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JENSEN–TYPE INEQUALITIES AND THEIR APPLICATIONS

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Abstract. In the present paper, we establish some entirely new Jensen-type discrete and integral inequalities. As applications of these results, we strengthen the well-known majorization theorem of Hardy, Littlewood and Pólya, and we also give a generalization of Andersson's inequality.

1. Introduction

The Jensen inequality for convex functions is one of the significant classical inequalities in mathematical analysis. Many other important inequalities such as the Hölder and Minkowski inequalities, inequalities between means, the Ky Fan inequality, etc can be obtained as particular cases of it.

Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be an arbitrary convex function, where I is an interval of real numbers. The Jensen inequality reads (see, for example, [4]) that if $f : [a,b] \to I$ is an integrable function and $w : [a,b] \to \mathbb{R}$ is an integrable non-negative weight function with $\int_a^b w(x)dx > 0$, the following inequality holds true

$$\Phi\Big(\frac{1}{\int_a^b w(x)dx}\int_a^b w(x)f(x)dx\Big) \leqslant \frac{1}{\int_a^b w(x)dx}\int_a^b w(x)\Phi(f(x))dx. \tag{1.1}$$

The Jensen inequality (1.1) changes its direction when Φ is a concave function. The discrete analogue of the inequality (1.1) is as follows:

$$\Phi\Big(\frac{1}{\sum_{i=1}^n w_i}\sum_{i=1}^n w_i x_i\Big) \leqslant \frac{1}{\sum_{i=1}^n w_i}\sum_{i=1}^n w_i\Phi(x_i), \tag{1.2}$$

where $w_i \geqslant 0, \sum_{i=1}^n w_i > 0$ and $x_i \in I$ for all $i = 1,\dots,n$. In the recent years, there is an increasing interest in the research on the Jensen and Jensen-type inequalities, see [5, 11, 8, 9, 14] and the references therein. Almost all of these works focus on refining, strengthening, reversing, generalizing or considering the bounds for the Jensen functional.

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Parrallel with those researches, an inspire work for the study of the Jensen and Jensen-type inequalities is Andersson's inequality. It says (see, for example, [1]) that if f_i are convex increasing functions defined on $[0,1]$ with $f_i(0)=0$ then

$$\int_0^1 \prod_{i=1}^n f_i(x)dx \geqslant \frac{2^n}{n+1}\prod_{i=1}^n \int_0^1 f_i(x)dx. \tag{1.3}$$

An interesting special case of (1.3) when taking f_i to be the same function f is the following inequality

$$\int_0^1 [f(x)]^n dx \geqslant \frac{2^n}{n+1}\left(\int_0^1 f(x)dx\right)^n \tag{1.4}$$

for any $n\in\mathbb{N}$. It is noticeable that the inequality (1.4) implies the classical Jensen inequality with respective to the convex function $\Phi(x)=x^n$; moreover, the coefficient $\frac{2^n}{n+1}$ is much greater than 1 when n is large and it is the best possible. In 2005, Mercer [12] proved that the inequality (1.4) was still valid for replacing n with $\alpha\in(-1,0]\cup[1,\infty)$. In other words, if f is an increasing convex function defined on $[0,1]$ with $f(0)=0$, the following inequality holds true

$$\int_0^1 \Phi(f(x))dx \geqslant \frac{2^\alpha}{\alpha+1}\Phi\Big(\int_0^1 f(x)dx\Big), \tag{1.5}$$

where $\Phi(x)=x^\alpha$ with $\alpha\in(-1,0]\cup[1,\infty)$ is the convex function. This inequality suggests a following natural question: Can the inequality (1.5) be generalized to other convex functions? In that case what does the coefficient $\frac{2^\alpha}{\alpha+1}$ change with respective to Φ?

Motivated by the mentioned results and the above questions, in this paper we give a generalization of the inequality (1.5) for convex functions which the coefficients in the expansion of its Bernstein polynomials satisfying some preset condition. The main result of the paper is given in Theorem 2.1. To obtain this theorem, we using some key tools consisting of the Bernstein polynomials of a continuous function and Chebyshev's inequality for monotonic tuples. A discrete version of Theorem 2.1 is also stated in Theorem 2.5. As applications of these results, we strengthen the well-known majorization theorem of Hardy, Littlewood and Pólya (see Theorem 3.2) and give a generalization of the Andersson inequality (see Theorem 3.4).

2. Main results

We begin this section with recalling some results of Bernstein polynomials. According to [10], the Bernstein polynomial B_n^f of order n of the function f defined on $[0,b]$ is given by

$$B_n^f(x)=\sum_{k=0}^n f\Big(k\frac{b}{n}\Big)\binom{n}{k}\Big(\frac{x}{b}\Big)^k\Big(1-\frac{x}{b}\Big)^{n-k}.$$

Furthermore, if f is a continuous function on $[0,b]$, the sequence of polynomials $\{B_n^f\}_{n=1}^\infty$ converges uniformly on $[0,b]$ to f, see [10, Theorem 1.1.1] for the details.

These polynomials have the expansion in powers of x of the form

$$B_n^f(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\Delta^k f(0)}{b^k} x^k = \sum_{k=0}^{n} c_{nk}^f x^k, \tag{2.1}$$

where $\Delta^k f(0)$ is the difference of kth order of the function f at $x=0$ given by

$$\Delta^k f(0) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f\Big(j\frac{b}{n}\Big).$$

Now, we are ready to state the main result of the paper.

THEOREM 2.1. *Let $f:[0,1]\to[0,b]$ be an integrable increasing function and $w:[0,1]\to[0,\infty)$ be a weight function with $\int_0^1 w(x)dx = 1$. Let $g:[0,1]\to[0,\infty)$ be also an integrable increasing function satisfying $\int_0^1 w(x)g(x)dx = 1$. Suppose that $g(x)$ and $g(x)c$ belong to the interval $[0,b]$ for all $x\in[0,1]$, where the symbol $c = \int_0^1 w(x)f(x)dx$. Let $\Phi:[0,b]\to\mathbb{R}$ be a continuously convex function such that the coefficients c_{nk}^{Φ} of the Bernstein polynomials B_n^{Φ} of the function Φ satisfy that, for any $n\in\mathbb{N}$,*

$$0 \leqslant \sum_{k=0}^{j} c_{nk}^{\Phi} \leqslant \sum_{k=0}^{n} c_{nk}^{\Phi}, \quad \forall j = 0,\ldots,n. \tag{2.2}$$

Then, if the function $\frac{f}{g}$ is also increasing on $(0,1)$ and $c \geqslant 1$, the following inequality holds true

$$\int_0^1 w(x)\Phi(f(x))dx \geqslant \frac{\int_0^1 w(x)\Phi(g(x))dx}{\Phi(1)} \Phi\Big(\int_0^1 w(x)f(x)dx\Big). \tag{2.3}$$

Conversely, if the function $\frac{f}{g}$ is decreasing on $(0,1)$ and $c \leqslant 1$, then the inequality (2.3) is reversed.

Proof. We show the result for the first case. To this end, let us first define the two functions f^* and F on $[0,1]$ by setting, for each $x\in[0,1]$,

$$f^*(x) = g(x)\int_0^1 w(t)f(t)dt \quad \text{and} \quad F(x) = \int_0^x w(t)[f(t)-f^*(t)]dt.$$

Clearly, we have $F(0)=F(1)=0$ by the hypothesis $\int_0^1 w(x)g(x)dt = 1$. Since $f^*(x) = cg(x)$, it is clear that f^* is increasing on $[0,1]$. On the other hand, by calculating directly, we obtain, for all $x\in(0,1)$,

$$F'(x) = w(x)[f(x)-f^*(x)] = w(x)g(x)\Big(\frac{f(x)}{g(x)} - \int_0^1 w(t)f(t)dt\Big).$$

It follows that the sign of F' depends only on the sign of the expression in the bracket. Therefore, the derivative F' changes sign exactly once in $(0,1)$ by the monotone of $\frac{f}{g}$

on $(0,1)$. Since $\frac{f}{g}$ is increasing on $(0,1)$, the derivative F' is negative near 0. This, along with $F(0)=F(1)=0$, leads to $F(x)\leqslant 0$ for all $x\in[0,1]$. This fact, together with the hypothesis $\int_0^1 w(x)g(x)dx=1$, yields

$$\int_x^1 w(t)[f(t)-f^*(t)]dt\geqslant 0,\quad x\in[0,1]. \tag{2.4}$$

Besides, by the gradient inequality for the convex function Φ (see, for example, [4]) and the non-negativity of w, we deduce that, for all $x\in[0,1]$,

$$w(x)\Phi(f(x))-w(x)\Phi(f^*(x))\geqslant w(x)\varphi(f^*(x))(f(x)-f^*(x)), \tag{2.5}$$

where φ is a subdifferential of Φ. Notice that, by the increase of φ on $[0,b]$ (see, for example, [4]) and of f^* on $[0,1]$, we get $\varphi(f^*(1))\geqslant\varphi(f^*(0))$. Moreover, due to the second mean-value theorem for the integral (see [2, p. 35]), together with the inequalities (2.4), (2.5), there is a $\xi\in[0,1]$ such that

$$\begin{aligned}
&\int_0^1 w(x)\Phi(f(x))dx-\int_0^1 w(x)\Phi(f^*(x))dx\\
\geqslant&\int_0^1 w(x)\varphi(f^*(x))(f(x)-f^*(x))dx\\
=&\varphi(f^*(0))\int_0^\xi w(x)[f(x)-f^*(x)]dx+\varphi(f^*(1))\int_\xi^1 w(x)[f(x)-f^*(x)]dx\\
\geqslant&\varphi(f^*(0))\int_0^\xi w(x)[f(x)-f^*(x)]dx+\varphi(f^*(0))\int_\xi^1 w(x)[f(x)-f^*(x)]dx\\
=&\varphi(f^*(0))\int_0^1 w(x)[f(x)-f^*(x)]dx=0,
\end{aligned}$$

which is equivalent to

$$\int_0^1 w(x)\Phi(f(x))dx\geqslant\int_0^1 w(x)\Phi(g(x)c)dx, \tag{2.6}$$

where $c=\int_0^1 w(t)f(t)dt$.

It remains to prove that the following inequality holds for all $c\geqslant 1$

$$\int_0^1 w(x)\Phi(g(x)c)dx\geqslant\frac{\Phi(c)}{\Phi(1)}\int_0^1 w(x)\Phi(g(x))dx. \tag{2.7}$$

Indeed, by using Jensen's inequality and the hypotheses on the functions g and w, we easily deduce that, for all $k\geqslant 1$,

$$1\leqslant\Big(\int_0^1 w(x)[g(x)]^k dx\Big)^{1/k}\leqslant\Big(\int_0^1 w(x)[g(x)]^{k+1}dx\Big)^{1/(k+1)}. \tag{2.8}$$

This inequality leads to $\int_0^1 w(x)[g(x)]^k dx \geqslant 1$ for all $k \geqslant 1$. This, together with (2.8), gives us that

$$\begin{aligned}\int_0^1 w(x)[g(x)]^k dx &\leqslant \left(\int_0^1 w(x)[g(x)]^{k+1} dx\right)^{\frac{k}{k+1}} \\ &\leqslant \left(\int_0^1 w(x)[g(x)]^{k+1} dx\right)^{\frac{k}{k+1}} \left(\int_0^1 w(x)[g(x)]^{k+1} dx\right)^{\frac{1}{k+1}} \\ &= \int_0^1 w(x)[g(x)]^{k+1} dx,\end{aligned}$$

that is, the sequence $\{\int_0^1 w(x)[g(x)]^k dx\}_{k=0}^n$ is monotonically increasing. Clearly, the sequence $\{c^k\}_{k=0}^n$ is also monotonically increasing because $c \geqslant 1$. On the other hand, we can write the Bernstein polynomials of order n of the function Φ as

$$B_n^{\Phi}(x) = \sum_{k=0}^n c_{nk}^{\Phi} x^k.$$

By the hypotheses of the theorem, the coefficients c_{nk}^{Φ} in the above expansion satisfy the condition (2.2). Thus, using Chebyshev's inequality for monotonic sequences (see, for example, [13]), we have

$$\begin{aligned}\int_0^1 w(x)B_n^{\Phi}(g(x)c)dx &= \sum_{k=0}^n c_{nk}^{\Phi} c^k \int_0^1 w(x)[g(x)]^k dx \\ &\geqslant \frac{1}{\sum_{k=0}^n c_{nk}^{\Phi}} \left(\sum_{k=0}^n c_{nk}^{\Phi} \int_0^1 w(x)[g(x)]^k dx\right)\left(\sum_{k=0}^n c_{nk}^{\Phi} c^k\right) \qquad (2.9) \\ &= \frac{B_n^{\Phi}(c)}{B_n^{\Phi}(1)} \int_0^1 w(x)B_n^{\Phi}(g(x))dx.\end{aligned}$$

Since the function Φ is continuous on $[0,b]$, the Bernstein polynomials B_n^{Φ} converges uniformly on $[0,b]$ to the function Φ by [10, Theorem 1.1.1]. Hence, by letting $n \to \infty$ in the above inequality (2.9), we obtain the inequality (2.7).

The other case is proved similarly, we omit the details.

We can use a similar argument as in the proof of Theorem 2.1 to obtain the following result.

COROLLARY 2.2. *Let $f,g,w : [0,1] \to [0,\infty)$ be integrable increasing functions, and $\Phi : \mathbb{R} \to \mathbb{R}$ be an increasing convex function. If the function $\frac{f}{g}$ is also increasing on $(0,1)$, we have the following inequality*

$$\int_0^1 w(x)\Phi(f(x))dx \geqslant \int_0^1 w(x)\Phi\Big(g(x)\int_0^1 f(t)dt\Big)dx. \qquad (2.10)$$

If the function $\frac{f}{g}$ is decreasing on $(0,1)$, then the inequality (2.10) is reversed.

Following [10, p. 13], the coefficients c_{nk}^{Φ} converges to $\frac{\Phi^{(k)}(0)}{k!}$ as $n \to \infty$ provided these derivatives exist. We thus deduce an immediate consequence from the proof Theorem 2.1 as follows.

COROLLARY 2.3. *Let f, g and w be as in Theorem 2.1. Suppose that $\Phi : [0,b] \to \mathbb{R}$ is a smooth convex function satisfying $\Phi^{(k)}(0) \geqslant 0$ for all $k \geqslant 0$. Then, if the function $\frac{f}{g}$ is increasing on $(0,1)$ and $c \geqslant 1$, the inequality (2.3) holds true. Conversely, if the function $\frac{f}{g}$ is decreasing on $(0,1)$ and $c \leqslant 1$, then the inequality (2.3) is reversed.*

REMARK 2.4. (i) Theorem 2.1 not only gives an affirmative answer to the first question, but also shows the best coefficient to the second question with respective to the convex function Φ and the function f. It should be pointed out that not all convex functions satisfy the condition (2.2). Indeed, it can check that the functions $\exp(x)$ and $x^{\alpha}, \alpha \geqslant 1$, obey (2.2), whereas the function $\frac{1}{1+x}$ does not verify (2.2).

(ii) We consider a specific case of Theorem 2.1 which the result is similar to (1.5). The hypothesis of the function f in inequality (1.5) implies that the function $\frac{f(x)}{x}$ is increasing on $[0,1]$. However, the function $\frac{f(x)}{x^{\beta}}, \beta > 1$, may still be increasing on $[0,1]$. Hence, if this case is true, we can take $w(x) = 1, g(x) = (\beta+1)x^{\beta}, \Phi(x) = x^{\alpha}, \alpha \geqslant 1$, and obtain the following inequality

$$\int_0^1 [f(x)]^{\alpha} dx \geqslant \frac{(1+\beta)^{\alpha}}{1+\alpha\beta} \left(\int_0^1 f(x) dx \right)^{\alpha}.$$

Notice that this inequality holds true without the condition $\int_0^1 f(x)dx \geqslant 1$ by Corollary 2.2; moreover, the coefficient $\frac{(1+\beta)^{\alpha}}{1+\alpha\beta}$ is greater than the coefficient $\frac{2^{\alpha}}{1+\alpha}$.

A discrete analogue of Theorem 2.1 is as follows.

THEOREM 2.5. *Let $\{y_k\}_{k=1}^n$ be an increasing sequence of real numbers in the interval $[0,b]$ and $\{w_k\}_{k=1}^n$ be a sequence of non-negative real numbers with $\sum_{k=1}^n w_k = 1$. Let $\{x_k\}_{k=1}^n$ be an increasing sequence of non-negative real numbers with $\sum_{k=1}^n w_k x_k = 1$. Suppose that $x_k, cx_k \in [0,b]$ for all $k = 1, \dots, n$, where $c = \sum_{k=1}^n w_k y_k \geqslant 0$. Let $\Phi : [0,b] \to \mathbb{R}$ be as in Theorem 2.1 or as in Corollary 2.3. Then, if $\{\frac{y_k}{x_k}\}_{k=1}^n$ is increasing and $c \geqslant 1$, the following inequality holds true*

$$\sum_{k=1}^n w_k \Phi(y_k) \geqslant \frac{\sum_{k=1}^n w_k \Phi(x_k)}{\Phi(\sum_{k=1}^n w_k x_k)} \Phi\Big(\sum_{k=1}^n w_k y_k \Big). \tag{2.11}$$

Conversely, if $\{\frac{y_k}{x_k}\}_{k=1}^n$ is decreasing and $c \leqslant 1$, then the inequality (2.11) is reversed.

3. Some applications

3.1. Applications to majorization inequalities

Majorization is a preorder relation between vectors on $\mathbb{R}^n$ originally introduced by Hardy, Littlewood and Pólya (see [7]). For each vector $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, let $\boldsymbol{x}^* = (x_1^*, \dots, x_n^*)$ be a vector obtained by rearranging the components of $\boldsymbol{x}$ in increasing order. Then for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we say that $\boldsymbol{x}$ is majorized by $\boldsymbol{y}$, written $\boldsymbol{x} \prec \boldsymbol{y}$, if

$$\sum_{i=1}^{k} x_i^* \geqslant \sum_{i=1}^{k} y_i^*, \quad k = 1, \dots, n,$$

with the equality occurs when $k = n$. This relation is characterized by Hardy, Littlewood and Pólya as follows.

THEOREM 3.1. (see [3]) *The following are equivalent for* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

(i) $\boldsymbol{x} \prec \boldsymbol{y}$;

(ii) $\sum_{i=1}^n \Phi(x_i) \leqslant \sum_{i=1}^n \Phi(y_i)$ *for all continuous convex functions* Φ *defined on* $\mathbb{R}$;

(iii) $\boldsymbol{x}$ *is in the convex hull of the set* $\{\boldsymbol{z} : \boldsymbol{z}^* = \boldsymbol{x}^*\}$ *in* $\mathbb{R}^n$;

(iv) *There is a doubly stochastic matrix* A *of order* n *such that* $\boldsymbol{x} = A\boldsymbol{y}$.

This result is called the well-known majorization theorem for vectors. Recall that a doubly stochastic matrix $A = (a_{ij})$ of order n is a square matrix of order n satisfying that each entry a_{ij} is non-negative and the sum of each row or of each column is unit. For two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we write $\boldsymbol{x} \leqslant \boldsymbol{y}$ if $x_i \leqslant y_i, i = 1, \dots, n$. We also use the symbol $\boldsymbol{e}$ to denote the vector $(1, \dots, 1) \in \mathbb{R}^n$.

We can now strengthen the majorization theorem above as follows.

THEOREM 3.2. *Let* $\Phi : [0, b] \to \mathbb{R}$ *be as in Theorem 2.5. Let* $\boldsymbol{x} = \{x_k\}_{k=1}^n$ *and* $\boldsymbol{y} = \{y_k\}_{k=1}^n$ *be two increasing sequences of real numbers from the interval* $[0, b]$ *such that the sequence* $\{\frac{y_k}{x_k}\}_{k=1}^n$ *is also increasing. If* $A = (a_{ij})$ *be a doubly stochastic matrix of order* n *satisfying that* $\boldsymbol{x} = A\boldsymbol{y}$ *and* $A\boldsymbol{y} \geqslant A\boldsymbol{x} \geqslant \boldsymbol{e}$, *then the following inequality holds true*

$$\sum_{k=1}^{n} \Phi(x_k) \leqslant c \sum_{k=1}^{n} \Phi(y_k),$$

where

$$c = \max_{1 \leqslant i \leqslant n} \frac{\Phi(\sum_{j=1}^n a_{ij} x_j)}{\sum_{j=1}^n a_{ij} \Phi(x_j)} = \frac{\Phi(1)}{\min\limits_{1 \leqslant i \leqslant n} \sum_{j=1}^n a_{ij} \Phi(x_j)}.$$

Proof. Clearly, the hypotheses of this theorem satisfy the conditions of Theorem 2.5. Moreover, we can write $x_i = \sum_{j=1}^n a_{ij} y_j$ for all $i = 1, \dots, n$ by $\boldsymbol{x} = A\boldsymbol{y}$. Hence, from Theorem 2.5 we infer that

$$\Phi(x_i) = \Phi\Big(\sum_{j=1}^{n} a_{ij} y_j\Big) \leqslant \frac{\Phi(\sum_{j=1}^n a_{ij} x_j)}{\sum_{j=1}^n a_{ij} \Phi(x_j)} \sum_{j=1}^{n} a_{ij} \Phi(y_j) \leqslant c \sum_{j=1}^{n} a_{ij} \Phi(y_j)$$

for all $i=1,\dots,n$. Adding these inequalities, we obtain

$$\sum_{i=1}^{n}\Phi(x_i)\leqslant c\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\Phi(y_j)=c\sum_{j=1}^{n}\Big(\sum_{i=1}^{n}a_{ij}\Big)\Phi(y_j)=c\sum_{j=1}^{n}\Phi(y_j),$$

which finishes the proof of the theorem.

REMARK 3.3. When Φ is a strict convex function and x_i's are different from each other, the constant c in Theorem 3.2 is smaller than 1.

3.2. Inequalities of Andersson type

Andersson [1] showed that if f_i are convex increasing functions defined on $[0,1]$ with $f_i(0)=0$ then

$$\int_0^1\prod_{i=1}^{n}f_i(x)dx\geqslant\frac{2^n}{n+1}\prod_{i=1}^{n}\int_0^1 f_i(x)dx.$$

This result was extended to the class of continuously differentiable functions f on $[0,1]$ such that $\frac{f(t)}{t}$ is increasing on $[0,1]$ with $f(0)=0$ by Fink [6] in 2003.

The main goal of this subsection is to give a generalization of the above results.

THEOREM 3.4. *Let $f_i,g:[0,1]\to[0,\infty)$ be integrable increasing functions on $[0,1]$ such that $\frac{f_i}{g}$ is also increasing on $[0,1]$ and $\int_0^1 g(x)dx=1$. Then, the following inequality holds true*

$$\int_0^1\prod_{i=1}^{n}f_i(x)dx\geqslant\int_0^1[g(x)]^n dx\prod_{i=1}^{n}\int_0^1 f_i(x)dx.$$

Proof. Since f_i are increasing on $[0,1]$, the product $\prod_{i=1}^{n-1}f_i(x)$ is also increasing on $[0,1]$. Hence, by applying Corollary 2.2 for the functions $w(x)=\prod_{i=1}^{n-1}f_i(x), f(x)=f_n(x)$ and $\Phi(x)=x$, we have

$$\int_0^1\prod_{i=1}^{n}f_i(x)dx\geqslant\int_0^1 g(x)\prod_{i=1}^{n-1}f_i(x)dx\int_0^1 f_n(x)dx. \tag{3.1}$$

Similarly, by applying Corollary 2.2 for the functions $w(x)=g(x)\prod_{i=1}^{n-2}f_i(x), f(x)=f_{n-1}(x)$ and $\Phi(x)=x$, we get

$$\int_0^1 g(x)\prod_{i=1}^{n-1}f_i(x)dx\geqslant\int_0^1[g(x)]^2\prod_{i=1}^{n-2}f_i(x)dx\int_0^1 f_{n-1}(x)dx. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\int_0^1\prod_{i=1}^{n}f_i(x)dx\geqslant\int_0^1[g(x)]^2\prod_{i=1}^{n-2}f_i(x)dx\prod_{i=n-1}^{n}\int_0^1 f_i(x)dx.$$

We continue this process until getting the desired inequality.

COROLLARY 3.5. *Let f_i be as in Theorem 3.4. Let $\mathscr{S}$ denote the set of all non-negative increasing functions g defined on $[0,1]$ such that $\int_0^1 g(x)dx = 1$ and each function $\frac{f_i}{g}$ is also increasing on $[0,1]$. Then, we have*

$$\int_0^1 \prod_{i=1}^n f_i(x)dx \geqslant \left(\sup_{g\in\mathscr{S}} \int_0^1 [g(x)]^n dx \right) \prod_{i=1}^n \int_0^1 f_i(x)dx.$$

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