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GENERALIZATION OF HEINZ OPERATOR INEQUALITIES VIA HYPERBOLIC FUNCTIONS

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Abstract. In this paper, we use the Taylor series of hyperbolic functions $\cosh x$ and $\sinh x$ to get some generalized inequalities for the Heinz operator means.

1. Introduction

Throughout this paper, $\mathfrak{B}^+$ denotes the set of all positive invertible operators on a Hilbert space $\mathscr{H}$. For $A,B \in \mathfrak{B}^+$ and $\nu \in [0,1]$, the weighted arithmetic operator mean $A\nabla_\nu B$ and geometric mean $A\sharp_\nu B$, are defined as follows:

$$A\nabla_\nu B = (1-\nu)A + \nu B,$$

$$A\sharp_\nu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}}.$$

We refer the reader to F. Kubo and T. Ando [5]. When $\nu = 1/2$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively. The Heinz operator mean is defined by

$$H_\nu(A,B) = \frac{A\sharp_\nu B + A\sharp_{1-\nu} B}{2},$$

where $A,B \in \mathfrak{B}^+$ and $\nu \in [0,1]$. It is easy to see that the Heinz operator mean interpolates the arithmetic-geometric operator mean inequality [4]:

$$A\sharp B \leqslant H_\nu(A,B) \leqslant A\nabla B.$$

In this paper, we study some operator inequalities related to Heinz means. Since A,B are positive and invertible, ν can be extended to $(-\infty,+\infty)$ in the definition of Arithmetic mean, Geometric mean and Heinz mean. For recent results treating the Heinz means, we refer the reader to [2, 3, 6, 7, 8].

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2. Main results

The main idea is that we can use the Taylor series of hyperbolic functions $\cosh x$ and $\sinh x$ to get some refinements of inequalities of the Heinz means. To be specific, if we let $\alpha = 1-2t$ and $x = (\log a - \log b)/2$, then we have

$$\cosh \alpha x = \frac{a^{1-t}b^t + a^t b^{1-t}}{2\sqrt{ab}} = \frac{H_t(a,b)}{\sqrt{ab}},$$

and

$$\frac{\sinh \alpha x}{\alpha x} = \frac{a^{1-t}b^t - a^t b^{1-t}}{(1-2t)(\log a - \log b)} \frac{1}{\sqrt{ab}}.$$

So by improving some inequalities of hyperbolic functions, we can get some refinements of Heinz means inequalities.

THEOREM 2.1. *Let $A,B \in \mathfrak{B}^+$, and $r,s,t \in \mathbb{R}$ with $t,r \neq 1/2$. If $|1-2r| \leqslant |1-2t|$, then*

$$\begin{aligned} &\left(1-\frac{(1-2s)^2}{(1-2r)^2}\right)A\sharp B + \frac{(1-2s)^2}{(1-2r)^2}H_r(A,B) \\ \leqslant &\left(1-\frac{(1-2s)^2}{(1-2t)^2}\right)A\sharp B + \frac{(1-2s)^2}{(1-2t)^2}H_t(A,B). \end{aligned} \tag{2.1}$$

Proof. We first show that the following inequality

$$\left(1-\frac{\beta^2}{\gamma^2}\right) + \frac{\beta^2}{\gamma^2}\cosh \gamma x \leqslant \left(1-\frac{\beta^2}{\alpha^2}\right) + \frac{\beta^2}{\alpha^2}\cosh \alpha x \quad (x \in \mathbb{R}) \tag{2.2}$$

holds for real numbers α, β, γ with $\alpha, \gamma \neq 0$ and $|\gamma| \leqslant |\alpha|$. By the Taylor series of $\cosh x$ we have

$$\begin{aligned} &\left[\left(1-\frac{\beta^2}{\alpha^2}\right) + \frac{\beta^2}{\alpha^2}\cosh \alpha x\right] - \left[\left(1-\frac{\beta^2}{\gamma^2}\right) + \frac{\beta^2}{\gamma^2}\cosh \gamma x\right] \\ = &\left[1-\frac{\beta^2}{\alpha^2} + \frac{\beta^2}{\alpha^2}\left(1+\frac{\alpha^2x^2}{2!}+\frac{\alpha^4x^4}{4!}+\cdots\right)\right] \\ &-\left[1-\frac{\beta^2}{\gamma^2} + \frac{\beta^2}{\gamma^2}\left(1+\frac{\gamma^2x^2}{2!}+\frac{\gamma^4x^4}{4!}+\cdots\right)\right] \\ = &\left[\frac{\beta^2}{\alpha^2}\left(\frac{\alpha^2x^2}{2!}+\frac{\alpha^4x^4}{4!}+\cdots\right)\right] - \left[\frac{\beta^2}{\gamma^2}\left(\frac{\gamma^2x^2}{2!}+\frac{\gamma^4x^4}{4!}+\cdots\right)\right] \\ = &\,\beta^2\left[(\alpha^2-\gamma^2)\frac{x^4}{4!} + (\alpha^4-\gamma^4)\frac{x^6}{6!}+\cdots\right] \\ \geqslant &\, 0. \end{aligned}$$

Then (2.2) holds. Now let $\alpha = 1-2t, \beta = 1-2s, \gamma = 1-2r$ and $x = (\log a - \log b)/2$. Then it follows that for $a, b > 0$,

$$\left(1 - \frac{(1-2s)^2}{(1-2r)^2}\right)\sqrt{ab} + \frac{(1-2s)^2}{(1-2r)^2} H_r(a,b) \leqslant \left(1 - \frac{(1-2s)^2}{(1-2t)^2}\right)\sqrt{ab} + \frac{(1-2s)^2}{(1-2t)^2} H_t(a,b).$$

Hence, for invertible positive operator X we have

$$\left(1 - \frac{(1-2s)^2}{(1-2r)^2}\right)X^{\frac{1}{2}} + \frac{(1-2s)^2}{(1-2r)^2} H_r(X,1) \leqslant \left(1 - \frac{(1-2s)^2}{(1-2t)^2}\right)X^{\frac{1}{2}} + \frac{(1-2s)^2}{(1-2t)^2} H_t(X,1).$$

Substituting X with $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, and multiplying both sides with $A^{\frac{1}{2}}$, we have the inequality (2.1). □

COROLLARY 2.2. *Under the assumptions of Theorem 2.1, if $t = 1$ then for $|1-2r| \leqslant 1$,*

$$\left(1 - \frac{(1-2s)^2}{(1-2r)^2}\right)A\sharp B + \frac{(1-2s)^2}{(1-2r)^2} H_r(A,B)$$
$$\leqslant \left(1 - (1-2s)^2\right)A\sharp B + (1-2s)^2 A\nabla B. \tag{2.3}$$

If $r = 1$ then for $1 \leqslant |1-2t|$,

$$\left(1 - (1-2s)^2\right)A\sharp B + (1-2s)^2 A\nabla B$$
$$\leqslant \left(1 - \frac{(1-2s)^2}{(1-2t)^2}\right)A\sharp B + \frac{(1-2s)^2}{(1-2t)^2} H_t(A,B). \tag{2.4}$$

If $s = r$ then for $|1-2s| \leqslant |1-2t|$,

$$H_s(A,B) \leqslant \left(1 - \frac{(1-2s)^2}{(1-2t)^2}\right)A\sharp B + \frac{(1-2s)^2}{(1-2t)^2} H_t(A,B). \tag{2.5}$$

If $s = t$ then for $|1-2r| \leqslant |1-2s|$,

$$\left(1 - \frac{(1-2s)^2}{(1-2r)^2}\right)A\sharp B + \frac{(1-2s)^2}{(1-2r)^2} H_r(A,B) \leqslant H_s(A,B). \tag{2.6}$$

Moreover the inequalities (2.3) and (2.4) are equivalent, and (2.5) and (2.6) are equivalent.

Now define a function $F_\nu : \mathbb{R}_+ \to \mathbb{R}, (\nu \in \mathbb{R})$ by

$$F_\nu(x) = \begin{cases} \dfrac{x^\nu - x^{1-\nu}}{\log x}, & x > 0, x \neq 1, \\ 2\nu - 1, & x = 1. \end{cases}$$

Then we have

THEOREM 2.3. *Let $r,s,t\in\mathbb{R}$ with $t,r\neq 1/2$. If $|1-2r|\leqslant|1-2t|$ and $A,B\in\mathfrak{B}^+$, then*

$$\left(1-\frac{(1-2s)^2}{(1-2r)^2}\right)A\sharp B+\frac{(1-2s)^2}{(1-2r)^2}\frac{1}{2r-1}A^{\frac12}F_r(A^{-\frac12}BA^{-\frac12})A^{\frac12}$$
$$\leqslant\left(1-\frac{(1-2s)^2}{(1-2t)^2}\right)A\sharp B+\frac{(1-2s)^2}{(1-2t)^2}\frac{1}{2t-1}A^{\frac12}F_t(A^{-\frac12}BA^{-\frac12})A^{\frac12}.$$

Proof. By a similar argument as in Theorem 2.1, for the case of $\sinh x/x$ we have

$$\left(1-\frac{\beta^2}{\gamma^2}\right)+\frac{\beta^2}{\gamma^2}\frac{\sinh\gamma x}{\gamma x}\leqslant\left(1-\frac{\beta^2}{\alpha^2}\right)+\frac{\beta^2}{\alpha^2}\frac{\sinh\alpha x}{\alpha x}$$

holds for real numbers α,β,γ with $\alpha,\gamma\neq 0$ and $|\gamma|\leqslant|\alpha|$. And then for $a,b>0$ and $r,s,t\in\mathbb{R}$ with $t,r\neq 1/2$ and $|1-2r|\leqslant|1-2t|$, we have

$$\left(1-\frac{(1-2s)^2}{(1-2r)^2}\right)+\frac{(1-2s)^2}{(1-2r)^2}\frac{a^{1-r}b^r-a^rb^{1-r}}{(1-2r)(\log a-\log b)}\frac{1}{\sqrt{ab}}$$
$$\leqslant\left(1-\frac{(1-2s)^2}{(1-2t)^2}\right)+\frac{(1-2s)^2}{(1-2t)^2}\frac{a^{1-t}b^t-a^tb^{1-t}}{(1-2t)(\log a-\log b)}\frac{1}{\sqrt{ab}}.$$

Hence the conclusions follow. □

THEOREM 2.4. *Let $r,s\in\mathbb{R}$ with $r,s\neq\frac12$. If*

$$\frac{(1-2s)^2}{(1-2r)^2}\geqslant\frac{5}{3},$$

then for $A,B\in\mathfrak{B}^+$,

$$\left(1-\frac13\frac{(1-2s)^2}{(1-2r)^2}\right)A\sharp B+\frac13\frac{(1-2s)^2}{(1-2r)^2}H_r(A,B)\leqslant\frac{1}{2s-1}A^{\frac12}F_s(A^{-\frac12}BA^{-\frac12})A^{\frac12}.$$

Proof. For real numbers β,γ with $\beta,\gamma\neq 0$, and $\beta^2/\gamma^2\geqslant 5/3$, we have

$$\begin{aligned}\frac{\sinh\beta x}{\beta x}&=1+\frac{\beta^2x^2}{3!}+\frac{\beta^4x^4}{5!}+\cdots\\
&=1-\frac13\frac{\beta^2}{\gamma^2}+\frac13\frac{\beta^2}{\gamma^2}+\frac13\frac{\beta^2}{\gamma^2}\frac{\gamma^2x^2}{2!}+\frac13\frac35\frac{\beta^2}{\gamma^2}\frac{\beta^2\gamma^2x^4}{4!}+\frac13\frac37\frac{\beta^2}{\gamma^2}\frac{\beta^4\gamma^2x^6}{6!}+\cdots\\
&\geqslant 1-\frac13\frac{\beta^2}{\gamma^2}+\frac13\frac{\beta^2}{\gamma^2}\left(1+\frac{\gamma^2x^2}{2!}+\frac{\gamma^4x^4}{4!}+\frac37\left(\frac53\right)^2\frac{\gamma^6x^6}{6!}+\cdots\right)\\
&\geqslant 1-\frac13\frac{\beta^2}{\gamma^2}+\frac13\frac{\beta^2}{\gamma^2}\left(1+\frac{\gamma^2x^2}{2!}+\frac{\gamma^4x^4}{4!}+\frac{\gamma^6x^6}{6!}+\cdots\right)\\
&=1-\frac13\frac{\beta^2}{\gamma^2}+\frac13\frac{\beta^2}{\gamma^2}\cosh\gamma x.\end{aligned}$$

Let $\beta = 1-2s, \gamma = 1-2r$, and $x = (\log a - \log b)/2$. It follows that

$$\left(1 - \frac{1}{3}\frac{(1-2s)^2}{(1-2r)^2}\right) + \frac{1}{3}\frac{(1-2s)^2}{(1-2r)^2}\frac{H_r(a,b)}{\sqrt{ab}} \leqslant \frac{a^{1-s}b^s - a^s b^{1-s}}{(1-2s)(\log a - \log b)}\frac{1}{\sqrt{ab}}.$$

Hence the conclusion follows. □

It has been proved that for $\beta^2 \leqslant \alpha^2/3$,

$$\cosh \beta x \leqslant \frac{\sinh \alpha x}{\alpha x}.$$

See [6]. Now we consider its converse version.

THEOREM 2.5. *Let $t, s \in \mathbb{R}$ with $t \neq \frac{1}{2}$. If $3(1-2s)^2 \geqslant (1-2t)^2$, then for $A, B \in \mathfrak{B}^+$,*

$$H_s(A,B) \geqslant \frac{1}{2t-1} A^{\frac{1}{2}} F_t(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

Proof. For $\beta^2 \geqslant \alpha^2/3$, one has

$$\cosh \beta x = 1 + \frac{\beta^2 x^2}{2!} + \frac{\beta^4 x^4}{4!} + \frac{\beta^6 x^6}{6!} + \cdots \geqslant 1 + \frac{\alpha^2 x^2}{3!} + \frac{\alpha^4 x^4}{5!} + \frac{\alpha^6 x^6}{7!} + \cdots = \frac{\sinh \alpha x}{\alpha x}.$$

Therefore,

$$\cosh \beta x \geqslant \frac{\sinh \alpha x}{\alpha x}.$$

Let $\alpha = 1-2t, \beta = 1-2s$, and $x = (\log a - \log b)/2$. Then it follows that for $a, b > 0$,

$$H_s(a,b) \geqslant \frac{a^{1-t}b^t - a^t b^{1-t}}{(1-2t)(\log a - \log b)}\frac{1}{\sqrt{ab}}.$$

Hence the conclusion for positive operators follows. □

REMARK 2.6. In the above Theorem, when $t = s$, we have

$$H_t(A,B) \geqslant \frac{1}{2t-1} A^{\frac{1}{2}} F_t(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$$

holds for $t \in \mathbb{R}, t \neq 1/2$. And we also get the condition for

$$H_s(a,b) \geqslant L(a,b)$$

is

$$s \leqslant \frac{1 - \frac{1}{\sqrt{3}}}{2} \quad \text{or} \quad s \geqslant \frac{1 + \frac{1}{\sqrt{3}}}{2},$$

where $L(a,b)$ is the logarithmic mean defined by

$$L(a,b) = \frac{a-b}{\log a - \log b} \quad \text{for} \quad a,b > 0.$$

Notice that for $3(1-2s)^2 \leqslant (1-2t)^2$, we have

$$H_s(A,B) \leqslant \frac{1}{2t-1} A^{\frac{1}{2}} F_t(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

See [7].

There is a basic monotonicity of Heinz means.

PROPOSITION 2.7. *Set $s,t \in \mathbb{R}$ satisfying $|1-2s| \leqslant |1-2t|$. If $A,B \in \mathfrak{B}^+$, then*

$$H_s(A,B) \leqslant H_t(A,B),$$

and

$$\frac{1}{2s-1} A^{\frac{1}{2}} F_s(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} \leqslant \frac{1}{2t-1} A^{\frac{1}{2}} F_t(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

Proof. By the monotonicity of $\cosh x$ and $\sinh x/x$ and using the same arguments as above, we can easily get the conclusions. □

LEMMA 2.8. *Consider the function*

$$H(t) = \frac{(t-1)^2 \log t}{t^2 - 1 - 2t \log t}$$

defined on $(1,\infty)$. Then $H(t)$ is strictly increasing on $(1,\infty)$, and

$$\lim_{t \to 1} H(t) = 3.$$

Proof. Firstly we have

$$H'(t) = \frac{(t-1)\left[(t-1)^2(t+1) + 2t(t-1)\log t - 2t(t+1)\log^2 t\right]}{t(t^2-1-2t\log t)^2}.$$

Let

$$f(t) = (t-1)^2(t+1) + 2t(t-1)\log t - 2t(t+1)\log^2 t.$$

Then $f(1) = 0$ and

$$f'(t) = 3(t-1)(t+1) - 6\log t - (4t+2)\log^2 t, \quad f'(1) = 0;$$

$$f''(t) = 6t - 6\frac{1}{t} - 4\log^2 t - (8t+4)\frac{1}{t}\log t, \quad f''(1) = 0;$$

$$f'''(t) = \frac{2 - 8t + 6t^2 + (4-8t)\log t}{t^2}, \quad f'''(1) = 0.$$

Now let

$$h(t) = 2 - 8t + 6t^2 + (4 - 8t)\log t.$$

Then $h(1) = 0$ and

$$h'(t) = 12t - 16 - 8\log t + 4\frac{1}{t}, \quad h'(1) = 0;$$

$$h''(t) = \frac{12t^2 - 8t - 4}{t^2}, \quad h''(1) = 0.$$

Hence $f(t)$ is strictly increasing on $(1,\infty)$ and $f(t) > 0$ on $(1,\infty)$. And since $t^2 - 1 - 2t\log t > 0$ on $(1,\infty)$, $H(t)$ is strictly increasing on $(1,\infty)$. Direct calculations show that

$$\lim_{t\to 1} H(t) = 3.$$

Hence $H(t) > 3$ on $(1,\infty)$. □

LEMMA 2.9. *Consider the equation*

$$\frac{x(\cosh x - 1)}{\sinh x - x} = 2p + 1 \qquad (x > 0). \tag{2.7}$$

Then for $p > 1$, *there is uniquely one solution* $x_p > 0$ *for the equation.*

Proof. According to Lemma 2.8, If $p > 1$, then there is uniquely one solution $t_p > 1$ for the equation

$$H(t) = 2p + 1.$$

Setting $t = \exp x$ $(x > 0)$, one has

$$H(t) = \frac{x(\cosh x - 1)}{\sinh x - x}. \tag{2.8}$$

Hence the conclusion follows. □

Now we consider the hyperbolic sine. Define

$$G(x) = \frac{\sinh x - x}{x^{2p+1}} \qquad x > 0,$$

where $p \geqslant 1$. Then we have

THEOREM 2.10. *Let* $p > 1, s \neq 1/2$, *and* $\mu \geqslant 1$. *If* $A, B \in \mathfrak{B}^+$ *with* $A \geqslant \mu B$ *or* $B \geqslant \mu A$, *then we have*

$$\frac{1}{2s-1} A^{\frac{1}{2}} F_s(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} \geqslant \left(1 + \frac{1}{2^{2p}} G(x_p) |1 - 2s|^{2p} (\log\mu)^{2p}\right) A\sharp B,$$

where x_p *is the solution of the equation*

$$\frac{x(\cosh x - 1)}{\sinh x - x} = 2p + 1.$$

Proof. Direct calculations show that

$$G'(x) = \frac{x^{2p+1}(\cosh x - 1) - (2p+1)x^{2p}(\sinh x - x)}{x^{4p+2}}.$$

If $G'(x) = 0$, then

$$x(\cosh x - 1) = (2p+1)(\sinh x - x),$$

i.e.,

$$\frac{x(\cosh x - 1)}{\sinh x - x} = 2p+1.$$

So if $x_p > 0$ is the unique solution (According to Lemma 2.9) of the equation (2.7), then $G(x)$ gets its minimum at x_p. Hence we have

$$\frac{\sinh \beta x}{\beta x} \geqslant 1 + G(x_p)(\beta x)^{2p}.$$

It follows that

$$\frac{a^{1-s}b^s - a^s b^{1-s}}{(1-2s)(\log a - \log b)} \frac{1}{\sqrt{ab}} \geqslant 1 + \frac{1}{2^{2p}} G(x_p)|1-2s|^{2p} |\log a - \log b|^{2p}.$$

And the result follows. $\square$

Now for $p \geqslant 1$, define

$$F(x) = \frac{\cosh x - 1}{x^{2p}}, \qquad x > 0.$$

Then we have

THEOREM 2.11. *Let $p \geqslant 1, \mu \geqslant 1$, and $A, B \in \mathfrak{B}^+$ with $A \geqslant \mu B$ or $B \geqslant \mu A$. If $y_p > 0$ satisfies the equation*

$$\frac{x \sinh x}{2(\cosh x - 1)} = p,$$

then for $s \in \mathbb{R}$,

$$H_s(A,B) \geqslant \left(1 + \frac{1}{2^{2p}} F(y_p)|1-2s|^{2p} (\log \mu)^{2p}\right) A\sharp B.$$

Proof. Notice that

$$F'(x) = \frac{x^{2p}\sinh x - 2p x^{2p-1}(\cosh x - 1)}{x^{4p}}.$$

Let $F'(x) = 0$. Then we have

$$x \sinh x - 2p(\cosh x - 1) = 0.$$

i.e.,

$$\frac{x\sinh x}{2(\cosh x-1)}=p. \tag{2.9}$$

If we set $x=\log t$ where $t>1$, then the equation (2.9) is equivalent to

$$\frac{t+1}{2(t-1)}\log t=p. \tag{2.10}$$

And there is only one solution $t_p>1$ for this equation and $F(\log t)$ gets its minimum at t_p according to [2]. So if some $y_p>0$ satisfies the equation (2.9), $F(x)$ get its minimum at y_p, and

$$F(y_p)=\frac{\cosh y_p-1}{y_p^{2p}}.$$

Hence,

$$\cosh\beta x\geqslant 1+F(y_p)\beta^{2p}x^{2p}.$$

Finally, we get

$$H_s(a,b)\geqslant\left(1+\frac{1}{2^{2p}}F(y_p)|1-2s|^{2p}|\log a-\log b|^{2p}\right)\sqrt{ab},$$

and the result follows. □

In particular, when $p\to 1$ we have $\frac{1}{2^{2p}}F(x_p)=\frac{1}{8}$, which can be verified by the following argument.

Since

$$\cosh\beta x=1+\frac{\beta^2x^2}{2!}+\frac{\beta^4x^4}{4!}+\cdots\geqslant 1+\frac{\beta^2x^2}{2!}.$$

Letting $\beta=1-2s$ and $x=(\log a-\log b)/2$ we obtain

$$\frac{H_s(a,b)}{\sqrt{ab}}\geqslant 1+\frac{1}{8}(1-2s)^2(\log a-\log b)^2,$$

which is equivalent to the case of $p=1$ in Theorem 2.11. That is,

$$H_s(A,B)\geqslant\left(1+\frac{1}{8}(1-2s)^2(\log\mu)^2\right)A\sharp B.$$

REMARK 2.12. Theorem 2.11 can be considered as another version and proof of Theorem 2.2 (1) of [2].

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