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MULTIDIMENSIONAL HARDY–TYPE INEQUALITIES ON TIME SCALES WITH VARIABLE EXPONENTS

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Abstract. A new Jensen inequality for multivariate superquadratic functions is derived and proved. The derived Jensen inequality is then employed to obtain the general Hardy-type integral inequality for superquadratic and subquadratic functions of several variables.

1. Introduction

Hardy's discrete inequality reads: if $p > 1$ and $\{a_k\}_{k=1}^{\infty}$ is a sequence of nonnegative real numbers, then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p. \tag{1.1}$$

Furthermore, G. H. Hardy [9] announced (without proof) that if $p > 1$ and the function f is nonnegative and integrable over the interval $(0,x)$, then

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f^p(x)dx. \tag{1.2}$$

Inequality (1.2) was finally proved by Hardy [10] in 1925. Thus, inequality (1.2) is usually referred to in the literature as the classical Hardy integral inequality while inequality (1.1) is its discrete analogue. The constant $\left(\frac{p}{p-1}\right)^p$ on the right hand sides of both inequalities (1.1) and (1.2) is the best possible.

Note that (1.1) follows from (1.2), which was pointed out by Hardy [9] but there he also informed that a proof of (1.1) was given to him already in a private letter from E. Landau in 1921. More information concerning the interesting prehistory of Hardy's inequality can be found in [15].

In the last five decades, the Hardy inequality (1.2) has been extensively studied and generalized. A lot of information as regarding applications, alternative proofs, variants, generalizations and refinements abound in the literature (see e.g. the books [11, 16, 17] and the references cited therein).

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In his PhD thesis, S. Hilger [12] (see also [6, 13, 14]) initiated the calculus of time scales in order to create a theory that will unify discrete and continuous analysis. This new concept has inspired researchers to study Hardy inequalities on time scales. The first known work in this direction is probably due to P. Řehák [19] who in 2005 derived Hardy integral inequality on time scales. Indeed, he showed that

$$\int_a^\infty \left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t\right)^p \Delta x < \left(\frac{p}{p-1}\right)^p \int_a^\infty f^p(x)\Delta x,$$

where $a > 0, p > 1$ and f is a nonnegative function.

For notations here and in the sequel see Section 2.

In 2001, R. P. Agarwal et al. [1] obtained the following Jensen's inequality on time scales

$$\Phi\left(\frac{1}{b-a}\int_a^b f(x)\Delta x\right) \leqslant \frac{1}{b-a}\int_a^b \Phi(f(x))\Delta x.$$

Moreover, T. Donchev et al. [8] employed the above result to derive the following Hardy-type inequality involving multivariate convex functions on time scales:

THEOREM 1.1. *Let* $(\Omega_1, \mathscr{M}, \mu_\Delta)$ *and* $(\Omega_2, \mathscr{L}, \lambda_\Delta)$ *be two time scale measure spaces and* $U \subset \mathbb{R}^n$ *be a closed convex set. Let* $K : \Omega_1 \to \mathbb{R}$ *be defined by* $K(x) := \int_{\Omega_2} k(x,y)\Delta y < \infty, \quad x \in \Omega_1$, *where* $k(x,y) \geqslant 0$ *is a kernel. Moreover, let* $\zeta : \Omega_1 \to \mathbb{R}$ *and the weight* $w = w(y)$ *be defined by*

$$w(y) := \int_{\Omega_1} \left(\frac{k(x,y)\zeta(x)}{K(x)}\right)\Delta x, \qquad y \in \Omega_2.$$

Then for each convex function Φ,

$$\int_{\Omega_1} \zeta(x)\Phi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\boldsymbol{f}(y)\Delta y\right)\Delta x \leqslant \int_{\Omega_2} w(y)\Phi(\boldsymbol{f}(y))\Delta y \tag{1.3}$$

holds for all λ_Δ*-integrable functions* $\boldsymbol{f} : \Omega_2 \to \mathbb{R}^n$ *such that* $\boldsymbol{f}(\Omega_2) \subset U \subset \mathbb{R}^n$.

In a recent paper, Oguntuase and Persson [18] presented a number of Hardy-type inequalities on time scales using superquadraticity technique which is based on the application of Jensen dynamic inequality. For some recent developments on Hardy-type inequalities on time scales and related results we refer interested reader to the book [3].

Motivated by the above results, our main aim in this paper is to first establish a Jensen inequality for multivariate superquadratic functions and then employ it to derive some new general Hardy-type inequalities for multivariate superquadratic functions involving more general kernels on arbitrary time scales.

The paper is organized as follows: In Section 2, we recall some basic notions, definitions and results on multivariate superquadratic functions on time scales. In Section 3 we state and prove our main results.

2. Preliminaries, definitions and some basic results

First, we recall that a time scale (or measure chain) $\mathbb{T}$ is an arbitrary nonempty closed subset of the real line $\mathbb{R}$ with the topology of the subspace $\mathbb{R}$. Examples of time scales are the real numbers $\mathbb{R}$ and the discrete time scale $\mathbb{Z}$. Since a time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and the backward jump operator by

$$\rho(t) = \inf\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is right-scattered and if $\rho(t) < t$ we say that t is left-scattered. The points that are both right-scattered and left-scattered are called isolated. If $\sigma(t) = t$, then t is said to be right-dense, and if $\rho(t) = t$ then t is said to be left-dense. The points that are simultaneously right-dense and left-dense are called dense. The mapping $\mu : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

is called the graininess function. If $\mathbb{T}$ has a left-scattered maximum M, then define $\mathbb{T}^k = \mathbb{T} \backslash \{M\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{R}$ be a function. Then we define the function $f^\sigma : \mathbb{T} = \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$. Also, for a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative is defined by

$$f^\Delta(t) := \lim_{s \to t, \sigma(s) \neq t} \frac{f^\sigma(s) - f(t)}{\sigma(s) - t}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exists (finite) at all left-dense points in $\mathbb{T}$. We refer interested readers to the books [2], [6] and [7] for more details concerning the calculus of time scales. Note that we have

$$\sigma(t) = t,\ \mu(t) = 0,\ f^\Delta = f',\ \int_a^b f(t)\Delta t = \int_a^b f(t)dt, \text{ when } \mathbb{T} = \mathbb{R},$$

$$\sigma(t) = t + 1,\ \mu(t) = 1,\ f^\Delta = \Delta f,\ \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t), \text{ when } \mathbb{T} = \mathbb{Z}.$$

The following Fubini's theorem on time scale in [5] will be needed in the proof of our results in Section 3:

LEMMA 2.1. *Let $(\Omega, \mathscr{M}, \mu_\Delta)$ and $(\Lambda, \mathscr{L}, \lambda_\Delta)$, be two finite dimensional time scale measures spaces. If $f : \Omega \times \Lambda \to \Re$ is a $\mu_\Delta \times \lambda_\Delta$-integrable function and define the function $\phi(y) = \int_\Omega f(x,y)\Delta x$ for a.e. $y \in \Lambda$ and $\varphi(x) = \int_\lambda f(x,y)\Delta y$ for a.e. $x \in \Omega$, then ϕ is $\lambda_\Delta-$integrable on Λ, φ is μ_Δ-integrable on Ω and*

$$\int_\Omega \Delta x \int_\Lambda f(x,y)\Delta y = \int_\Lambda \Delta y \int_\Omega f(x,y)\Delta x. \tag{2.1}$$

Moreover, M. Anwar et al. [4] result on the Jensen inequality for convex functions in several variables on time scales will also be needed.

THEOREM 2.2. *Let* $(\Omega_1,\Sigma_1,\mu_\Delta)$ *and* $(\Omega_2,\Sigma_2,\lambda_\Delta)$ *be two time scale measure spaces. Suppose* $U\subset\mathbb{R}^n$ *is a closed convex set and* $\Phi\in C(U,\mathbb{R})$ *is convex. Moreover, let* $k:\Omega_1\times\Omega_2\to\mathbb{R}$ *be nonnegative such that* $k(x,.)$ *is* $\lambda_\Delta-integrable$*. Then*

$$\Phi\left(\frac{\int_{\Omega_2}k(x,y)\mathbf{f}(y)\Delta y}{\int_{\Omega_2}k(x,y)\Delta y}\right)\leqslant\frac{\int_{\Omega_2}k(x,y)\Phi(\mathbf{f}(y))\Delta y}{\int_{\Omega_2}k(x,y)\Delta y} \tag{2.2}$$

holds for all functions $\mathbf{f}:\Omega_2\to U$*, where* $f_j(y)$ *are* μ_{Δ_2}*-integrable for all* $j\in\{1,2,\ldots,n\}$*, and* $\int_{\Omega_2}k(x,y)\mathbf{f}(y)\Delta(y)$ *denotes the n-tuple*

$$\left(\int_{\Omega_2}k(x,y)f_1(y)\Delta(y),\int_{\Omega_2}k(x,y)f_2(y)\Delta(y),\ldots,\int_{\Omega_2}k(x,y)f_n(y)\Delta(y)\right).$$

In the sequel, we make the following definitions, assumptions and notations.

(A1.) $\Omega_1=\Omega_2=[\mathbf{a},\mathbf{l})=[a_1,l_1)_{\mathbb{T}}\times[a_2,l_2)_{\mathbb{T}}\ldots\times[a_n,l_n)_{\mathbb{T}}$, where $0\leqslant a_i<l_i\leqslant\infty$.

(A2.) $\mathbf{a}<\mathbf{b}$ if componentwise $a_i<b_i,\ i=1,2,\ldots,n.$

(A3.) $k:[\mathbf{a},\mathbf{l})\times[\mathbf{a},\mathbf{l})\to\mathbb{R}_+$ is such that

$$k(\mathbf{x},\mathbf{y})=\begin{cases}1 & \text{if } \mathbf{a}\leqslant\mathbf{y}<\sigma(\mathbf{x})\leqslant\mathbf{l},\\ 0 & otherwise,\end{cases} \tag{2.3}$$

that is

$$k(x_1,..,x_n,y_1,\ldots,y_n)=\begin{cases}1 & \text{if } a_i\leqslant y_i<\sigma(x_i)\leqslant l_i, i=1,\ldots,n\\ 0 & otherwise,\end{cases} \tag{2.4}$$

(A4.) $\Phi(u)=u^p, p>1.$

REMARK 2.3. Under the assumptions (A1- A4), for $m=1,$ Theorem 2.2 yields the inequality

$$\left(\frac{1}{\prod_{i=1}^{n}(\sigma(x_i)-a_i)}\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_1}^{\sigma(x_1)}f(y_1,\ldots,y_n)\Delta y_1\ldots\Delta y_n\right)^p$$
$$\leqslant\frac{1}{\prod_{i=1}^{n}(\sigma(x_i)-a_i)}\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_1}^{\sigma(x_1)}f^p(y_1,\ldots,y_n))\Delta y_1\ldots\Delta y_n. \tag{2.5}$$

We will also need the following Lemmas for the proof of our main results in the paper.

LEMMA 2.4. *Let* $\beta > 0$ *and* $a, b, l \in \mathbb{T}$ *be such that* $0 \leqslant a < b \leqslant l$.

(i) *If* $\beta > 1$, *then*

$$\int_b^l (s-a)^{\beta-1}\Delta s \leqslant \frac{1}{\beta}\left[(t-a)^{\beta}-(b-a)^{\beta}\right] \leqslant \int_b^l (\sigma(s)-a)^{\beta-1}\Delta s. \tag{2.6}$$

(ii) *If* $\beta < 1$, *then*

$$\int_b^l (s-a)^{\beta-1}\Delta s \geqslant \frac{1}{\beta}\left[(l-a)^{\beta}-(b-a)^{\beta}\right] \geqslant \int_b^l (\sigma(s)-a)^{\beta-1}\Delta s. \tag{2.7}$$

Proof. For case (i), let $\beta > 1$. Then by applying Keller's chain [6], we find that

$$\begin{aligned}\left((t-a)^{\beta}\right)^{\Delta} &= \beta\int_0^1 [h(\sigma(t)-a)+(1-h)(t-a)]^{\beta-1}\,dh \\ &\geqslant \beta\int_0^1 [h(t-a)+(1-h)(t-a)]^{\beta-1}\,dh \\ &= \beta(t-a)^{\beta-1}.\end{aligned}$$

Integrating, we obtain

$$\int_b^l (t-a)^{\beta-1}\Delta t \leqslant \frac{1}{\beta}\left[l-a)^{\beta}-(b-a)^{\beta}\right]. \tag{2.8}$$

On the other hand,

$$\begin{aligned}\left((t-a)^{\beta}\right)^{\Delta} &= \beta\int_0^1 [h(\sigma(t)-a)+(1-h)(t-a)]^{\beta-1}\,dh \\ &\leqslant \beta\int_0^1 [h(\sigma(t)-a)+(1-h)(\sigma(t)-a)]^{\beta-1}\,dh \\ &= \beta(\sigma(t)-a)^{\beta-1},\end{aligned}$$

yielding

$$\frac{1}{\beta}\left[(l-a)^{\beta}-(b-a)^{\beta}\right] \leqslant \int_b^l (\sigma(t)-a)^{\beta-1}\Delta t. \tag{2.9}$$

Finally, combining inequalities (2.8) and (2.9) yields the desired result.
(ii). For the case $\beta < 1$, the proof is similar to the proof of (i), except that the inequalities signs are reversed. □

LEMMA 2.5. *Let* $n \in \mathbb{N}$. *If* $0 \leqslant x_i \leqslant y_i$, *for* $1 \leqslant i \leqslant n$. *Then*

$$\prod_{i=1}^{n}(y_i - x_i) \leqslant \prod_{i=1}^{n} y_i - \prod_{i=1}^{n} x_i. \tag{2.10}$$

Proof. The proof is performed by induction and just noting that

$$\begin{aligned}(y_2-x_2)(y_1-x_1) &= y_2y_1 - x_2x_1 - x_2(y_1-x_1) - x_1(y_2-x_2) \\ &\leqslant y_2y_1 - x_2x_1. \quad \square\end{aligned}$$

3. Multidimensional Hardy-type inequalities for convex functions on time scales

Our first main result reads:

THEOREM 3.1. *Let $\boldsymbol{0} \leqslant \boldsymbol{a} < \boldsymbol{b} < \infty$. Let the functions $p, \beta : [\mathbf{a}, \mathbf{b})_{\mathbb{T}} \to \mathbb{R}$ be defined, respectively, by*

$$p(\boldsymbol{x}) = \begin{cases} p_o, & \mathbf{0} \leqslant \mathbf{x} \leqslant \mathbf{b}, \\ p_1, & \mathbf{x} > \mathbf{b}, \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} \beta_o, & 0 \leqslant x \leqslant b, \\ \beta_1, & x > b. \end{cases} \tag{3.1}$$

Moreover, assume that $p_o, p_1 \in \mathbb{R} \setminus \{0\}$ are such that $p_o \geqslant 1, p_1 \geqslant 1$ or $p_o \geqslant 1, p_1 < 0$ or $p_o < 0, p_1 \geqslant 1$ or $p_o < 0, p_1 < 0$. If $f : [\boldsymbol{a}, \boldsymbol{l}] \to \mathbb{R}$ is non-negative Δ-integrable and $f \in C_{rd}([\boldsymbol{a}, \boldsymbol{l}], \mathbb{R})$ for which

$$\begin{aligned} &\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f^{p(x)}(y_1, \ldots, y_n) \left(\prod_{i=1}^{n} \frac{1}{\beta(x)} \left((y_i - a_i)^{-\beta(x)} \right) \right) \\ &\times \left[1 - \prod_{i=1}^{n} \left(\frac{y_i - a_i}{l_i - a_i} \right)^{\beta(x)} \right] \Delta y_1 \ldots \Delta y_n < \infty, \end{aligned} \tag{3.2}$$

then

$$\begin{aligned} &\int_{a_1}^{l_1} \cdots \int_{a_n}^{l_n} \left(\frac{1}{\prod_{i=1}^{n} (\sigma(x_i) - a_i)} \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_1}^{\sigma(x_1)} f(y_1, \ldots, y_n) \Delta y_1 \ldots \Delta y_n \right)^{p(\boldsymbol{x})} \\ &\times \prod_{i=1}^{n} (\sigma(x_i) - a_i)^{-\beta(x)} \Delta x_1 \ldots \Delta x_n \\ &\leqslant \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f^{p(x)}(y_1, \ldots, y_n) \left(\prod_{i=1}^{n} \frac{1}{\beta(x)} \left((y_i - a_i)^{-\beta(x)} \right) \right) \\ &\times \left[1 - \prod_{i=1}^{n} \left(\frac{y_i - a_i}{l_i - a_i} \right)^{\beta(x)} \right] \Delta y_1 \ldots \Delta y_n + I_o, \end{aligned} \tag{3.3}$$

where $I_o = 0$ if $l \leqslant b$ (so that $\beta(x) = \beta_o$ and $p(x) = p_o$) and

$$\begin{aligned} I_o = &\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f^{p_1}(y_1, \ldots, y_n) \prod_{i=1}^{n} \left[\frac{1}{\beta_1} \left((y_i - a_i)^{-\beta_1} - (l_i - a_i)^{-\beta_1} \right) \right] \Delta y_1 \ldots \Delta y_n \\ &- \int_{a_1}^{b_o} \cdots \int_{a_n}^{b_n} f^{p_o}(y_1, \ldots, y_n) \prod_{i=1}^{n} \left[\frac{1}{\beta_o} \left((y_i - a_i)^{-\beta_o} - (l_i - a_i)^{-\beta_o} \right) \right] \Delta y_1 \ldots \Delta y_n. \end{aligned} \tag{3.4}$$

If $0 < p(x) \leqslant 1$, then (3.3) holds in the reverse direction.

Proof. Let $b \geqslant l$. By applying Jensen's inequality (see Remark 2.3), Lemma 2.1 and Lemma 2.4, we find that

$$\int_{a_1}^{l_1} \cdots \int_{a_n}^{l_n} \left(\frac{1}{\prod\limits_{i=1}^{n} (\sigma(x_i) - a_i)} \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_1}^{\sigma(x_1)} f(y_1, \ldots, y_n) \Delta y_1 \ldots \Delta y_n \right)^{p(\mathbf{x})}$$
$$\times \prod_{i=1}^{n} (\sigma(x_i) - a_i)^{-\beta(x)} \Delta x_1 \ldots \Delta x_n$$
$$\leqslant \int_{a_1}^{l_1} \cdots \int_{a_n}^{l_n} \left[\frac{1}{\prod\limits_{i=1}^{n} (\sigma(x_i) - a_i)} \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_1}^{\sigma(x_1)} f^{p_o}(y_1, \ldots, y_n) \Delta y_1 \ldots \Delta y_n \right]$$
$$\times \prod_{i=1}^{n} (\sigma(x_i) - a_i)^{-\beta_o} \Delta x_1 \ldots \Delta x_n$$
$$\leqslant \int_{a_1}^{l_1} \cdots \int_{a_n}^{l_n} f^{p_o}(y_1, \ldots, y_n) \left[\int_{y_1}^{l_1} \cdots \int_{y_n}^{l_n} \prod_{i=1}^{n} (\sigma(x_i) - a_i)^{-(\beta_o+1)} \Delta x_1 \ldots \Delta x_n \right]$$
$$\times \Delta y_1 \ldots \Delta y_n$$
$$\leqslant \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f^{p(x)}(y_1, \ldots, y_n) \left(\prod_{i=1}^{n} \frac{1}{\beta(x)} \left((y_i - a_i)^{-\beta(x)} \right) \right)$$
$$\times \left[1 - \prod_{i=1}^{n} \left(\frac{y_i - a_i}{l_i - a_i} \right)^{\beta(x)} \right] \Delta y_1 \ldots \Delta y_n.$$

Hence, (3.3) is proved for this case.

Next, let $b \leqslant l$. By applying Jensen's inequality (see Remark 2.3) and Lemma 2.1, we find that

$$\int_{a_1}^{l_1} \cdots \int_{a_n}^{l_n} \left(\frac{1}{\prod\limits_{i=1}^{n} (\sigma(x_i) - a_i)} \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_1}^{\sigma(x_1)} f(y_1, \ldots, y_n) \Delta y_1 \ldots \Delta y_n \right)^{p(\mathbf{x})}$$
$$\times \prod_{i=1}^{n} (\sigma(x_i) - a_i)^{-\beta(x_i)} \Delta x_1 \ldots \Delta x_n$$
$$\leqslant \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left(\frac{1}{\prod\limits_{i=1}^{n} (\sigma(x_i) - a_i)} \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_1}^{\sigma(x_1)} f(y_1, \ldots, y_n) \Delta y_1 \ldots \Delta y_n \right)^{p_o}$$
$$\times \prod_{i=1}^{n} (\sigma(x_i) - a_i)^{-\beta_o} \Delta x_1 \ldots \Delta x_n$$

$$
\begin{aligned}
&+\int_{b_1}^{l_1}\cdots\int_{b_n}^{l_n}\left(\frac{1}{\prod_{i=1}^{n}(\sigma(x_i)-a_i)}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}f(y_1,\ldots,y_n)\Delta y_1\ldots\Delta y_n\right)^{p_1}\\
&\times\prod_{i=1}^{n}(\sigma(x_i)-a_i)^{-\beta_1}\Delta x_1\ldots\Delta x_n\\
&+\int_{b_1}^{l_1}\cdots\int_{b_n}^{l_n}\left(\frac{1}{\prod_{i=1}^{n}(\sigma(x_i)-a_i)}\int_{b_1}^{\sigma(x_1)}\cdots\int_{b_n}^{\sigma(x_n)}f(y_1,\ldots,y_n)\Delta y_1\ldots\Delta y_n\right)^{p_1}\\
&\times\prod_{i=1}^{n}(\sigma(x_i)-a_i)^{-\beta_1}\Delta x_1\ldots\Delta x_n\\
\leqslant&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}f^{p_o}(y_1,\ldots,y_n)\left(\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n}\prod_{i=1}^{n}(\sigma(x_i)-a_i)^{-\beta_o}\Delta x_1\ldots\Delta x_n\right)\Delta y_1\ldots\Delta y_n\\
&+\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}f^{p_1}(y_1,\ldots,y_n)\left(\int_{b_1}^{l_1}\cdots\int_{b_n}^{l_n}\prod_{i=1}^{n}(\sigma(x_i)-a_i)^{-\beta_o}\Delta x_1\ldots\Delta x_n\right)\Delta y_1\ldots\Delta y_n\\
&+\int_{b_1}^{l_1}\cdots\int_{b_n}^{l_n}f^{p_1}(y_1,\ldots,y_n)\left(\int_{y_1}^{l_1}\cdots\int_{y_n}^{l_n}\prod_{i=1}^{n}(\sigma(x_i)-a_i)^{-\beta_o}\Delta x_1\ldots\Delta x_n\right)\Delta y_1\ldots\Delta y_n\\
:=&I. \qquad (3.5)
\end{aligned}
$$

By Lemma 2.4 and Lemma 2.5, we find that

$$
\begin{aligned}
I\leqslant&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}f^{p_o}(y_1,\ldots,y_n)\prod_{i=1}^{n}\left[\frac{1}{\beta_o}\left((y_i-a_i)^{-\beta_o}-(b_i-a_i)^{-\beta_o}\right)\right]\Delta y_1\ldots\Delta y_n\\
&+\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}f^{p_1}(y_1,\ldots,y_n)\prod_{i=1}^{n}\left[\frac{1}{\beta_1}\left((b_i-a_i)^{-\beta_1}-(l_i-a_i)^{-\beta_1}\right)\right]\Delta y_1\ldots\Delta y_n\\
&+\int_{b_1}^{l_1}\cdots\int_{b_n}^{l_n}f^{p_1}(y_1,\ldots,y_n)\prod_{i=1}^{n}\left[\frac{1}{\beta_1}\left((y_i-a_i)^{-\beta_1}-(l_i-a_i)^{-\beta_1}\right)\right]\Delta y_1\ldots\Delta y_n\\
\leqslant&\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}f^{p_o}(y_1,\ldots,y_n)\left(\prod_{i=1}^{n}\frac{1}{\beta_o}\left((y_i-a_i)^{-\beta_o}\right)\right)\left[1-\prod_{i=1}^{n}\left(\frac{y_i-a_i}{l_i-a_i}\right)^{\beta_o}\right]\\
&\cdot\Delta y_1\ldots\Delta y_n\\
&+\int_{b_1}^{l_1}\cdots\int_{b_n}^{l_n}f^{p_1}(y_1,\ldots,y_n)\left(\prod_{i=1}^{n}\frac{1}{\beta_1}\left((y_i-a_i)^{-\beta_1}\right)\right)\left[1-\prod_{i=1}^{n}\left(\frac{y_i-a_i}{l_i-a_i}\right)^{\beta_1}\right]\\
&\cdot\Delta y_1\ldots\Delta y_n\\
&+\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}f^{p_1}(y_1,\ldots,y_n)\prod_{i=1}^{n}\left[\frac{1}{\beta_1}\left((y_i-a_i)^{-\beta_1}-(l_i-a_i)^{-\beta_1}\right)\right]\Delta y_1\ldots\Delta y_n
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{a_1}^{b_o}\cdots\int_{a_n}^{b_n} f^{p_o}(y_1,\ldots,y_n)\prod_{i=1}^{n}\left[\frac{1}{\beta_o}\left((y_i-a_i)^{-\beta_1}-(l_i-a_i)^{-\beta_o}\right)\right]\Delta y_1\ldots\Delta y_n\\
&=\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} f^{p(x)}(y_1,\ldots,y_n)\left(\prod_{i=1}^{n}\frac{1}{\beta(x)}\left((y_i-a_i)^{-\beta(x)}\right)\right)\\
&\quad\times\left[1-\prod_{i=1}^{n}\left(\frac{y_i-a_i}{l_i-a_i}\right)^{\beta(x)}\right]\Delta y_1\ldots\Delta y_n+I_o. \qquad (3.6)
\end{aligned}
$$

By combining the inequalities (3.5) with (3.6) the inequality (3.3) follows so that the proof is complete. □

The next result concerns the dual version of Theorem 3.1 when the Hardy operator

$$
H: f(x_1,..,x_n)\longrightarrow\frac{1}{\prod_{i=1}^{n}(\sigma(x_i)-a_i)}\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_1}^{\sigma(x_1)} f(y_1,\ldots,y_n)\Delta y_1\ldots\Delta y_n
$$

is replaced by the dual Hardy operator

$$
H^*: f(x_1,..,x_n)\longrightarrow\prod_{i=1}^{n}(\sigma(x_i)-a_i)\int_{\sigma(x_1)}^{\infty}\cdots\int_{\sigma(x_n)}^{\infty}\frac{f(y_1,\ldots,y_n)\Delta y_1\ldots\Delta y_n}{\prod_{i=1}^{n}(\sigma(y_i)-a_i)(y_i-a_i)}.
$$

Our next main result concerning the dual Hardy operator H^* reads:

THEOREM 3.2. *Let $\boldsymbol{0}\leqslant\boldsymbol{a}<\boldsymbol{b}<\infty$. Let the functions $p,\beta:[\mathbf{a},\mathbf{b})_{\mathbb{T}}\to\mathbb{R}$ be defined, respectively, by*

$$
p(\boldsymbol{x})=\begin{cases}p_o, & \mathbf{0}\leqslant\mathbf{x}\leqslant\mathbf{b},\\ p_1, & \mathbf{x}>\mathbf{b},\end{cases}\quad ,\beta(x)=\begin{cases}\beta_o, & 0\leqslant x\leqslant b,\\ \beta_1, & x>b.\end{cases} \qquad (3.7)
$$

Moreover, assume that $p_o,p_1\in\mathbb{R}\setminus\{0\}$ are such that $p_o\geqslant 1, p_1\geqslant 1$ or $p_o\geqslant 1, p_1<0$ or $p_o<0, p_1\geqslant 1$ or $p_o<0, p_1<0$. If $f:[\boldsymbol{a},\boldsymbol{l}]\to\mathbb{R}$ is non-negative Δ-integrable and $f\in C_{rd}([\boldsymbol{a},\boldsymbol{l}],\mathbb{R})$ for which

$$
\begin{aligned}
&\int_{l_1}^{\infty}\cdots\int_{l_n}^{\infty} f^{p_1}(y_1,\ldots,y_n)\left(\prod_{i=1}^{n}\frac{(y_i-a_i)^{\beta(y)}}{\beta(y)}\right)\left[1-\prod_{i=1}^{n}\left(\frac{l_i-a_i}{y_i-a_i}\right)^{\beta(y)}\right]\\
&\times\frac{\Delta y_1\ldots\Delta y_n}{\prod_{i=1}^{n}(\sigma(y_i)-a_i)(y_i-a_i)}<\infty, \qquad (3.8)
\end{aligned}
$$

then

$$
\int_{l_1}^{\infty}\cdots\int_{l_n}^{\infty}\left(\prod_{i=1}^{n}(\sigma(x_i)-a_i)\int_{\sigma(x_1)}^{\infty}\cdots\int_{\sigma(x_n)}^{\infty}\left(\frac{f(y_1,\ldots,y_n)}{\prod_{i=1}^{n}(\sigma(y_i)-a_i)(y_i-a_i)}\right)\Delta y_1\ldots\Delta y_n\right)^{p(x)}
$$

$$\times\left(\prod_{i=1}^{n}(x_i-a_i)\right)^{\beta(x)-1}\frac{\Delta x_1\dots\Delta x_n}{\prod_{i=1}^{n}(\sigma(x_i)-a_i)}$$

$$\leqslant\int_{l_1}^{\infty}\dots\int_{l_n}^{\infty}f^{p_1}(y_1,\dots,y_n)\left(\prod_{i=1}^{n}\frac{(y_i-a_i)^{\beta(y)}}{\beta(y)}\right)\left[1-\prod_{i=1}^{n}\left(\frac{l_i-a_i}{y_i-a_i}\right)^{\beta(y)}\right]$$

$$\times\frac{\Delta y_1\dots\Delta y_n}{\prod_{i=1}^{n}(\sigma(y_i)-a_i)(y_i-a_i)}+I_o. \tag{3.9}$$

where $I_o=0$ *if* $l\leqslant b$ *(so that* $\beta(x)=\beta_o$ *and* $p(x)=p_o$*) and*

$$I_o=\int_{l_1}^{\infty}\dots\int_{l_n}^{\infty}\frac{f^{p_1}(y_1,\dots,y_n)}{\prod_{i=1}^{n}(\sigma(y_i)-a_i)(y_i-a_i)}\prod_{i=1}^{n}\left[\frac{1}{\beta_1}\left((y_i-a_i)^{\beta_1}-(l_i-a_i)^{\beta_1}\right)\right]\Delta y_1\dots\Delta y_n$$

$$-\int_{l_1}^{\infty}\dots\int_{l_n}^{\infty}\frac{f^{p_o}(y_1,\dots,y_n)}{\prod_{i=1}^{n}(\sigma(y_i)-a_i)(y_i-a_i)}\prod_{i=1}^{n}\left[\frac{1}{\beta_o}\left((y_i-a_i)^{\beta_o}-(l_i-a_i)^{\beta_o}\right)\right]\Delta y_1\dots\Delta y_n. \tag{3.10}$$

If $0<p(x)\leqslant 1$*, then (3.9) holds in the reverse direction.*

Proof. Let $b\geqslant l$. Applying Jensen's inequality (see Remark 2.3) and Lemma 2.1, we obtain that

$$\int_{l_1}^{\infty}\dots\int_{l_n}^{\infty}\left(\prod_{i=1}^{n}(\sigma(x_i)-a_i)\int_{\sigma(x_1)}^{\infty}\dots\int_{\sigma(x_n)}^{\infty}\left(\frac{f(y_1,\dots,y_n)}{\prod_{i=1}^{n}(\sigma(y_i)-a_i)(y_i-a_i)}\right)\Delta y_1\dots\Delta y_n\right)^{p(x)}$$

$$\times\left(\prod_{i=1}^{n}(x_i-a_i)\right)^{\beta(x)-1}\frac{\Delta x_1\dots\Delta x_n}{\prod_{i=1}^{n}(\sigma(x_i)-a_i)}$$

$$\leqslant\int_{l_1}^{\infty}\dots\int_{l_n}^{\infty}\int_{\sigma(x_1)}^{\infty}\dots\int_{\sigma(x_n)}^{\infty}\left(\frac{f^{p_1}(y_1,\dots,y_n)\Delta y_1\dots\Delta y_n}{\prod_{i=1}^{n}(\sigma(y_i)-a_i)(y_i-a_i)}\right)$$

$$\times\left(\prod_{i=1}^{n}(x_i-a_i)\right)^{\beta_1-1}\Delta x_1\dots\Delta x_n$$

$$\leqslant\int_{l_1}^{\infty}\dots\int_{l_n}^{\infty}\frac{f^{p_1}(y_1,\dots,y_n)}{\prod_{i=1}^{n}(\sigma(y_i)-a_i)(y_i-a_i)}\left[\int_{l_1}^{y_1}\dots\int_{l_n}^{y_n}\left(\prod_{i=1}^{n}(x_i-a_i)\right)^{\beta_1-1}\Delta x_1\dots\Delta x_n\right]$$

$$
\begin{aligned}
&\times \Delta y_1 \dots \Delta y_n \\
&\leqslant \int_{l_1}^{\infty} \dots \int_{l_n}^{\infty} f^{p_1}(y_1,\dots,y_n) \left(\prod_{i=1}^{n} \frac{(y_i - a_i)^{\beta(y)}}{\beta(y)} \right) \left[1 - \prod_{i=1}^{n} \left(\frac{l_i - a_i}{y_i - a_i} \right)^{\beta(y)} \right] \\
&\times \frac{\Delta y_1 \dots \Delta y_n}{\prod_{i=1}^{n} (\sigma(y_i) - a_i)(y_i - a_i)}. \qquad (3.11)
\end{aligned}
$$

Finally, let $b \leqslant l$. Also the proof of this case is completely analogous to the corresponding part of the proof of Theorem 3.1 so we leave out the details. □

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