{(1 - 2r)^2} \frac{H_r(a, b)}{\sqrt{ab}} \leq \frac{a^{1-s}b^s - a^s b^{1-s}}{(1 - 2s)(\log a - \log b)} \frac{1}{\sqrt{ab}}.
\]
Hence the conclusion follows. □

It has been proved that for $\beta^2 \leq \alpha^2/3$,
\[
\cosh \beta x \leq \frac{\sinh \alpha x}{\alpha x}.
\]
See [6]. Now we consider its converse version.

**Theorem 2.5.** Let $t, s \in \mathbb{R}$ with $t \neq \frac{1}{2}$. If $3(1 - 2s)^2 \geq (1 - 2t)^2$, then for $A, B \in \mathcal{B}^+$,
\[
H_s(A, B) \geq \frac{1}{2t - 1} A^\frac{t}{2} F_t(A^{-\frac{t}{2}} BA^{-\frac{t}{2}}) A^\frac{t}{2}.
\]

**Proof.** For $\beta^2 \geq \alpha^2/3$, one has
\[
\cosh \beta x = 1 + \frac{\beta^2 x^2}{2!} + \frac{\beta^4 x^4}{4!} + \frac{\beta^6 x^6}{6!} + \cdots \geq 1 + \frac{\alpha^2 x^2}{2!} + \frac{\alpha^4 x^4}{4!} + \frac{\alpha^6 x^6}{6!} + \cdots = \frac{\sinh \alpha x}{\alpha x}.
\]
Therefore,
\[
\cosh \beta x \geq \frac{\sinh \alpha x}{\alpha x}.
\]
Let $\alpha = 1 - 2t, \beta = 1 - 2s$, and $x = (\log a - \log b)/2$. Then it follows that for $a, b > 0$,
\[
H_s(a, b) \geq \frac{a^{1-t}b^t - a^t b^{1-t}}{(1 - 2t)(\log a - \log b)} \frac{1}{\sqrt{ab}}.
\]
Hence the conclusion for positive operators follows. □

**Remark 2.6.** In the above Theorem, when $t = s$, we have
\[
H_t(A, B) \geq \frac{1}{2t - 1} A^\frac{t}{2} F_t(A^{-\frac{t}{2}} BA^{-\frac{t}{2}}) A^\frac{t}{2}
\]
holds for $t \in \mathbb{R}, t \neq 1/2$. And we also get the condition for
\[
H_s(a, b) \geq L(a, b)
\]
is
\[
s \leq \frac{1 - \frac{1}{\sqrt{3}}}{2} \quad \text{or} \quad s \geq \frac{1 + \frac{1}{\sqrt{3}}}{2},
\]
where \( L(a, b) \) is the logarithmic mean defined by
\[
L(a, b) = \frac{a - b}{\log a - \log b} \quad \text{for} \quad a, b > 0.
\]

Notice that for \( 3(1 - 2s)^2 \leq (1 - 2t)^2 \), we have
\[
H_s(A, B) \leq \frac{1}{2t - 1} A^\frac{1}{2} F_t(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}) A^\frac{1}{2}.
\]

See [7].

There is a basic monotonicity of Heinz means.

**Proposition 2.7.** Set \( s, t \in \mathbb{R} \) satisfying \( |1 - 2s| \leq |1 - 2t| \). If \( A, B \in \mathcal{B}^+ \), then
\[
H_s(A, B) \leq H_t(A, B),
\]
and
\[
\frac{1}{2s - 1} A^\frac{1}{2} F_s(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}) A^\frac{1}{2} \leq \frac{1}{2t - 1} A^\frac{1}{2} F_t(A^{-\frac{1}{2}} BA^{-\frac{1}{2}}) A^\frac{1}{2}.
\]

**Proof.** By the monotonicity of \( \cosh x \) and \( \sinh x/x \) and using the same arguments as above, we can easily get the conclusions. \( \square \)

**Lemma 2.8.** Consider the function
\[
H(t) = \frac{(t - 1)^2 \log t}{t^2 - 1 - 2t \log t}
\]
defined on \((1, \infty)\). Then \( H(t) \) is strictly increasing on \((1, \infty)\), and
\[
\lim_{t \to 1} H(t) = 3.
\]

**Proof.** Firstly we have
\[
H'(t) = \frac{(t - 1) \left[ (t - 1)^2 (t + 1) + 2t(t - 1) \log t - 2t(t + 1) \log^2 t \right]}{t(t^2 - 1 - 2t \log t)^2}.
\]
Let
\[
f(t) = (t - 1)^2 (t + 1) + 2t(t - 1) \log t - 2t(t + 1) \log^2 t.
\]
Then \( f(1) = 0 \) and
\[
f'(t) = 3(t - 1)(t + 1) - 6 \log t - (4t + 2) \log^2 t, \quad f'(1) = 0;
\]
\[
f''(t) = 6t - \frac{1}{t} - 4 \log^2 t - (8t + 4) \frac{1}{t} \log t, \quad f''(1) = 0;
\]
\[
f'''(t) = \frac{2 - 8t + 6t^2 + (4 - 8t) \log t}{t^2}, \quad f'''(1) = 0.
\]
Now let
\[ h(t) = 2 - 8t + 6t^2 + (4 - 8t) \log t. \]
Then \( h(1) = 0 \) and
\[ h'(t) = 12t - 16 - 8 \log t + \frac{4}{t}, \quad h'(1) = 0; \]
\[ h''(t) = \frac{12t^2 - 8t - 4}{t^2}, \quad h''(1) = 0. \]
Hence \( f(t) \) is strictly increasing on \((1, \infty)\) and \( f(t) > 0 \) on \((1, \infty)\). And since \( t^2 - 1 - 2t \log t > 0 \) on \((1, \infty)\), \( H(t) \) is strictly increasing on \((1, \infty)\). Direct calculations show that
\[ \lim_{t \to 1} H(t) = 3. \]
Hence \( H(t) > 3 \) on \((1, \infty)\). □

**Lemma 2.9.** Consider the equation
\[ \frac{x(\cosh x - 1)}{\sinh x - x} = 2p + 1 \quad (x > 0). \] (2.7)
Then for \( p > 1 \), there is uniquely one solution \( x_p > 0 \) for the equation.

**Proof.** According to Lemma 2.8, if \( p > 1 \), then there is uniquely one solution \( t_p > 1 \) for the equation
\[ H(t) = 2p + 1. \]
Setting \( t = \exp x \) \((x > 0)\), one has
\[ H(t) = \frac{x(\cosh x - 1)}{\sinh x - x}. \] (2.8)
Hence the conclusion follows. □

Now we consider the hyperbolic sine. Define
\[ G(x) = \frac{\sinh x - x}{x^{2p+1}} \quad x > 0, \]
where \( p \geq 1 \). Then we have

**Theorem 2.10.** Let \( p > 1, s \neq 1/2, \) and \( \mu \geq 1 \). If \( A, B \in \mathfrak{B}^+ \) with \( A \geq \mu B \) or \( B \geq \mu A \), then we have
\[ \frac{1}{2s-1} A^{s} F \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{s} \geq \left( 1 + \frac{1}{2s} G(x_p) \right) \frac{1}{1 - 2 |s|^2 (\log \mu)^2} A^{s} B, \]
where \( x_p \) is the solution of the equation
\[ \frac{x(\cosh x - 1)}{\sinh x - x} = 2p + 1. \]
Proof. Direct calculations show that

\[ G'(x) = \frac{x^{2p+1}(\cosh x - 1) - (2p + 1)x^{2p}(\sinh x - x)}{x^{4p+2}}. \]

If \( G'(x) = 0 \), then

\[ x(\cosh x - 1) = (2p + 1)(\sinh x - x), \]

i.e.,

\[ \frac{x(\cosh x - 1)}{\sinh x - x} = 2p + 1. \]

So if \( x_p > 0 \) is the unique solution (According to Lemma 2.9) of the equation (2.7), then \( G(x) \) gets its minimum at \( x_p \). Hence we have

\[ \frac{\sinh \beta x}{\beta x} \geq 1 + G(x_p) (\beta x)^{2p}. \]

It follows that

\[ \frac{a^{1-s} b^s - a^s b^{1-s}}{(1 - 2s)(\log a - \log b)} \frac{1}{\sqrt{ab}} \geq 1 + \frac{1}{2^{2p}} G(x_p) |1 - 2s|^{2p} |\log a - \log b|^{2p}. \]

And the result follows. \( \square \)

Now for \( p \geq 1 \), define

\[ F(x) = \frac{\cosh x - 1}{x^{2p}}, \quad x > 0. \]

Then we have

**Theorem 2.11.** Let \( p \geq 1, \mu \geq 1 \), and \( A, B \in \mathcal{B}^+ \) with \( A \geq \mu B \) or \( B \geq \mu A \). If \( y_p > 0 \) satisfies the equation

\[ \frac{x \sinh x}{2(\cosh x - 1)} = p, \]

then for \( s \in \mathbb{R} \),

\[ H_s(A, B) \geq \left( 1 + \frac{1}{2^{2p}} F(y_p) |1 - 2s|^{2p} (\log \mu)^{2p} \right) A^{\#} B. \]

Proof. Notice that

\[ F'(x) = \frac{x^{2p} \sinh x - 2px^{2p-1}(\cosh x - 1)}{x^{4p}}. \]

Let \( F'(x) = 0 \). Then we have

\[ x \sinh x - 2p(\cosh x - 1) = 0. \]
i.e.,

\[
\frac{x \sinh x}{2(\cosh x - 1)} = p. \tag{2.9}
\]

If we set \( x = \log t \) where \( t > 1 \), then the equation (2.9) is equivalent to

\[
\frac{t + 1}{2(t - 1)} \log t = p. \tag{2.10}
\]

And there is only one solution \( t_p > 1 \) for this equation and \( F(\log t) \) gets its minimum at \( t_p \) according to [2]. So if some \( y_p > 0 \) satisfies the equation (2.9), \( F(x) \) get its minimum at \( y_p \), and

\[
F(y_p) = \frac{\cosh y_p - 1}{y_p^2}. \]

Hence,

\[
\cosh \beta x \geq 1 + F(y_p)\beta^2 x^2.
\]

Finally, we get

\[
H_s(a, b) \geq \left(1 + \frac{1}{2p} F(y_p)|1 - 2s|^{2p} \log |a - \log b|^{2p}\right) \sqrt{ab},
\]

and the result follows. \( \Box \)

In particular, when \( p \to 1 \) we have \( \frac{1}{2p} F(x_p) = \frac{1}{8} \), which can be verified by the following argument.

Since

\[
\cosh \beta x = 1 + \frac{\beta^2 x^2}{2!} + \frac{\beta^4 x^4}{4!} + \cdots \geq 1 + \frac{\beta^2 x^2}{2!}.
\]

Letting \( \beta = 1 - 2s \) and \( x = (\log a - \log b)/2 \) we obtain

\[
\frac{H_s(a, b)}{\sqrt{ab}} \geq 1 + \frac{1}{8}(1 - 2s)^2(\log a - \log b)^2,
\]

which is equivalent to the case of \( p = 1 \) in Theorem 2.11. That is,

\[
H_s(A, B) \geq \left(1 + \frac{1}{8}(1 - 2s)^2(\log \mu)^2\right) A^\# B.
\]

**Remark 2.12.** Theorem 2.11 can be considered as another version and proof of Theorem 2.2 (1) of [2].

**Acknowledgement.** The author is grateful to the anonymous referee whose suggestions are quite helpful to improve the paper. The author acknowledges support from the Natural Science Foundation of the Jiangsu Higher Education Institutions of China, Grant No: 18KJB110033.
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(Received October 5, 2018)

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