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SOME GENERAL GRADIENT ESTIMATES FOR TWO NONLINEAR PARABOLIC EQUATIONS ALONG RICCI FLOW

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Abstract. In this paper, by maximum principle and cutoff function, we investigate gradient estimates for positive solutions to two nonlinear parabolic equations along the Ricci flow. As applications, the related Harnack inequalities for positive solutions to the nonlinear parabolic equations along the Ricci flow are derived. These results can be regarded as generalizations of the results of Li-Yau, J. Y. Li, Hamilton and Li-Xu to a more general nonlinear parabolic equation along the Ricci flow. Our results also improve the estimates of S. P. Liu, J. Sun and Y. Y. Yang to the nonlinear parabolic equation along the Ricci flow.

1. Introduction

Beginning with the pioneering work of Li and Yau [14], gradient estimates are also known as differential Harnack inequalities, which have tremendous impact in geometric analysis, as shown for example in [14, 15, 16]. Indeed, they have very important applications in singularity analysis, especially for the Perelman's breaking work [22, 23] on the Poincaré conjecture.

We first simply introduce research progress associated with this article.

Let (M^n,g) be a complete Riemannian manifold. Li and Yau [14] established a famous gradient estimate for positive solutions to the following heat equation

$$u_t = \Delta u \tag{1.1}$$

on (M^n,g), which is described as

THEOREM A. *(Li-Yau [14])* Let (M^n,g) be a complete Riemannian manifold. Suppose that on the ball $B_R\times(0,T)$ ($B_R := \{(x,t)|d(x,x_0,t)\leqslant R\}$), $Ricci(B_R)\geqslant -K$. Then for any $\alpha>1$,

$$\sup_{B_{2R}}\left(\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}\right)\leqslant\frac{C\alpha^2}{R^2}\left(\frac{\alpha^2}{\alpha^2-1}+\sqrt{K}R\right)+\frac{n\alpha^2K}{\alpha-1}+\frac{n\alpha^2}{2t}. \tag{1.2}$$

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In general, on a complete Riemannian manifold, if $Ricci(M) \geqslant -k$, by letting $R \to \infty$ in (1.2), one has

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leqslant \frac{n\alpha^2 k}{2(\alpha-1)} + \frac{n\alpha^2}{2t}. \tag{1.3}$$

In 1991, Li [15] generalized Li and Yau's estimates to the nonlinear parabolic equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x,t) + h(x,t)u^{\alpha}(x,t) = 0 \tag{1.4}$$

on (M^n, g). Hamilton in [8] generalized the constant α of Li and Yau's result to the function $\alpha(t) = e^{2Kt}$. Sun [27] also obtained a gradient estimate with different coefficients. Li and Xu in [17] further promoted Li and Yau's result, and found two new functions $\alpha(t)$. Recently, the first author and Zhang in [28] further generalized Li and Xu's results to the nonlinear parabolic equation (1.4). Related results can be referred to [6, 11, 33].

Motivated by [1, 8, 17, 27], in this paper, we investigate two nonlinear parabolic equations

$$\partial_t u(x,t) = \Delta u(x,t) + h(x,t)u^l(x,t) \tag{1.5}$$

and

$$\partial_t u(x,t) = \Delta u(x,t) + au(x,t)\log u(x,t) \tag{1.6}$$

along the Ricci flow, where function $h(x,t) \geqslant 0$ is defined on $M^n \times [0,T]$, which is C^2 in the first variable and C^1 in the second variable, and T is a positive constant and $l, a \in \mathbb{R}$, respectively.

Assume M^n is an n-dimensional manifold without boundary, and let $(M^n, g(t))_{t\in[0,T]}$ be an n-dimensional complete manifold with metric $g(t)$ evolving by the Ricci flow

$$\frac{\partial g(t)}{\partial t} = -2R_{ij}, \quad (x,t) \in M^n \times [0,T]. \tag{1.7}$$

Recently, there are a number of gradient estimates on manifolds along the Ricci flow, see for the example [9, 10] and others, because the Ricci flow is a powerful tool in analyzing the structure of manifolds.

In 2008, Kuang and Zhang [11] proved a gradient estimate for positive solutions to the conjugate heat equation along the Ricci flow on a closed manifold. In 2009, Liu [18] derived a gradient estimate for positive solutions to the heat equation along the Ricci flow. Afterwards, Sun[26] generalized Liu's results to a general geometric flow. In 2010, Bailesteanu, Cao and Pulemotov [2] established some gradient estimates for positive solutions to the heat equation along the Ricci flow. In 2016, Li and Zhu [19] generalized J. Y. Li's [15] estimates along the Ricci flow. Recently, Cao and Zhu [4] derived some Aronson and Bénilan estimates for porous medium equation

$$u_t = \Delta u^m, \quad m > 1$$

along the Ricci flow. Li, Bai and Zhang [13] studied fast diffusion equation

$$u_t = \Delta u^m, \quad 0 < m < 1$$

along the Ricci flow. Zhao and Fang [32] generalized Yang's result [30] to the Ricci flow.

This paper is organized as follows: We prove gradient estimates for the equation (1.5) in Section 2 and gradient estimates for the equation (1.6) in Section 3. We derive related Harnack inequalities in Section 4. As a special case, we deduce gradient estimates and Harnack inequalities to the heat equation in Section 5. Detailed calculation of some specific functions $\alpha(t)$, $\varphi(t)$ and $\gamma(t)$ are given in Section 6.

2. Gradient estimates to the equation (1.5)

In this section, we will derive some new gradient estimates for positive solutions to equation (1.5) along the Ricci flow.

2.1. Main results

Firstly, we introduce three C^1 functions $\alpha(t)$, $\varphi(t)$ and $\gamma(t)$: $(0,+\infty)\to(0,+\infty)$. Suppose that three C^1 functions $\alpha(t)$, $\varphi(t)$ and $\gamma(t)$ satisfy the following conditions:

(C1) $\alpha(t)>1$.

(C2) $\alpha(t)$ and $\varphi(t)$ satisfy the following system

$$\begin{cases} \dfrac{2\varphi}{n}-2\alpha K \geqslant (\dfrac{2\varphi}{n}-\alpha')\dfrac{1}{\alpha}, \\ \dfrac{2\varphi}{n}-\alpha'>0, \\ \dfrac{\varphi^2}{n}+\alpha\varphi'\geqslant 0. \end{cases}$$

(C3) $\gamma(t)$ satisfies

$$\frac{\gamma'}{\gamma}-(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}\leqslant 0.$$

(C4) $\gamma(t)$ is non-decreasing, and $\alpha(t)$ is also non-decreasing or is bounded uniformly. Here $\alpha'=\frac{d\alpha}{dt}$, $\varphi'=\frac{d\varphi}{dt}$ and $\gamma'=\frac{d\gamma}{dt}$.

We state our results as follows.

THEOREM 2.1. *Let $(M^n,g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.7). Suppose that there exist three functions $\alpha(t)$, $\varphi(t)$ and $\gamma(t)$ satisfy conditions (C1), (C2), (C3) and (C4).*

Given $x_0\in M^n$ and $R>0$, let u be a positive solution of the equation (1.5) in the cube $B_{2R,T}:=\{(x,t)|d(x,x_0,t)\leqslant 2R, 0\leqslant t\leqslant T\}$. Assume that $|\mathrm{Ric}(x,t)|\leqslant K$ for some $K>0$ on $B_{2R,T}$. Let $h(x,t)$ be a function defined on $M^n\times[0,T]$ which is C^1 in t and C^2 in x, satisfying $|\nabla h|^2\leqslant\delta_2 h$ and $\Delta h\geqslant-\delta_3$ on $B_{2R,T}$ for some positive constants δ_2 and δ_3.

(1) $l \leqslant 1$. *If* $\frac{\gamma\alpha^4}{\alpha-1} \leqslant C_1$ *for some constant* C_1, *then*

$$\begin{aligned}
&\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha h(x,t)u^{l-1} \\
\leqslant & C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + K\right) + \frac{Cn^2}{R^2\gamma} + n^{\frac{3}{2}}\alpha^2 K \\
&+ \alpha\sqrt{n\overline{u}_1\delta_3} + \frac{n\alpha^2[\alpha(3-2l)-1]}{\alpha-1}\overline{u}_1\delta_1 + \sqrt{\frac{2-l}{2}}\alpha^{\frac{3}{2}}\sqrt{n\overline{u}_1\delta_2} + \alpha\varphi,
\end{aligned}$$

where $C = C(n, C_1)$ *is a constant.*

If $\frac{\gamma}{\alpha-1} \leqslant C_2$ *for some constant* C_2, *then*

$$\begin{aligned}
&\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha h(x,t)u^{l-1} \\
\leqslant & C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + K\right) + \frac{Cn^2\alpha^4}{R^2\gamma} + n^{\frac{3}{2}}\alpha^2 K \\
&+ \alpha\sqrt{n\overline{u}_1\delta_3} + \frac{n\alpha^2[\alpha 3-2l-1]}{\alpha-1}\overline{u}_1\delta_1 + \sqrt{\frac{2-l}{2}}\alpha^{\frac{3}{2}}\sqrt{n\overline{u}_1\delta_2} + \alpha\varphi,
\end{aligned}$$

where $C = C(n, C_2)$ *is a constant and set*

$$\overline{u}_1 := \max_{B_{2R,T}} u^{l-1}, \quad \delta_1 := \max_{B_{2R,T}} h(x,t).$$

(2) $l > 1$. *If* $\frac{\gamma\alpha^4}{\alpha-1} \leqslant C_1$ *for some constant* C_1, *then*

$$\begin{aligned}
&\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha h(x,t)u^{l-1} \\
\leqslant & C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + K\right) + \frac{Cn^2}{R^2\gamma} + n^{\frac{3}{2}}\alpha^2 K + n\alpha^2(l-1)\delta_1\overline{u}_2 \\
&+ \alpha\sqrt{\frac{n(l\alpha-1)\overline{u}_2\delta_2}{l-1}} + \alpha^{\frac{3}{2}}\sqrt{n(l-1)\delta_1\varphi} + \alpha^{\frac{3}{2}}\sqrt{n\delta_3\overline{u}_2} + \alpha\varphi,
\end{aligned}$$

where $C = C(n, C_1)$ *is a constant.*

If $\frac{\gamma}{\alpha-1} \leqslant C_2$ *for some constant* C_2, *then*

$$\begin{aligned}
&\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha h(x,t)u^{l-1} \\
\leqslant & C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + K\right) + \frac{Cn^2\alpha^4}{R^2\gamma} + n^{\frac{3}{2}}\alpha^2 K + n\alpha^2(l-1)\delta_1\overline{u}_2 \\
&+ \alpha\sqrt{\frac{n(l\alpha-1)\overline{u}_2\delta_2}{l-1}} + \alpha^{\frac{3}{2}}\sqrt{n(l-1)\delta_1\varphi} + \alpha^{\frac{3}{2}}\sqrt{n\delta_3\overline{u}_2} + \alpha\varphi,
\end{aligned}$$

where $C = C(n, C_2)$ *is a constant and set*

$$\overline{u}_2 := \max_{B_{2R,T}} u^{l-1}, \quad \delta_1 := \max_{B_{2R,T}} h(x,t).$$

Let us list some examples to illustrate that Theorem 2.1 holds for special cases and see appendix in Section 6 for detailed calculation process.

COROLLARY 2.1. *Suppose that* $(M^n, g(t))_{t\in[0,T]}$ *satisfies the hypotheses of Theorem* 2.1*. Then the following special estimates are valid.*

1. *Li-Yau type:*

$$\alpha(t) = constant, \quad \varphi(t) = \frac{\alpha n}{t} + \frac{nK\alpha^2}{\alpha - 1}, \gamma(t) = t^\theta \quad with \quad 0 < \theta \leqslant 2.$$

If $l \leqslant 1$*, then*

$$\begin{aligned}
&\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha h(x,t)u^{l-1} \\
\leqslant& C\alpha^2\left[\frac{1}{R^2}(1+\sqrt{K}R) + \frac{\alpha^2}{\alpha-1}\frac{1}{R^2} + K\right] + \alpha\varphi \\
&+ n^{\frac{3}{2}}\alpha^2 K + \alpha\sqrt{n\overline{u}_1\delta_3} + \frac{n\alpha^2[\alpha(3-2l)-1]}{\alpha-1}\overline{u}_1\delta_1 + \sqrt{\frac{2-l}{2}}\alpha^{\frac{3}{2}}\sqrt{n\overline{u}_1\delta_2}.
\end{aligned}$$

If $l > 1$*, then*

$$\begin{aligned}
&\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha h(x,t)u^{l-1} \\
\leqslant& C\alpha^2\left[\frac{1}{R^2}(1+\sqrt{K}R) + \frac{\alpha^2}{\alpha-1}\frac{1}{R^2} + K\right] + \alpha\varphi \\
&+ n^{\frac{3}{2}}\alpha^2 K + n\alpha^2(l-1)\delta_1\overline{u}_2 + \alpha\sqrt{\frac{n(l\alpha-1)\overline{u}_2\delta_2}{l-1}} \\
&+ \alpha^{\frac{3}{2}}\sqrt{n(l-1)\delta_1\varphi} + \alpha^{\frac{3}{2}}\sqrt{n\delta_3\overline{u}_2}.
\end{aligned}$$

2. *Hamilton type:*

$$\alpha(t) = e^{2Kt}, \quad \varphi(t) = \frac{n}{t}e^{4Kt}, \quad \gamma(t) = te^{2Kt}.$$

If $l \leqslant 1$*, then*

$$\begin{aligned}
&\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha h(x,t)u^{l-1} \\
\leqslant& C\alpha^2\left[\frac{1}{R^2}(1+\sqrt{K}R) + K\right] + \frac{C\alpha^4}{R^2 te^{2Kt}} + \alpha\varphi
\end{aligned}$$

$$+n^{\frac{3}{2}}\alpha^2K+\alpha\sqrt{n\bar{u}_1\delta_3}+\frac{n\alpha^2[\alpha(3-2l)-1]}{\alpha-1}\bar{u}_1\delta_1+\sqrt{\frac{2-l}{2}}\alpha\sqrt{n\bar{u}_1\delta_2}.$$

If $l>1$, then

$$\begin{aligned}&\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}+\alpha h(x,t)u^{l-1}\\ \leqslant & C\alpha^2\left[\frac{1}{R^2}(1+\sqrt{K}R)+K\right]+\frac{C\alpha^4}{R^2te^{2Kt}}+\alpha\varphi\\ &+n^{\frac{3}{2}}\alpha^2K+n\alpha^2(l-1)\delta_1\bar{u}_2+\alpha\sqrt{\frac{n(l\alpha-1)\bar{u}_2\delta_2}{l-1}}\\ &+\alpha^{\frac{3}{2}}\sqrt{n(l-1)\delta_1\varphi}+\alpha^{\frac{3}{2}}\sqrt{n\delta_3\bar{u}_2}.\end{aligned}$$

3. *Li-Xu type:*

$$\alpha(t)=1+\frac{\sinh(Kt)\cosh(Kt)-Kt}{\sinh^2(Kt)},\quad \varphi(t)=2nK[1+\coth(Kt)],$$
$$\gamma(t)=\tanh(Kt).$$

If $l\leqslant 1$, then

$$\begin{aligned}&\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}+\alpha h(x,t)u^{l-1}\\ \leqslant & C\left[\frac{1}{R^2}(1+\sqrt{K}R)+K\right]+\frac{C}{R^2\tanh(Kt)}+\alpha\varphi\\ &+n^{\frac{3}{2}}\alpha^2K+\alpha\sqrt{n\bar{u}_1\delta_3}+\frac{n\alpha^2[\alpha(3-2l)-1]}{\alpha-1}\bar{u}_1\delta_1+\sqrt{\frac{2-l}{2}}\alpha\sqrt{n\bar{u}_1\delta_2}.\end{aligned}$$

If $l>1$, then

$$\begin{aligned}&\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}+\alpha h(x,t)u^{l-1}\\ \leqslant & C\alpha^2\left[\frac{1}{R^2}(1+\sqrt{K}R)+K\right]+\frac{C}{R^2\tanh(Kt)}+\alpha\varphi\\ &+n^{\frac{3}{2}}\alpha^2K+n\alpha^2(l-1)\delta_1\bar{u}_2+\alpha\sqrt{\frac{n(l\alpha-1)\bar{u}_2\delta_2}{l-1}}\\ &+\alpha^{\frac{3}{2}}\sqrt{n(l-1)\delta_1\varphi}+\alpha^{\frac{3}{2}}\sqrt{n\delta_3\bar{u}_2},\end{aligned}$$

where $\alpha(t)$ is bounded uniformly.

4. *Linear Li-Xu type:*

$$\alpha(t)=1+2Kt,\varphi(t)=\frac{n}{t}+nK(1+2Kt+\mu Kt),\gamma(t)=Kt\quad \textit{with}\quad \mu\geqslant\frac{1}{4}.$$

If $l \leqslant 1$, then

$$
\begin{aligned}
&\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} + \alpha h(x,t) u^{l-1} \\
\leqslant & C\alpha^2 \left[\frac{1}{R^2}(1+\sqrt{K}R) + K \right] + \frac{C\alpha^4}{R^2 K t} + \alpha \varphi \\
&+ n^{\frac{3}{2}} \alpha^2 K + \alpha \sqrt{n \overline{u}_1 \delta_3} + \frac{n\alpha^2 [\alpha(3-2l)-1]}{\alpha - 1} \overline{u}_1 \delta_1 + \sqrt{\frac{2-l}{2}} \alpha \sqrt{n \overline{u}_1 \delta_2}.
\end{aligned}
$$

If $l > 1$, then

$$
\begin{aligned}
&\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} + \alpha h(x,t) u^{l-1} \\
\leqslant & C\alpha^2 \left[\frac{1}{R^2}(1+\sqrt{K}R) + K \right] + \frac{C\alpha^4}{R^2 K t} + \alpha \varphi \\
&+ n^{\frac{3}{2}} \alpha^2 K + n\alpha^2 (l-1) \delta_1 \overline{u}_2 + \alpha \sqrt{\frac{n(l\alpha - 1)\overline{u}_2 \delta_2}{l-1}} \\
&+ \alpha^{\frac{3}{2}} \sqrt{n(l-1)\delta_1 \varphi} + \alpha^{\frac{3}{2}} \sqrt{n \delta_3 \overline{u}_2}.
\end{aligned}
$$

REMARK 2.1. The above results can be regard as some generalizations of the cases of Li-Yau [14], J. Y. Li [15], Hamilton [8] and Li-Xu [17] to the Ricci flow. Our results also generalize the estimates of S. P. Liu [18] and J. Sun [26] to the nonlinear parabolic equation along the Ricci flow. If $l < 0$ and c is a constant, there is an analog of (1.5) (see [31]), namely

$$\Delta u + c u^l = 0.$$

So our result also generalize Yang's result in [31].

The local estimates in Theorem 2.1 imply global estimates.

COROLLARY 2.2. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.7). Assume that $|\mathrm{Ric}(x,t)| \leqslant K$ for some $K > 0$ and all $(x,t) \in M^n \times [0,T]$. Let $u(x,t)$ be a positive solution to equation (1.5) on $M^n \times [0,T]$. Let $h(x,t)$ be a function defined on $M^n \times [0,T]$ which is C^1 in t and C^2 in x, satisfying $|\nabla h|^2 \leqslant \delta_2 h$ and $\Delta h \geqslant -\delta_3$ on $M^n \times [0,T]$ for some positive constants δ_2 and δ_3.*

If $l \leqslant 1$ and for $(x,t) \in M^n \times (0,T]$, then

$$
\begin{aligned}
&\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} + \alpha h(x,t) u^{l-1} \\
\leqslant & \alpha\varphi + C\alpha \left[\alpha K + \sqrt{\overline{u}_1 \delta_3} + \frac{\alpha[\alpha(3-2l)-1]}{\alpha-1} \overline{u}_1 \delta_1 + \sqrt{\frac{2-l}{2}} \sqrt{\overline{u}_1 \delta_2} \right],
\end{aligned}
$$

where C is a positive constant depending only on n and set

$$\overline{u}_1 := \max_{M^n \times [0,T]} u^{l-1}, \quad \delta_1 := \max_{M^n \times [0,T]} h(x,t).$$

If $l>1$ and for $(x,t)\in M^n\times(0,T]$, then

$$\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}+\alpha h(x,t)u^{l-1}\leqslant\alpha\varphi$$
$$+C\alpha\Big[\alpha K+(l-1)\alpha\overline{u}_2\delta_1+\sqrt{\frac{(l\alpha-1)\overline{u}_2\delta_2}{l-1}}+\alpha^{\frac{1}{2}}\sqrt{(l-1)\delta_1\varphi}+\alpha^{\frac{1}{2}}\sqrt{\overline{u}_2\delta_3}\Big],$$

where C is a positive constant depending only on n and set

$$\overline{u}_1:=\max_{M^n\times[0,T]}u^{l-1},\quad \delta_1:=\max_{M^n\times[0,T]}h(x,t).$$

We can derive a gradient estimate for an any positive solution to the following nonlinear parabolic equation along the Ricci flow on a closed manifold without any curvature conditions. The method of the proof is inspired by Hamilton [10], Shi [24] and Liu [18].

THEOREM 2.2. *Let $(M^n,g(x,t))_{t\in[0,T]}$ be a solution to the Ricci flow (1.7) on a closed manifold. If u is a positive solution to equation*

$$\partial_t u=\Delta u+h(t)u^l,$$

where $h(t)$ is a C^1 function and $h(t)\leqslant 0$. Then for $l\geqslant 1$, we have

$$|\nabla u(x,t)|^2\leqslant\frac{1}{2t}\left(\max_{x\in M^n}u^2(x,0)-u^2(x,t)\right)\quad for\quad (x,t)\in M^n\times[0,T].\tag{2.1}$$

2.2. Auxilliary lemma

To prove main results, we need a lemma.

Let $f=\log u$. Then

$$f_t=\Delta f+|\nabla f|^2+hu^{l-1}.\tag{2.2}$$

Let $F=|\nabla f|^2-\alpha f_t+\alpha hu^{l-1}-\alpha\varphi$, where $\alpha=\alpha(t)>1$ and $\varphi=\varphi(t)>0$.

LEMMA 2.1. *Suppose that $(M^n,g(t))_{t\in[0,T]}$ satisfies the hypotheses of Theorem 2.1. We also assume that $\alpha(t)>1$ and $\varphi(t)>0$ satisfy the following system*

$$\begin{cases}\dfrac{2\varphi}{n}-2\alpha K\geqslant(\dfrac{2\varphi}{n}-\alpha')\dfrac{1}{\alpha},\\ \dfrac{2\varphi}{n}-\alpha'>0,\\ \dfrac{\varphi^2}{n}+\alpha\varphi'\geqslant 0,\end{cases}\tag{2.3}$$

and $\alpha(t)$ is non-decreasing. Then

$$(\Delta-\partial_t)F\geqslant|f_{ij}+\frac{\varphi}{n}\delta_{ij}|^2+(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}F-\alpha^2n^2K^2-2\nabla f\nabla F$$

$$
\begin{aligned}
&+2h(\alpha-1)(l-1)u^{l-1}|\nabla f|^2+\alpha(l-1)^2hu^{l-1}|\nabla f|^2\\
&+2[(\alpha-1)+\alpha(l-1)]u^{l-1}\nabla f\cdot\nabla h\\
&+\alpha(l-1)hu^{l-1}\Delta f+\alpha u^{l-1}\Delta h. \qquad (2.4)
\end{aligned}
$$

Proof. By directly computing, we have

$$
\begin{aligned}
\Delta F &= \Delta|\nabla f|^2-\alpha\Delta(f_t)+\alpha\Delta(hu^{l-1})\\
&= 2|f_{ij}|^2+2f_jf_{iij}+2R_{ij}f_if_j-\alpha\Delta(f_t)+\alpha h\Delta(u^{l-1})\\
&\quad+\alpha u^{l-1}\Delta h+2\alpha\nabla h\nabla u^{l-1}\\
&= 2\Big(|f_{ij}|^2+\alpha R_{ij}f_{ij}\Big)+2f_jf_{iij}+2R_{ij}f_if_j-\alpha(\Delta f)_t\\
&\quad+\alpha h\Delta(u^{l-1})+\alpha u^{l-1}\Delta h+2\alpha\nabla h\nabla u^{l-1},
\end{aligned}
$$

where we have used Bochner's formula and

$$\Delta(f_t)=(\Delta f)_t-2R_{ij}f_{ij}.$$

Applying Young's inequality

$$R_{ij}f_{ij}\leqslant|R_{ij}f_{ij}|\leqslant\frac{\alpha}{2}|R_{ij}|^2+\frac{1}{2\alpha}|f_{ij}|^2,$$

we conclude for $|R_{ij}|\leqslant K$,

$$
\begin{aligned}
\Delta F &\geqslant |f_{ij}|^2-\alpha^2|R_{ij}|^2+2f_jf_{iij}+2R_{ij}f_if_j-\alpha(\Delta f)_t\\
&\quad+\alpha h\Delta(u^{l-1})+\alpha u^{l-1}\Delta h++2\alpha\nabla h\nabla u^{l-1}\\
&\geqslant |f_{ij}|^2-\alpha^2n^2K^2+2f_jf_{iij}+2R_{ij}f_if_j-\alpha(\Delta f)_t\\
&\quad+\alpha h\Delta(u^{l-1})+\alpha u^{l-1}\Delta h+2\alpha\nabla h\nabla u^{l-1}. \qquad (2.5)
\end{aligned}
$$

On the other hand, we infer

$$
\begin{aligned}
\partial_tF &= (|\nabla f|^2)_t-\alpha f_{tt}-\alpha' f_t+\alpha' hu^{l-1}+\alpha h(u^{l-1})_t\\
&\quad+\alpha u^{l-1}h_t-\alpha\varphi'-\alpha'\varphi\\
&= 2\nabla f\nabla(f_t)+2R_{ij}f_if_j-\alpha f_{tt}-\alpha' f_t+\alpha' hu^{l-1}+\alpha u^{l-1}h_t\\
&\quad+\alpha h(u^{l-1})_t-\alpha\varphi'-\alpha'\varphi, \qquad (2.6)
\end{aligned}
$$

where we used the fact that

$$(|\nabla f|^2)_t=2\nabla f\cdot\nabla(f_t)+2\mathrm{Ric}(\nabla f,\nabla f). \qquad (2.7)$$

We follow from (2.5) and (2.6),

$$
\begin{aligned}
(\Delta-\partial_t)F &\geqslant |f_{ij}|^2-\alpha^2n^2K^2+2\nabla f\nabla(\Delta f)-\alpha(\Delta f)_t+\alpha h\Delta(u^{l-1})\\
&\quad+\alpha u^{l-1}\Delta h+2\alpha\nabla h\nabla u^{l-1}-2\nabla f\nabla(f_t)+\alpha f_{tt}+\alpha' f_t
\end{aligned}
$$

$$
\begin{aligned}
&-\alpha' hu^{l-1} - \alpha h(u^{l-1})_t - \alpha u^{l-1}h_t + \alpha\varphi' + \alpha'\varphi \\
=& |f_{ij}|^2 - \alpha^2 n^2 K^2 + 2\nabla f\nabla(\Delta f) - \alpha(f_t - |\nabla f|^2 - hu^{l-1})_t \\
&+\alpha h\Delta(u^{l-1}) + \alpha u^{l-1}\Delta h + 2\alpha\nabla h\nabla u^{l-1} - 2\nabla f\nabla(f_t) + \alpha f_{tt} \\
&+\alpha' f_t - \alpha' hu^{l-1} - \alpha h(u^{l-1})_t - \alpha u^{l-1}h_t + \alpha\varphi' + \alpha'\varphi \\
=& |f_{ij}|^2 - \alpha^2 n^2 K^2 + 2\nabla f\nabla(\Delta f) + \alpha(|\nabla f|^2)_t + \alpha h\Delta(u^{l-1}) \\
&+\alpha u^{l-1}\Delta h + 2\alpha\nabla h\nabla u^{l-1} - 2\nabla f\nabla(f_t) + \alpha' f_t \\
&-\alpha' hu^{l-1} + \alpha\varphi' + \alpha'\varphi.
\end{aligned}
$$

By using the formula (2.7) and (2.2), we have

$$
\begin{aligned}
(\Delta - \partial_t)F \geqslant& |f_{ij}|^2 - \alpha^2 n^2 K^2 + 2\nabla f\nabla(\Delta f) + 2\alpha\nabla f\nabla(f_t) \\
&+2\alpha R_{ij}f_i f_j + \alpha h\Delta(u^{l-1}) + \alpha u^{l-1}\Delta h + 2\alpha\nabla h\nabla u^{l-1} \\
&-2\nabla f\nabla(f_t) + \alpha' f_t - \alpha' hu^{l-1} + \alpha\varphi' + \alpha'\varphi \\
=& |f_{ij}|^2 + 2\alpha R_{ij}f_i f_j - \alpha^2 n^2 K^2 - 2\nabla f\nabla F \\
&+2(\alpha-1)\nabla f\nabla(hu^{l-1}) + \alpha h\Delta(u^{l-1}) + \alpha u^{l-1}\Delta h \\
&+2\alpha\nabla h\nabla u^{l-1} + \alpha' f_t - \alpha' hu^{l-1} + \alpha\varphi' + \alpha'\varphi.
\end{aligned} \tag{2.8}
$$

Applying the following two equations

$$
\begin{aligned}
\nabla(u^{l-1}) &= (l-1)u^{l-1}\nabla f, \\
\Delta(u^{l-1}) &= (l-1)^2 u^{l-1}|\nabla f|^2 + (l-1)u^{l-1}\Delta f,
\end{aligned}
$$

to (2.8), we have

$$
\begin{aligned}
(\Delta - \partial_t)F \geqslant& |f_{ij}|^2 + 2\alpha R_{ij}f_i f_j - \alpha^2 n^2 K^2 + 2\nabla f\nabla F + \alpha u^{l-1}\Delta h \\
&+2h(\alpha-1)(l-1)u^{l-1}|\nabla f|^2 + 2\big[(\alpha-1) + \alpha(l-1)\big]u^{l-1}\nabla f\cdot\nabla h \\
&+h\alpha(l-1)^2 u^{l-1}|\nabla f|^2 + h\alpha(l-1)u^{l-1}\Delta f \\
&+\alpha' f_t - \alpha' hu^{l-1} + \alpha\varphi' + \alpha'\varphi.
\end{aligned} \tag{2.9}
$$

Further applying unit matrix $(\delta_{ij})_{n\times n}$ and (2.9), we derive

$$
\begin{aligned}
(\Delta - \partial_t)F \geqslant& |f_{ij} + \frac{\varphi}{n}\delta_{ij}|^2 - 2\alpha K|\nabla f|^2 - \alpha^2 n^2 K^2 + 2\nabla f\nabla F \\
&+2h(\alpha-1)(l-1)u^{l-1}|\nabla f|^2 + 2\big[(\alpha-1) + \alpha(l-1)\big]u^{l-1}\nabla f\cdot\nabla h \\
&+h\alpha(l-1)^2 u^{l-1}|\nabla f|^2 + h\alpha(l-1)u^{l-1}\Delta f + \alpha u^{l-1}\Delta h \\
&+\alpha' f_t - \alpha' hu^{l-1} + \alpha\varphi' + \alpha'\varphi - \frac{\varphi^2}{n} - 2\frac{\varphi}{n}\Delta f.
\end{aligned}
$$

Applying (2.2), we have

$$
(\Delta - \partial_t)F \geqslant |f_{ij} + \frac{\varphi}{n}\delta_{ij}|^2 + (\frac{2\varphi}{n} - 2\alpha K)|\nabla f|^2 - (\frac{2\varphi}{n} - \alpha')f_t
$$

$$
\begin{aligned}
&+(\frac{2\varphi}{n}-\alpha')hu^{l-1}-(\frac{2\varphi}{n}-\alpha')\frac{\alpha\varphi}{\alpha}-\alpha^2n^2K^2-2\nabla f\nabla F\\
&+2h(\alpha-1)(l-1)u^{l-1}|\nabla f|^2+2\big[(\alpha-1)+\alpha(l-1)\big]u^{l-1}\nabla f\cdot\nabla h\\
&+h\alpha(l-1)^2u^{l-1}|\nabla f|^2+h\alpha(l-1)u^{l-1}\Delta f+\alpha u^{l-1}\Delta h\\
&+\alpha\varphi'+\alpha'\varphi-\frac{\varphi^2}{n}+(\frac{2\varphi}{n}-\alpha')\frac{\alpha\varphi}{\alpha}.
\end{aligned}\tag{2.10}
$$

Therefore, (2.4) is derived from (2.3) and (2.10). The proof is complete.

2.3. Proof of Theorem 2.1 and 2.2

In this section, we will prove the Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Let $G=\gamma(t)F$ and $\gamma(t)>0$ be non-decreasing. Then

$$
\begin{aligned}
(\Delta-\partial_t)G=&\gamma(\Delta-\partial_t)F-\gamma' F\\
\geqslant&\gamma|f_{ij}+\frac{\varphi}{n}\delta_{ij}|^2+(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}G-\gamma\alpha^2n^2K^2-2\nabla f\nabla G\\
&+2h\gamma(\alpha-1)(l-1)u^{l-1}|\nabla f|^2+2[(\alpha-1)+\alpha(l-1)]\gamma u^{l-1}\nabla f\cdot\nabla h\\
&+h\gamma\alpha(l-1)^2u^{l-1}|\nabla f|^2+h\gamma\alpha(l-1)u^{l-1}\Delta f+\alpha\gamma u^{l-1}\Delta h-\gamma' F\\
=&\gamma|f_{ij}+\frac{\varphi}{n}g_{ij}|^2+\Big[(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}-\frac{\gamma'}{\gamma}\Big]G-\gamma\alpha^2n^2K^2-2\nabla f\nabla G\\
&+2h\gamma(\alpha-1)(l-1)u^{l-1}|\nabla f|^2+2\big[(\alpha-1)+\alpha(l-1)\big]\gamma u^{l-1}\nabla f\cdot\nabla h\\
&+\gamma\alpha(l-1)^2hu^{l-1}|\nabla f|^2+\gamma\alpha(l-1)hu^{l-1}\Delta f+\alpha\gamma u^{l-1}\Delta h.
\end{aligned}\tag{2.11}
$$

Now let $\varphi(r)$ be a C^2 function on $[0,\infty)$ such that

$$
\varphi(r)=\begin{cases}1, & if\ r\in[0,1],\\ 0, & if\ r\in[2,\infty),\end{cases}
$$

and

$$
0\leqslant\varphi(r)\leqslant 1,\quad \varphi'(r)\leqslant 0,\quad \varphi''(r)\leqslant 0,\quad \frac{|\varphi'(r)|}{\varphi(r)}\leqslant C,
$$

where C is an absolute constant. Let define by

$$
\phi(x,t)=\varphi(d(x,x_0,t))=\varphi\left(\frac{d(x,x_0,t)}{R}\right)=\varphi\left(\frac{\rho(x,t)}{R}\right),
$$

where $\rho(x,t)=d(x,x_0,t)$. By using the maximum principle, the argument of Calabi [2] allows us to suppose that the function $\phi(x,t)$ with support in $B_{2R,T}$, is C^2 at the maximum point. By utilizing the Laplacian comparison theorem, we deduce that

$$
\frac{|\nabla\phi|^2}{\phi}\leqslant\frac{C}{R^2},\quad -\Delta\phi\leqslant\frac{C}{R^2}(1+\sqrt{K}R),\tag{2.12}
$$

For any $0 \leqslant T_1 \leqslant T$, let $H = \phi G$ and (x_1, t_1) be the point in B_{2R,T_1} at which H attains its maximum value. We can suppose that the value is positive, because otherwise the proof is trivial. Then at the point (x_1, t_1), we infer

$$\left.\begin{aligned} 0 = \nabla(\phi G) = G\nabla\phi + \phi\nabla G, \\ \Delta(\phi G) \leqslant 0, \\ \partial_t(\phi G) \geqslant 0. \end{aligned}\right\} \tag{2.13}$$

By the evolution formula of the geodesic length along the Ricci flow [6], we calculate

$$\begin{aligned} \phi_t G = & -G\phi'\left(\frac{\rho}{R}\right)\frac{1}{R}\frac{d\rho}{dt} = G\phi'\left(\frac{\rho}{R}\right)\int_{\gamma_{t_1}} \mathrm{Ric}(S,S)ds \\ \leqslant & G\phi'\left(\frac{\rho}{R}\right)\frac{1}{R}K\rho \leqslant G\phi'\left(\frac{\rho}{R}\right)K_2 \leqslant G\sqrt{C}K, \end{aligned}$$

where γ_{t_1} is the geodesic connecting x and x_0 along the metric $g(t_1)$, S is the unite tangent vector to γ_{t_1}, and ds is the element of the arc length.

All the following computations are at the point (x_1, t_1). It is not difficult to find that

$$\begin{aligned} |f_{ij} + \frac{\varphi}{n}\delta_{ij}|^2 \geqslant & \frac{1}{n}\Big(\mathbf{tr}|f_{ij} + \frac{\varphi}{n}\delta_{ij}|\Big)^2 \\ = & \frac{1}{n}\Big(\Delta f + \varphi\Big)^2 \\ = & \frac{1}{n}\Big[-\frac{1}{\alpha}F - \frac{1}{\alpha}(\alpha-1)|\nabla f|^2\Big]^2 \\ = & \frac{1}{\alpha^2 n}\Big[\frac{G}{\gamma} + (\alpha-1)|\nabla f|^2\Big]^2, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \Delta f = & f_t - |\nabla f|^2 - hu^{l-1} \\ = & -\frac{F}{\alpha} - \frac{\alpha-1}{\alpha}|\nabla f|^2 - \varphi < 0. \end{aligned} \tag{2.15}$$

To obtain main results, two cases will be shown.

Case 1 $l \leqslant 1$.

From (2.15), we have $\Delta f \leqslant 0$. Then by substituting it into (2.11), we obtain

$$\begin{aligned} (\Delta - \partial_t)G = & \gamma(\Delta - \partial_t)F - \gamma' F \\ \geqslant & \gamma|f_{ij} + \frac{\varphi}{n}\delta_{ij}|^2 + \Big[(\frac{2\varphi}{n} - \alpha')\frac{1}{\alpha} - \frac{\gamma'}{\gamma}\Big]G - \gamma\alpha^2 n^2 K^2 - 2\nabla f\nabla G \\ & + 2h\gamma(\alpha-1)(l-1)u^{l-1}|\nabla f|^2 + \alpha\gamma u^{l-1}\Delta h \\ & + 2\big[(\alpha-1) + \alpha(l-1)\big]\gamma u^{l-1}\nabla f \cdot \nabla h, \end{aligned}$$

where we drop one term $h\gamma\alpha(l-1)^2u^{l-1}|\nabla f|^2$. Using (2.14), we infer

$$
\begin{aligned}
0 \geqslant & (\Delta-\partial_t)(\phi G) \\
= & G\Big(\Delta\phi - 2\frac{|\nabla\phi|^2}{\phi}\Big) + \phi(\Delta-\partial_t)G - G\phi_t \\
\geqslant & G\Big(\Delta\phi - 2\frac{|\nabla\phi|^2}{\phi}\Big) + \frac{\phi\gamma}{\alpha^2 n}\Big[\frac{G}{\gamma} + (\alpha-1)|\nabla f|^2\Big]^2 \\
& + \Big[(\frac{2\varphi}{n} - \alpha')\frac{1}{\alpha} - \frac{\gamma'}{\gamma}\Big]\phi G - \gamma\phi\alpha^2 n^2 K^2 - 2\phi\nabla f\nabla G \\
& + 2h\phi\gamma(\alpha-1)(l-1)u^{l-1}|\nabla f|^2 + \phi\alpha\gamma u^{l-1}\Delta h \\
& + 2\big[(\alpha-1)+\alpha(l-1)\big]\phi\gamma u^{l-1}\nabla f\cdot\nabla h - G\sqrt{C}K. \qquad (2.16)
\end{aligned}
$$

Multiply ϕ to inequality (2.16), we have

$$
\begin{aligned}
0 \geqslant\ & \phi G\Big[\Delta\phi - 2\frac{|\nabla\phi|^2}{\phi} + (\frac{2\varphi}{n}-\alpha')\frac{\phi}{\alpha} - \frac{\gamma'}{\gamma}\phi\Big] + \frac{\phi^2\gamma}{\alpha^2 n}\Big[\frac{G}{\gamma} + (\alpha-1)|\nabla f|^2\Big]^2 \\
& - \gamma\phi^2\alpha^2 n^2 K^2 - 2\phi^2\nabla f\nabla G + 2h\phi^2\gamma(\alpha-1)(l-1)u^{l-1}|\nabla f|^2 \\
& + 2\big[(\alpha-1)+\alpha(l-1)\big]\phi^2\gamma u^{l-1}\nabla f\cdot\nabla h + \phi^2\alpha\gamma u^{l-1}\Delta h - \phi G\sqrt{C}K \\
=\ & \phi G\Big[\Delta\phi - 2\frac{|\nabla\phi|^2}{\phi} + (\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha} - \frac{\gamma'}{\gamma}\Big] + \frac{\phi^2 G^2}{\alpha^2 n\gamma} + \frac{\phi^2(\alpha-1)^2\gamma}{n\alpha^2}|\nabla f|^4 \\
& + \frac{2\phi^2(\alpha-1)}{n\alpha^2}G|\nabla f|^2 - \gamma\phi^2\alpha^2 n^2 K^2 + 2\phi G\nabla\phi\nabla f \\
& + 2h\phi^2\gamma(\alpha-1)(l-1)u^{l-1}|\nabla f|^2 + \phi^2\alpha\gamma u^{l-1}\Delta h \\
& + 2\big[(\alpha-1)+\alpha(l-1)\big]\phi^2\gamma u^{l-1}\nabla f\cdot\nabla h - \phi G\sqrt{C}K.
\end{aligned}
$$

Using the Cauchy inequality

$$
\nabla f\cdot\nabla h \geqslant -|\nabla f||\nabla h| \geqslant -h|\nabla f|^2 - \frac{|\nabla h|^2}{4h},
$$

$$
\nabla f\cdot\nabla h \leqslant h|\nabla f|^2 + \frac{|\nabla h|^2}{4h},
$$

we conclude

$$
\begin{aligned}
0 \geqslant & \phi G\Big[\Delta\phi - 2\frac{|\nabla\phi|^2}{\phi} + (\frac{2\varphi}{n}-\alpha')\frac{\phi}{\alpha} - \frac{\gamma'}{\gamma}\phi\Big] + \frac{\phi^2 G^2}{\alpha^2 n\gamma} + \frac{\phi^2(\alpha-1)^2\gamma}{n\alpha^2}|\nabla f|^4 \\
& + \frac{2\phi^2(\alpha-1)}{n\alpha^2}G|\nabla f|^2 - \gamma\phi^2\alpha^2 n^2 K^2 + 2\phi G\nabla\phi\nabla f \\
& - 2h\phi^2\gamma(\alpha-1)(1-l)u^{l-1}|\nabla f|^2 - 2\big[(\alpha-1)+\alpha(1-l)\big]\phi^2\gamma u^{l-1}h|\nabla f|^2 \\
& - \frac{1}{2}\big[(\alpha-1)+\alpha(1-l)\big]\phi^2\gamma u^{l-1}\frac{|\nabla h|^2}{h} + \phi^2\alpha\gamma u^{l-1}\Delta h - \phi G\sqrt{C}K
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant \phi G\Big[\Delta\phi - 2\frac{|\nabla\phi|^2}{\phi} + (\frac{2\varphi}{n} - \alpha')\frac{\phi}{\alpha} - \frac{\gamma'}{\gamma}\phi\Big] + \frac{\phi^2 G^2}{\alpha^2 n\gamma} + \frac{\phi^2(\alpha-1)^2\gamma}{n\alpha^2}|\nabla f|^4 \\
&\quad + \frac{2\phi^2(\alpha-1)}{n\alpha^2}G|\nabla f|^2 - \gamma\phi^2\alpha^2 n^2 K^2 + 2\phi G\nabla\phi\nabla f \\
&\quad - 2\big[\alpha(3-2\alpha)-1\big]\phi^2\gamma u^{l-1}h|\nabla f|^2 - \frac{1}{2}\big[(\alpha-1)+\alpha(1-l)\big]\phi^2\gamma u^{l-1}\frac{|\nabla h|^2}{h} \\
&\quad + \phi^2\alpha\gamma u^{l-1}\Delta h - \phi G\sqrt{C}K, \qquad (2.17)
\end{aligned}
$$

where we used the fact that $(\alpha-1)(1-l)+(\alpha-1)+\alpha(1-l)\leqslant \alpha(3-2l)-1$. Further using the inequality $Ax^2+Bx\geqslant -\frac{B^2}{4A}$ with $A>0$, we have

$$
\frac{2\phi^2(\alpha-1)}{n\alpha^2}G|\nabla f|^2 + 2\phi G\nabla\phi\nabla f \geqslant -\frac{n\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi}\phi G,
$$

and

$$
\begin{aligned}
&\frac{\phi^2(\alpha-1)^2\gamma}{n\alpha^2}|\nabla f|^4 - 2\big[\alpha(3-2l)-1\big]\phi^2\gamma u^{l-1}h|\nabla f|^2 \\
\geqslant & -\frac{n\alpha^2\big[\alpha(3-2l)-1\big]^2}{(\alpha-1)^2}\gamma\phi^2 u^{2(l-1)}h^2 \\
\geqslant & -\frac{n\alpha^2\big[\alpha(3-2\alpha)-1\big]^2}{(\alpha-1)^2}\gamma\phi^2\overline{u}_1^2\delta_1^2.
\end{aligned}
$$

Substituting above two inequalities into (2.17), we deduce that

$$
\begin{aligned}
0 \geqslant\ & \phi G\Big[\Delta\phi - 2\frac{|\nabla\phi|^2}{\phi} + (\frac{2\varphi}{n} - \alpha')\frac{\phi}{\alpha} - \frac{\gamma'}{\gamma}\phi - \frac{n\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi} - \sqrt{C}K\Big] \\
& + \frac{\phi^2 G^2}{\alpha^2 n\gamma} - \gamma\phi^2\alpha^2 n^2 K^2 - \frac{n\alpha^2\big[\alpha(3-2l)-1\big]^2}{(\alpha-1)^2}\gamma\phi^2\overline{u}_1^2\delta_1^2 \\
& - \frac{1}{2}\big[(\alpha-1)+\alpha(1-l)\big]\phi^2\gamma\overline{u}_1\delta_2 - \phi^2\alpha\gamma\overline{u}_1\delta_3.
\end{aligned}
$$

Applying (2.12), we infer

$$
\begin{aligned}
0 \geqslant\ & \Big[-\frac{C}{R^2}(1+\sqrt{k}R) - \frac{2C}{R^2} + (\frac{2\varphi}{n} - \alpha')\frac{\phi}{\alpha} - \frac{\gamma'}{\gamma}\phi \\
& - \frac{n\alpha^2}{2(\alpha-1)}\frac{C}{R^2} - \sqrt{C}K\Big]\phi G + \frac{\phi^2 G^2}{\alpha^2 n\gamma} - \gamma\phi^2\alpha^2 n^2 K^2 \\
& - \frac{n\alpha^2\big[\alpha(3-2l)-1\big]^2}{(\alpha-1)^2}\gamma\phi^2\overline{u}_1^2\delta_1^2 \\
& - \frac{1}{2}\big[(\alpha-1)+\alpha(1-l)\big]\phi^2\gamma\overline{u}_1\delta_2 - \phi^2\alpha\gamma\overline{u}_1\delta_3. \qquad (2.18)
\end{aligned}
$$

For the inequality $Ax^2 - Bx \leqslant C$, one has $x \leqslant \frac{B}{A} + \left(\frac{C}{A}\right)^{\frac{1}{2}}$, where $A,B,C > 0$. By using this inequality to (2.18) and then we arrive at

$$\begin{aligned}\phi G(x,T_1) &\leqslant (\phi G)(x_1,t_1)\\ &\leqslant \Bigg\{ n\gamma\alpha^2\left[\frac{C}{R^2}(1+\sqrt{K}R) + \frac{n\alpha^2}{2(\alpha-1)}\frac{C}{R^2} + \sqrt{C}K\right]\\ &\quad + n\gamma\alpha^2\left[\frac{\gamma'}{\gamma} - (\frac{2\varphi}{n} - \frac{\alpha'}{\alpha})\frac{1}{\alpha}\right] + n^{\frac{3}{2}}\gamma\alpha^2\phi K\\ &\quad + \frac{n\alpha^2\left[\alpha(3-2l)-1\right]}{\alpha-1}\phi\gamma\bar{u}_1\delta_1 + \alpha\phi\gamma\sqrt{n\bar{u}_1\delta_3}\\ &\quad + \sqrt{\frac{\left[(\alpha-1)+\alpha(1-l)\right]}{2}}\alpha\phi\gamma\sqrt{n\bar{u}_1\delta_2}\Bigg\}(x_1,t_1).\end{aligned}$$

If γ is nondecreasing which satisfies the system

$$\begin{cases} \dfrac{\gamma'}{\gamma} - (\dfrac{2\varphi}{n} - \alpha')\dfrac{1}{\alpha} \leqslant 0, \\ \dfrac{\gamma\alpha^4}{\alpha-1} \leqslant C_1. \end{cases} \tag{2.19}$$

Recall that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1 < T_1$. Hence, we have

$$\begin{aligned}\phi G(x,T_1) &\leqslant (\phi G)(x_1,t_1)\\ &\leqslant n\gamma(T_1)\alpha^2(T_1)\left[\frac{C}{R^2}\left(1+\sqrt{K}R\right) + K\right] + \frac{n^2C}{R^2}\\ &\quad + n^{\frac{3}{2}}\gamma(T_1)\alpha^2(T_1)\phi K + \phi\alpha(T_1)\gamma(T_1)\sqrt{n\bar{u}_1\delta_3}\\ &\quad + \frac{n\alpha^2(T_1)\left[\alpha(T_1)(3-2l)-1\right]}{\alpha(T_1)-1}\phi\gamma(T_1)\bar{u}_1\delta_1\\ &\quad + \sqrt{\frac{\left[(\alpha(T_1)-1)+\alpha(T_1)(1-l)\right]}{2}}\alpha(T_1)\phi\gamma(T_1)\sqrt{n\bar{u}_1\delta_2}.\end{aligned}$$

Hence, we have for $\phi \equiv 1$ on $B_{R,T}$,

$$\begin{aligned}F(x,T_1) &\leqslant n\alpha^2(T_1)\left[\frac{C}{R^2}\left(1+\sqrt{K}R\right) + CK\right] + \frac{n^2C}{R^2\gamma(T_1)}\\ &\quad + n^{\frac{3}{2}}\alpha^2(T_1)K + \alpha(T_1)\sqrt{n\bar{u}_1\delta_3}\\ &\quad + \frac{n\alpha^2(T_1)\left[\alpha(T_1)(3-2l)-1\right]}{\alpha(T_1)-1}\bar{u}_1\delta_1\\ &\quad + \sqrt{\frac{\left[(\alpha(T_1)-1)+\alpha(T_1)(1-l)\right]}{2}}\alpha(T_1)\sqrt{n\bar{u}_1\delta_2}.\end{aligned}$$

If γ is nondecreasing which satisfies the system

$$\begin{cases} \frac{\gamma'}{\gamma}-(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}\leqslant 0, \\ \frac{\gamma}{\alpha-1}\leqslant C_2. \end{cases} \tag{2.20}$$

Recall that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1<T_1$. Hence, we have

$$\begin{aligned}\phi G(x,T_1)&\leqslant(\phi G)(x_1,t_1)\\ &\leqslant n\gamma(T_1)\alpha^2(T_1)\left[\frac{C}{R^2}\left(1+\sqrt{K}R\right)+\frac{n\alpha^2}{\alpha-1}\frac{C}{R^2}+CK\right]\\ &\quad+n^{\frac{3}{2}}\gamma(T_1)\alpha^2(T_1)\phi K+\phi\alpha(T_1)\gamma(T_1)\sqrt{n\bar{u}_1\delta_3}\\ &\quad+\frac{n\alpha^2(T_1)\left[\alpha(T_1)(3-2l)-1\right]}{\alpha(T_1)-1}\phi\gamma(T_1)\bar{u}_1\delta_1\\ &\quad+\sqrt{\frac{\left[(\alpha(T_1)-1)+\alpha(T_1)(1-l)\right]}{2}}\alpha(T_1)\phi\gamma(T_1)\sqrt{n\bar{u}_1\delta_2}.\end{aligned}$$

Hence, we have for $\phi\equiv 1$ on $B_{R,T}$,

$$\begin{aligned}F(x,T_1)\leqslant{}& n\alpha^2(T_1)\left[\frac{C}{R^2}\left(1+\sqrt{K}R\right)+K\right]+\frac{n^2C\alpha^4}{R^2\gamma(T)}\\ &+n^{\frac{3}{2}}\gamma(T_1)\alpha^2(T_1)K+\alpha(T_1)\sqrt{n\bar{u}_1\delta_3}\\ &+\frac{n\alpha^2(T_1)\left[\alpha(T_1)(3-2l)-1\right]}{\alpha(T_1)-1}\bar{u}_1\delta_1\\ &+\sqrt{\frac{\left[(\alpha(T_1)-1)+\alpha(T_1)(1-l)\right]}{2}}\alpha(T_1)\sqrt{n\bar{u}_1\delta_2}.\end{aligned}$$

Because T_1 is arbitrary in $0<T_1<T$ and $\alpha-1+\alpha(1-l)\leqslant\alpha(2-l)$. Thus the conclusion is valid.

Case 2 $l>1$.

Substituting (2.15) into (2.11), we have

$$\begin{aligned}(\Delta-\partial_t)G\geqslant{}&\gamma|f_{ij}+\frac{\varphi}{n}\delta_{ij}|^2+\left[(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}-\frac{\gamma'}{\gamma}\right]G-\gamma\alpha^2n^2K^2\\ &-2\nabla f\nabla G+2(l\alpha-1)\gamma u^{l-1}\nabla f\cdot\nabla h-h(l-1)u^{l-1}G\\ &-h\gamma(l-1)u^{l-1}\alpha\varphi+h\gamma(l-1)(l\alpha-1)u^{l-1}|\nabla f|^2+\alpha\gamma u^{l-1}\Delta h.\end{aligned}$$

Using (2.14), we infer

$$\begin{aligned}0&\geqslant(\Delta-\partial_t)(\phi G)\\ &=G\Big(\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}\Big)+\phi(\Delta-\partial_t)G-\gamma G\phi_t\end{aligned}$$

$$
\begin{aligned}
&\geqslant G\Big(\Delta\phi - 2\frac{|\nabla\phi|^2}{\phi}\Big) + \frac{\phi\gamma}{\alpha^2 n}\Big[\frac{G}{\gamma} + (\alpha-1)|\nabla f|^2\Big]^2 \\
&\quad + \Big[(\frac{2\varphi}{2} - \alpha')\frac{1}{\alpha} - \frac{\gamma'}{\gamma}\Big]\phi G - \gamma\phi\alpha^2 n^2 K^2 - 2\phi\nabla f\nabla G \\
&\quad + 2(l\alpha-1)\phi\gamma u^{l-1}\nabla f\cdot\nabla h - h(l-1)u^{l-1}G - h\gamma(l-1)u^{l-1}\phi\alpha\varphi \\
&\quad + \phi\alpha\gamma u^{l-1}\Delta h + h\phi\gamma(l-1)(l\alpha-1)u^{l-1}|\nabla f|^2 - G\sqrt{C}K. \qquad (2.21)
\end{aligned}
$$

Multiply ϕ to (2.21), and we have

$$
\begin{aligned}
0 &\geqslant \phi G\Big[\Delta\phi - 2\frac{|\nabla\phi|^2}{\phi} + (\frac{2\varphi}{n} - \alpha')\frac{\phi}{\alpha} - \frac{\gamma'}{\gamma}\phi\Big] + \frac{\phi^2\gamma}{\alpha^2 n}\Big[\frac{G}{\gamma} + (\alpha-1)|\nabla f|^2\Big]^2 \\
&\quad - \gamma\phi^2\alpha^2 n^2 K^2 - 2\phi^2\nabla f\nabla G + 2(l\alpha-1)\phi^2\gamma u^{l-1}\nabla f\cdot\nabla h \\
&\quad + \phi^2\alpha\gamma u^{l-1}\Delta h - h(l-1)u^{l-1}\phi G - h\gamma(l-1)u^{l-1}\phi^2\alpha\varphi \\
&\quad + h\phi^2\gamma(l-1)(l\alpha-1)u^{l-1}|\nabla f|^2 - \phi G\sqrt{C}K \\
&\geqslant \phi G\Big[\Delta\phi - 2\frac{|\nabla\phi|^2}{\phi} + (\frac{2\varphi}{n} - \alpha')\frac{\phi}{\alpha} - \frac{\gamma'}{\gamma}\phi\Big] + \frac{\phi^2 G^2}{\alpha^2 n\gamma} \\
&\quad + \frac{2\phi^2(\alpha-1)}{n\alpha^2}G|\nabla f|^2 - \gamma\phi^2\alpha^2 n^2 K^2 + 2\phi G\nabla\phi\nabla f \\
&\quad + 2(l\alpha-1)\phi^2\gamma u^{l-1}\nabla f\cdot\nabla h + \phi^2\alpha\gamma u^{l-1}\Delta h \\
&\quad - h(l-1)u^{l-1}\phi G - h\gamma(l-1)u^{l-1}\phi^2\alpha\varphi \\
&\quad + h\phi^2\gamma(l-1)(l\alpha-1)u^{l-1}|\nabla f|^2 - \phi G\sqrt{C}K, \qquad (2.22)
\end{aligned}
$$

where we dropped the term $\frac{\phi^2(\alpha-1)^2\gamma}{n\alpha^2}|\nabla f|^4$.

Further using the inequality $Ax^2 + Bx \geqslant -\frac{B^2}{4A}$ with $A > 0$, we have

$$
\frac{2\phi^2(\alpha-1)}{n\alpha^2}G|\nabla f|^2 + 2\phi G\nabla\phi\nabla f \geqslant -\frac{n\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi}\phi G,
$$

and

$$
\begin{aligned}
&h\phi^2\gamma(l-1)(l\alpha-1)u^{l-1}|\nabla f|^2 + 2(l\alpha-1)\phi^2\gamma u^{l-1}\nabla f\cdot\nabla h \\
\geqslant &h\phi^2\gamma(l-1)(l\alpha-1)u^{l-1}|\nabla f|^2 - 2(l\alpha-1)\phi^2\gamma u^{l-1}|\nabla f|\cdot|\nabla h| \\
\geqslant &-\frac{l\alpha-1}{l-1}\gamma\phi^2 u^{l-1}\frac{|\nabla h|^2}{h} \\
\geqslant &-\frac{l\alpha-1}{l-1}\gamma\phi^2\overline{u}_2\delta_2.
\end{aligned}
$$

Substituting above two inequalities into (2.22), we deduce that

$$
0 \geqslant \phi G\Big[\Delta\phi - 2\frac{|\nabla\phi|^2}{\phi} + (\frac{2\varphi}{n} - \alpha')\frac{\phi}{\alpha} - \frac{\gamma'}{\gamma}\phi - \frac{n\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi}
$$

$$
\begin{aligned}
&-\delta_1(l-1)\overline{u}_2-\sqrt{C}K\Big]+\frac{\phi^2G^2}{\alpha^2n\gamma}-\gamma\phi^2\alpha^2n^2K^2\\
&-\frac{l\alpha-1}{l-1}\gamma\phi^2\overline{u}_2\delta_2-\gamma(l-1)\overline{u}_2\phi^2\delta_1\alpha\varphi-\phi^2\alpha\gamma\overline{u}_2\delta_3.
\end{aligned}
$$

Applying (2.12), we infer

$$
\begin{aligned}
0\geqslant&\Big[-\frac{C}{R^2}(1+\sqrt{k}R)-\frac{2C}{R^2}+(\frac{2\varphi}{n}-\alpha')\frac{\phi}{\alpha}-\frac{\gamma'}{\gamma}\phi-\frac{n\alpha^2}{2(\alpha-1)}\frac{C}{R^2}\\
&-\delta_1(l-1)\overline{u}_2-\sqrt{C}K\Big]\phi G+\frac{\phi^2G^2}{\alpha^2n\gamma}-\gamma\phi^2\alpha^2n^2K^2\\
&-\frac{l\alpha-1}{l-1}\gamma\phi^2\overline{u}_2\delta_2-\gamma(l-1)\overline{u}_2\phi^2\delta_1\alpha\varphi-\phi^2\alpha\gamma\overline{u}_2\delta_3.
\end{aligned}
\tag{2.23}
$$

For the inequality $Ax^2-2Bx\leqslant C$, one has $x\leqslant \frac{2B}{A}+\left(\frac{C}{A}\right)^{\frac{1}{2}}$, where $A,B,C>0$. By using this inequality to (2.23) and then we arrive at

$$
\begin{aligned}
\phi G(x,T_1)&\leqslant(\phi G)(x_1,t_1)\\
&\leqslant\Big\{n\gamma\alpha^2\left[\frac{C}{R^2}(1+\sqrt{K}R)+\frac{n\alpha^2}{2(\alpha-1)}\frac{C}{R^2}+\delta_1(l-1)\overline{u}_2+\sqrt{C}K\right]\\
&\quad+n\gamma\alpha^2\left[\frac{\gamma'}{\gamma}-(\frac{2\varphi}{n}-\frac{\alpha'}{\alpha})\frac{1}{\alpha}\right]+n^{\frac{3}{2}}\gamma\phi\alpha^2K+\alpha\phi\gamma\sqrt{\frac{n(l\alpha-1)\overline{u}_2\delta_2}{l-1}}\\
&\quad+\alpha^{\frac{3}{2}}\gamma\phi\sqrt{n(l-1)\delta_1\varphi}+\alpha^{\frac{3}{2}}\gamma\phi\sqrt{n\delta_3\overline{u}_2}\Big\}(x_1,t_1).
\end{aligned}
$$

If γ is nondecreasing which satisfies the system

$$
\begin{cases}
\dfrac{\gamma'}{\gamma}-(\dfrac{2\varphi}{n}-\alpha')\dfrac{1}{\alpha}\leqslant 0,\\
\dfrac{\gamma\alpha^4}{\alpha-1}\leqslant C_1.
\end{cases}
\tag{2.24}
$$

Recall that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1<T_1$. Hence, we have

$$
\begin{aligned}
\phi G(x,T_1)&\leqslant(\phi G)(x_1,t_1)\\
&\leqslant n\gamma(T_1)\alpha^2(T_1)\left[\frac{C}{R^2}\left(1+\sqrt{K}R\right)+K\right]+\frac{n^2C}{R^2}\\
&\quad+n\gamma(T_1)\alpha^2(T_1)(l-1)\delta_1\overline{u}_2+n^{\frac{3}{2}}\gamma\phi(T_1)\alpha^2(T_1)K\\
&\quad+\alpha(T_1)\phi\gamma(T_1)\sqrt{\frac{n(l\alpha(T_1)-1)\overline{u}_2\delta_2}{l-1}}\\
&\quad+\alpha^{\frac{3}{2}}(T_1)\gamma(T_1)\phi\sqrt{n(l-1)\delta_1\varphi}+\alpha^{\frac{3}{2}}(T_1)\gamma(T_1)\phi\sqrt{n\delta_3\overline{u}_2}.
\end{aligned}
$$

Hence, we have for $\phi\equiv 1$ on $B_{R,T}$,

$$
F(x,T_1)\leqslant n\alpha^2(T_1)\left[\frac{C}{R^2}\left(1+\sqrt{K}R\right)+K\right]+\frac{n^2C}{R^2}
$$

$$+n\alpha^2(T_1)(l-1)\delta_1\overline{u}_2+n^{\frac{3}{2}}\alpha^2(T_1)K$$
$$+\alpha(T_1)\sqrt{\frac{n(l\alpha(T_1)-1)\overline{u}_2\delta_2}{l-1}}$$
$$+\alpha^{\frac{3}{2}}(T_1)\sqrt{n(l-1)\delta_1\varphi}+\alpha^{\frac{3}{2}}(T_1)\sqrt{n\delta_3\overline{u}_2}.$$

If γ is nondecreasing which satisfies the system

$$\begin{cases} \frac{\gamma'}{\gamma}-(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}\leqslant 0, \\ \frac{\gamma}{\alpha-1}\leqslant C_2. \end{cases} \tag{2.25}$$

Recall that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1 < T_1$. Hence, we have

$$\begin{aligned} \phi G(x,T_1) &\leqslant (\phi G)(x_1,t_1) \\ &\leqslant n\gamma(T_1)\alpha^2(T_1)\left[\frac{C}{R^2}\left(1+\sqrt{K}R\right)+\frac{Cn\alpha^2(T)}{R^2}+CK\right] \\ &\quad +n\gamma(T_1)\alpha^2(T_1)(l-1)\delta_1\overline{u}_2+n^{\frac{3}{2}}\gamma(T_1)\phi\alpha^2(T_1)K \\ &\quad +\alpha(T_1)\phi\gamma(T_1)\sqrt{\frac{n(l\alpha(T_1)-1)\overline{u}_2\delta_2}{l-1}} \\ &\quad +\alpha^{\frac{3}{2}}(T_1)\gamma(T_1)\phi\sqrt{n(l-1)\delta_1\varphi}+\alpha^{\frac{3}{2}}(T_1)\gamma(T_1)\phi\sqrt{n\delta_3\overline{u}_2}. \end{aligned}$$

Hence, we have for $\phi\equiv 1$ on $B_{R,T}$,

$$\begin{aligned} F(x,T_1) \leqslant\ & n\alpha^2(T_1)\left[\frac{C}{R^2}\left(1+\sqrt{K}R\right)+\frac{Cn\alpha^2(T)}{R^2}+CK\right] \\ &+n\alpha^2(T_1)(l-1)\delta_1\overline{u}_2+n^{\frac{3}{2}}\alpha^2(T_1)K \\ &+\alpha(T_1)\sqrt{\frac{n(l\alpha(T_1)-1)\overline{u}_2\delta_2}{l-1}} \\ &+\alpha^{\frac{3}{2}}(T_1)\sqrt{n(l-1)\delta_1\varphi}+\alpha^{\frac{3}{2}}(T_1)\sqrt{n\delta_3\overline{u}_2}. \end{aligned}$$

Because T_1 is arbitrary in $0<T_1<T$, the conclusion is valid.

Proof of Theorem 2.2. Since $u_t=\Delta u+h(t)u^l$, we have

$$\begin{aligned} \partial_t(|\nabla u|^2) &= 2\mathrm{Ric}(\nabla u,\nabla u)+2<\nabla u,\nabla(u_t)> \\ &= 2\mathrm{Ric}(\nabla u,\nabla u)+2<\nabla u,\nabla(\Delta u)>+2<\nabla u,\nabla(h(t)u^l)>. \end{aligned}$$

Applying the Bochner's formula, the above equation becomes

$$\partial_t(|\nabla u|^2)=\Delta(|\nabla u|^2)-2|\nabla^2 u|^2+2<\nabla u,\nabla(h(t)u^l)>. \tag{2.26}$$

Besides,

$$\partial_t(u^2)=\Delta(u^2)-2|\nabla u|^2+2h(t)u^{l+1}. \tag{2.27}$$

Let $\overline{F}=t|\nabla u|^2+Xu^2$, where X decided later. Then combining (2.26) with (2.27), we obtain

$$\begin{aligned}\partial_t\overline{F} &= |\nabla u|^2+t[\Delta(|\nabla u|^2)-2|\nabla^2 u|^2+2<\nabla u,\nabla(h(t)u^l)>]\\ &\quad +X[\Delta(u^2)-2|\nabla u|^2+2h(t)u^{l+1}]\\ &= |\nabla u|^2+t[\Delta(|\nabla u|^2)-2|\nabla^2 u|^2+2h(t)(l-1)u^{l-2}|\nabla u|^2\\ &\quad +X[\Delta(u^2)-2|\nabla u|^2+2h(t)u^{l+1}]\\ &\leqslant \Delta\overline{F}+(1-2X)|\nabla u|^2. \end{aligned} \tag{2.28}$$

Selecting $X=\frac{1}{2}$ and using the maximum principle, we infer

$$\overline{F}(x,t)\leqslant \max_{x\in M^n}\overline{F}(x,0)=\frac{1}{2}\max_{x\in M^n}u^2(x,0),$$

which implies the theorem is valid.

3. Gradient estimates for the equation (1.6)

Recall that (M^n,g) is called a gradient Ricci soliton if there is a smooth function f on M^n such that for some constant $c\in\mathbb{R}$, which satisfies

$$Rc=cg+D^2 f, \tag{3.1}$$

where $D^2 f$ is the Hessian of f. Letting $u=e^f$, after some computation applying (3.1) as done in [21], we get

$$\Delta u+2cu\log u=(A_0-nc)u \quad in \quad M^n$$

for some constant A_0, where n is the dimension of M^n. In [21], Ma proved a local gradient estimate of positive solutions to the equation

$$\Delta u+au\log u+bu=0 \quad in \quad M^n,$$

where $a>0$ and $b\in\mathbb{R}$ are constants for complete noncompact manifolds with a fixed metric and curvature locally bounded below. In [30], Yang generalized Ma's result and derived a local gradient estimates for positive solutions to the equation

$$u_t=\Delta u+au\log u+bu \quad in \quad M^n\times(0,T],$$

where $a,b\in\mathbb{R}$ are constants for complete noncompact manifolds with a fixed metric and curvature locally bounded below. Replacing u by $e^{b/a}u$, the above equation becomes

$$u_t=\Delta u+au\log u. \tag{3.2}$$

One can find in [29, 30] some related results for equation (3.2) on manifolds.

In this section, we consider the nonlinear parabolic equation (1.6) along the Ricci flow.

3.1. Main results

Our main results state as follows.

THEOREM 3.1. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.7). Assume that $|\mathrm{Ric}(x,t)| \leqslant K$ for some $K > 0$ and all $t \in [0,T]$. Suppose that there exist three functions $\alpha(t)$, $\varphi(t)$ and $\gamma(t)$ which satisfy the following conditions (C1), (C2), (C3) and (C4).*

Given $x_0 \in M$ and $R > 0$, let u be a positive solution of the nonlinear parabolic equation

$$\partial_t u = \Delta u + au\log u$$

in the cube $B_{2R,T} := \{(x,t)|d(x,x_0,t) \leqslant 2R, 0 \leqslant t \leqslant T\}$, where a is a constant.

(1) For $a \leqslant 0$. If $\frac{\gamma\alpha^4}{\alpha-1} \leqslant C_1$ for some constant C_1, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a\log u \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + K\right) + \frac{Cn^2}{R^2\gamma} + n^{\frac{3}{2}}\alpha^2 K + n|a|\alpha^2 + \alpha\varphi,$$

where $C = C(n, C_1)$ is a constant.

If $\frac{\gamma}{\alpha-1} \leqslant C_2$ for some constant C_2, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a\log u \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + K\right) + \frac{Cn^2\alpha^4}{R^2\gamma} + n^{\frac{3}{2}}\alpha^2 K + n|a|\alpha^2 + \alpha\varphi,$$

where $C = C(n, C_2)$ is a constant.

(2) For $a > 0$. If $\frac{\gamma\alpha^4}{\alpha-1} \leqslant C_1$ for some constant C_1, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a\log u \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + a + K\right) + \frac{n^2 C}{R^2\gamma} + n^{\frac{3}{2}}\alpha^2 K + \alpha\varphi,$$

where $C = C(n, C_1)$ is a constant.

If $\frac{\gamma}{\alpha-1} \leqslant C_2$ for some constant C_2, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a\log u \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + a + K\right) + \frac{n^2 C\alpha^4}{R^2\gamma} + n^{\frac{3}{2}}\alpha^2 K + \alpha\varphi,$$

where $C = C(n, C_2)$ is a constant.

COROLLARY 3.1. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow* (1.7). *Assume that $|\mathrm{Ric}(x,t)| \leqslant K$ for some $K>0$ and all $t\in[0,T]$. Given $x_0 \in M$ and $R>0$, let u be a positive solution of the nonlinear parabolic equation* (1.6) *in the cube $B_{2R,T} := \{(x,t) | d(x,x_0,t) \leqslant 2R, 0 \leqslant t \leqslant T\}$. Then the following special estimates are valid.*

1. *Li-Yau type:*

$$\alpha(t) = constant, \quad \varphi(t) = \frac{\alpha n}{t} + \frac{nK\alpha^2}{\alpha-1}, \gamma(t) = t^\theta \quad with \quad 0 < \theta \leqslant 2.$$

If $a \leqslant 0$, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a \log u \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + \frac{\alpha^2}{\alpha-1}\frac{1}{R^2} + K\right) + n^{\frac{3}{2}}\alpha^2 K + n|a|\alpha^2 + \alpha\varphi.$$

If $a > 0$, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a \log u \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + \frac{\alpha^2}{\alpha-1}\frac{1}{R^2} + a + K\right) + n^{\frac{3}{2}}\alpha^2 K + \alpha\varphi.$$

2. *Hamilton type:*

$$\alpha(t) = e^{2Kt}, \quad \varphi(t) = \frac{n}{t}e^{4Kt}, \quad \gamma(t) = te^{2Kt}.$$

If $a \leqslant 0$, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a \log u \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + K\right) + \frac{C\alpha^4}{R^2 te^{2Kt}} + \alpha\varphi + n^{\frac{3}{2}}\alpha^2 K + n|a|\alpha^2 + \alpha\varphi.$$

If $a > 0$, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a \log u \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + a + K\right) + \frac{C\alpha^4}{R^2 te^{2Kt}} + n^{\frac{3}{2}}\alpha^2 K + \alpha\varphi.$$

3. *Li-Xu type:*

$$\alpha(t) = 1 + \frac{\sinh(Kt)\cosh(Kt) - Kt}{\sinh^2(Kt)}, \quad \varphi(t) = 2nK[1 + \coth(Kt)],$$

$$\gamma(t) = \tanh(Kt).$$

If $a \leqslant 0$, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a \log u \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + K\right) + \frac{C}{R^2\tanh(Kt)} + n^{\frac{3}{2}}\alpha^2 K + n|a|\alpha^2 + \alpha\varphi.$$

If $a > 0$, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a \log u \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + a + K\right) + \frac{C}{R^2\tanh(Kt)} + n^{\frac{3}{2}}\alpha^2 K + \alpha\varphi.$$

4. Linear Li-Xu type:

$$\alpha(t) = 1 + 2Kt, \varphi(t) = \frac{n}{t} + nK(1 + 2Kt + \mu Kt), \gamma(t) = Kt \quad \text{with} \quad \mu \geqslant \frac{1}{4}.$$

If $a \leqslant 0$, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a \log u \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + K\right) + \frac{C\alpha^4}{R^2 Kt} + n^{\frac{3}{2}}\alpha^2 K + n|a|\alpha^2 + \alpha\varphi.$$

If $a > 0$, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a \log u \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + a + K\right) + \frac{C\alpha^4}{R^2 Kt} + n^{\frac{3}{2}}\alpha^2 K + \alpha\varphi.$$

The local estimates above imply global estimates.

COROLLARY 3.2. *Let $(M^n, g(0))$ be a complete noncompact Riemannian manifold without boundary, and assume $g(t)$ evolves by Ricci flow in such a way that $|\mathrm{Ric}| \leqslant K$ for $t \in [0,T]$. Let $u(x,t)$ be a positive solution to the equation* (1.6). *If $l \in \mathbb{R}$ and for $(x,t) \in M^n \times (0,T]$, then*

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} + \alpha a \log u \leqslant C\alpha^2(K + |a|) + \alpha\varphi.$$

REMARK 3.1. The above results may be regarded as generalizations the gradient estimates of Yang [30] to the Ricci flow.

3.2. Auxiliary lemma

To prove the Theorem 3.1, the following lemma is needed.

Let $f=\log u$. Then
$$(\Delta-\partial_t)f=-|\nabla f|^2-af. \tag{3.3}$$
Let $F=|\nabla f|^2-\alpha f_t+\alpha af-\alpha\varphi$, where $\alpha=\alpha(t)$ and $\varphi=\varphi(t)$. Then
$$\begin{aligned}\Delta f &= f_t-af-|\nabla f|^2\\ &= -\frac{F}{\alpha}-(\frac{\alpha-1}{\alpha})|\nabla f|^2-\varphi.\end{aligned} \tag{3.4}$$

LEMMA 3.1. *We assume that* $\alpha(t)>1$ *and* $\varphi(t)>0$ *satisfy the system (2.3). Then*
$$\begin{aligned}(\Delta-\partial_t)F \geqslant\ & |f_{ij}+\frac{\varphi}{n}\delta_{ij}|^2+(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}F-\alpha^2n^2K^2-2\nabla f\nabla F\\ &+2a(\alpha-1)|\nabla f|^2+a\alpha\Delta f.\end{aligned} \tag{3.5}$$

Proof. A computation is shown that
$$\begin{aligned}\Delta F &= \Delta|\nabla f|^2-\alpha\Delta(f_t)+\alpha a\Delta f\\ &= 2|f_{ij}|^2+2f_jf_{iij}+2R_{ij}f_if_j-\alpha\Delta(f_t)+\alpha a\Delta f\\ &= 2\Big(|f_{ij}|^2+\alpha R_{ij}f_{ij}\Big)+2f_jf_{iij}+2R_{ij}f_if_j-\alpha(\Delta f)_t+\alpha a\Delta f\\ &\geqslant 2|f_{ij}|^2-2\alpha|R_{ij}||f_{ij}|+2f_jf_{iij}+2R_{ij}f_if_j-\alpha(\Delta f)_t+\alpha a\Delta f\\ &\geqslant 2|f_{ij}|^2-(\alpha^2|R_{ij}|^2+|f_{ij}|^2)+2f_jf_{iij}+2R_{ij}f_if_j-\alpha(\Delta f)_t+\alpha a\Delta f\\ &\geqslant |f_{ij}|^2-\alpha^2|R_{ij}|^2+2f_jf_{iij}+2R_{ij}f_if_j-\alpha(\Delta f)_t+\alpha a\Delta f\\ &\geqslant |f_{ij}|^2-\alpha^2n^2K^2+2f_jf_{iij}+2R_{ij}f_if_j-\alpha(\Delta f)_t+\alpha a\Delta f,\end{aligned} \tag{3.6}$$

and
$$\begin{aligned}\partial_tF &= (|\nabla f|^2)_t-\alpha f_{tt}-\alpha' f_t+\alpha' af+\alpha af_t-\alpha\varphi'-\alpha'\varphi\\ &= 2\nabla f\nabla(f_t)+2R_{ij}f_if_j-\alpha f_{tt}-\alpha' f_t+\alpha' af\\ &\quad+\alpha af_t-\alpha\varphi'-\alpha'\varphi.\end{aligned} \tag{3.7}$$

We follow that from (3.6) and (3.7)
$$\begin{aligned}(\Delta-\partial_t)F \geqslant\ & |f_{ij}|^2-\alpha^2n^2K^2+2\nabla f\nabla(\Delta f)-\alpha(\Delta f)_t+\alpha a\Delta d-2\nabla f\nabla(f_t)\\ &+\alpha f_{tt}+\alpha' f_t-\alpha' af-\alpha af_t+\alpha\varphi'+\alpha'\varphi\\ =\ & |f_{ij}|^2-\alpha^2n^2K^2+2\nabla f\nabla(\Delta f)-\alpha(f_t-|\nabla f|^2-af)_t\\ &+\alpha a\Delta f-2\nabla f\nabla(f_t)+\alpha f_{tt}+\alpha' f_t\\ &-\alpha' af-\alpha cf_t+\alpha\varphi'+\alpha'\varphi\end{aligned}$$

$$
\begin{aligned}
&= |f_{ij}|^2 - \alpha^2 n^2 K^2 + 2\nabla f \nabla(\Delta f) + \alpha(|\nabla f|^2)_t + \alpha a \Delta f \\
&\quad -2\nabla f \nabla(f_t) + \alpha' f_t - \alpha' a f + \alpha\varphi' + \alpha'\varphi \\
&= |f_{ij}|^2 - \alpha^2 n^2 K^2 + 2\nabla f \nabla(\Delta f) + 2\alpha \nabla f \nabla(f_t) + 2\alpha R_{ij} f_i f_j \\
&\quad +\alpha a \Delta f - 2\nabla f \nabla(f_t) + \alpha' f_t - \alpha' a f + \alpha\varphi' + \alpha'\varphi \\
&= |f_{ij}|^2 + 2\alpha R_{ij} f_i f_j - \alpha^2 n^2 K^2 + 2\nabla f \nabla F + 2a(\alpha-1)(l-1)|\nabla f|^2 \\
&\quad +\alpha a \Delta f + \alpha' f_t - \alpha' a f + \alpha\varphi' + \alpha'\varphi. \qquad (3.8)
\end{aligned}
$$

Further, by utilizing the unit matrix $(\delta_{ij})_{n\times n}$ and (3.8), we obtain

$$
\begin{aligned}
(\Delta-\partial_t)F \geqslant\ & |f_{ij} + \frac{\varphi}{n}\delta_{ij}|^2 - 2\alpha K|\nabla f|^2 - \alpha^2 n^2 K^2 + 2\nabla f \nabla F \\
& +2a(\alpha-1)|\nabla f|^2 + a\alpha\Delta f + \alpha' f_t - \alpha' a f + \alpha\varphi' + \alpha'\varphi \\
& -\frac{\varphi^2}{n} - 2\frac{\varphi}{n}\Delta f \\
=\ & |f_{ij} + \frac{\varphi}{n}\delta_{ij}|^2 + (\frac{2\varphi}{n} - 2\alpha K)|\nabla f|^2 - (\frac{2\varphi}{n} - \alpha')f_t \\
& +(\frac{2\varphi}{n} - \alpha')cu^{l-1} - \alpha^2 n^2 K^2 - 2\nabla f \nabla F + 2a(\alpha-1)|\nabla f|^2 \\
& +a\alpha\Delta f + \alpha\varphi' + \alpha'\varphi - \frac{\varphi^2}{n} + (\frac{2\varphi}{n} - \alpha')\frac{\alpha\varphi}{\alpha}. \qquad (3.9)
\end{aligned}
$$

3.3. The proof of Theorem

In this section, we will prove Theorem 3.1 .

Proof of Theorem 3.1. Let $G = \gamma(t)F$ and $\gamma(t) > 0$ be non-decreasing. Then

$$
\begin{aligned}
(\Delta-\partial_t)G &= \gamma(\Delta-\partial_t)F - \gamma' F \\
&\geqslant \gamma|f_{ij} + \frac{\varphi}{n}g_{ij}|^2 + (\frac{2\varphi}{n} - \alpha')\frac{1}{\alpha}G - \gamma\alpha^2 n^2 K^2 - 2\nabla f \nabla G \\
&\quad +2a\gamma(\alpha-1)|\nabla f|^2 + a\gamma\alpha\Delta f - \gamma' F \\
&= \gamma|f_{ij} + \frac{\varphi}{n}g_{ij}|^2 + \Big[(\frac{2\varphi}{n} - \alpha')\frac{1}{\alpha} - \frac{\gamma'}{\gamma}\Big]G - \gamma\alpha^2 n^2 K^2 \\
&\quad -2\nabla f \nabla G + 2a\gamma(\alpha-1)|\nabla f|^2 + a\gamma\alpha\Delta f. \qquad (3.10)
\end{aligned}
$$

Now, let $\varphi(r)$ be a C^2 function on $[0,\infty)$ such that

$$
\varphi(r) = \begin{cases} 1, & if\ r \in [0,1], \\ 0, & if\ r \in [2,\infty), \end{cases}
$$

and

$$
0 \leqslant \varphi(r) \leqslant 1, \quad \varphi'(r) \leqslant 0, \quad \varphi''(r) \leqslant 0, \quad \frac{|\varphi'(r)|}{\varphi(r)} \leqslant C,
$$

where C is an absolute constant. Define by

$$\phi(x,t)=\varphi(d(x,x_0,t))=\varphi\left(\frac{d(x,x_0,t)}{R}\right)=\varphi\left(\frac{\rho(x,t)}{R}\right),$$

where $\rho(x,t)=d(x,x_0,t)$. By using maximum principle, the argument of Calabi [2] allows us to suppose that the function $\phi(x,t)$ with support in $B_{2R,T}$, is C^2 at the maximum point. By utilizing the Laplacian theorem, we deduce that

$$\frac{|\nabla\phi|^2}{\phi}\leqslant\frac{C}{R^2},\quad -\Delta\phi\leqslant\frac{C}{R^2}(1+\sqrt{K}R). \tag{3.11}$$

For any $0\leqslant T_1\leqslant T$, let $H=\phi G$ and (x_1,t_1) be the point in B_{2R,T_1} at which H attains its maximum value. We can suppose that H is positive, because otherwise the proof is trivial. Then at the point (x_1,t_1), we infer

$$\left.\begin{aligned}0=\nabla(\phi G)=G\nabla\phi+\phi\nabla G,\\ \Delta(\phi G)\leqslant 0,\\ \partial_t(\phi G)\geqslant 0.\end{aligned}\right\} \tag{3.12}$$

By the evolution formula of the geodesic length along the Ricci flow [6], we calculate

$$\begin{aligned}\phi_t G=&-G\phi'\left(\frac{\rho}{R}\right)\frac{1}{R}\frac{d\rho}{dt}=G\phi'\left(\frac{\rho}{R}\right)\int_{\gamma_{t_1}}\mathrm{Ric}(S,S)ds\\ \leqslant& G\phi'\left(\frac{\rho}{R}\right)\frac{1}{R}K\rho\leqslant G\phi'\left(\frac{\rho}{R}\right)K_2\leqslant G\sqrt{C}K,\end{aligned}$$

where γ_{t_1} is the geodesic connecting x and x_0 along the metric $g(t_1)$, S is the unite tangent vector to γ_{t_1}, and ds is the element of the arc length.

All the following computations are at the point (x_1,t_1). Since

$$\begin{aligned}|f_{ij}+\frac{\varphi}{n}\delta_{ij}|^2\geqslant&\frac{1}{n}\Big(\mathbf{tr}|f_{ij}+\frac{\varphi}{n}\delta_{ij}|\Big)^2\\ =&\frac{1}{n}\Big(\Delta f+\varphi\Big)^2\\ =&\frac{1}{n}\Big[-\frac{1}{\alpha}F-\frac{1}{\alpha}(\alpha-1)|\nabla f|^2\Big]^2\\ =&\frac{1}{\alpha^2 n}\Big[\frac{G}{\gamma}+(\alpha-1)|\nabla f|^2\Big]^2,\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\Delta f=&f_t-|\nabla f|^2-af\\ =&-\frac{F}{\alpha}-\frac{\alpha-1}{\alpha}|\nabla f|^2-\varphi<0.\end{aligned} \tag{3.14}$$

Case 1 $a \leqslant 0$. Combining (3.14) with (3.10), we have

$$(\Delta-\partial_t)G \geqslant \gamma|f_{ij}+\frac{\varphi}{n}\delta_{ij}|^2+\Big[(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}-\frac{\gamma'}{\gamma}\Big]G-\gamma\alpha^2n^2K^2$$
$$-2\nabla f\nabla G+2a\gamma(\alpha-1)|\nabla f|^2.$$

Using (3.12) and (3.13), we infer

$$\begin{aligned}
0 &\geqslant (\Delta-\partial_t)(\phi G)\\
&= G\Big(\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}\Big)+\phi(\Delta-\partial_t)G-\gamma G\phi_t\\
&\geqslant G\Big(\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}\Big)+\frac{\phi\gamma}{\alpha^2 n}\Big[\frac{G}{\gamma}+(\alpha-1)|\nabla f|^2\Big]^2\\
&\quad+\Big[(\frac{2\varphi}{2}-\alpha')\frac{1}{\alpha}-\frac{\gamma'}{\gamma}\Big]\phi G-\gamma\phi\alpha^2n^2K^2-2\phi\nabla f\nabla G\\
&\quad+2a\phi\gamma(\alpha-1)|\nabla f|^2-G\sqrt{C}K.
\end{aligned} \tag{3.15}$$

Multiply ϕ to (3.15), and we have

$$\begin{aligned}
0 &\geqslant \phi G\Big[\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}+(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}-\frac{\gamma'}{\gamma}\Big]+\frac{\phi^2\gamma}{\alpha^2 n}\Big[\frac{G}{\gamma}+(\alpha-1)|\nabla f|^2\Big]^2\\
&\quad-\gamma\phi^2\alpha^2n^2K^2-2\phi^2\nabla f\nabla G+2a\phi^2\gamma(\alpha-1)|\nabla f|^2-\phi G\sqrt{C}K\\
&\geqslant \phi G\Big[\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}+(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}-\frac{\gamma'}{\gamma}\Big]+\frac{\phi^2G^2}{\alpha^2 n\gamma}+\frac{\phi^2(\alpha-1)^2\gamma}{n\alpha^2}|\nabla f|^4\\
&\quad+\frac{2\phi^2(\alpha-1)}{n\alpha^2}G|\nabla f|^2-\gamma\phi^2\alpha^2n^2K^2+2\phi G\nabla\phi\nabla f\\
&\quad+2a\phi^2\gamma(\alpha-1)|\nabla f|^2-\phi G\sqrt{C}K.
\end{aligned} \tag{3.16}$$

Using the fact

$$\frac{2\phi^2(\alpha-1)}{n\alpha^2}G|\nabla f|^2+2\phi G\nabla\phi\nabla f \geqslant -\frac{n\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi}\phi G,$$

and

$$\frac{\phi^2(\alpha-1)^2\gamma}{n\alpha^2}|\nabla f|^4+2a\phi^2\gamma(\alpha-1)|\nabla f|^2 \geqslant -na^2\alpha^2\gamma\phi^2,$$

to (3.16), we deduce that

$$\begin{aligned}
0 &\geqslant \phi G\Big[\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}+(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}-\frac{\gamma'}{\gamma}-\frac{n\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi}-\sqrt{C}K\Big]\\
&\quad+\frac{\phi^2G^2}{\alpha^2 n\gamma}-\gamma\phi^2\alpha^2n^2K^2-na^2\alpha^2\gamma\phi^2\\
&\geqslant \Big[-\frac{C}{R^2}(1+\sqrt{k}R)-\frac{2C}{R^2}+(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}-\frac{\gamma'}{\gamma}-\frac{n\alpha^2}{2(\alpha-1)}\frac{C}{R^2}-\sqrt{C}K\Big]\phi G
\end{aligned}$$

$$+\frac{\phi^2 G^2}{\alpha^2 n\gamma}-\gamma\phi^2\alpha^2 n^2 K^2-na^2\alpha^2\gamma\phi^2.$$

For the inequality $Ax^2-2Bx\leqslant C$, one has $x\leqslant\frac{2B}{A}+\left(\frac{C}{A}\right)^{\frac{1}{2}}$, where $A,B,C>0$. Hence, we infer

$$\begin{aligned}\phi G(x,T_1)&\leqslant(\phi G)(x_1,t_1)\\&\leqslant\Big\{n\gamma\alpha^2\left[\frac{C}{R^2}(1+\sqrt{K}R)+\frac{n\alpha^2}{2(\alpha-1)}\frac{C}{R^2}+\sqrt{C}K\right]\\&\quad+n\gamma\alpha^2\left[\frac{\gamma'}{\gamma}-(\frac{2\varphi}{n}-\frac{\alpha'}{\alpha})\frac{1}{\alpha}\right]\\&\quad+n^{\frac{3}{2}}\gamma\alpha^2\phi K+na\alpha^2\gamma\phi\Big\}(x_1,t_1).\end{aligned}$$

If γ is nondecreasing which satisfies the system

$$\begin{cases}\dfrac{\gamma'}{\gamma}-(\dfrac{2\varphi}{n}-\alpha')\dfrac{1}{\alpha}\leqslant 0,\\ \dfrac{\gamma\alpha^4}{\alpha-1}\leqslant C_1.\end{cases}$$

Recall that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1<T_1$. Hence, we have

$$\begin{aligned}\phi G(x,T_1)&\leqslant(\phi G)(x_1,t_1)\\&\leqslant n\gamma(T_1)\alpha^2(T_1)\left[\frac{C}{R^2}\left(1+\sqrt{K}R\right)+K\right]+\frac{n^2C}{R^2}\\&\quad+n^{\frac{3}{2}}\gamma(T_1)\alpha^2(T_1)K+na\alpha^2(T_1)\gamma(T_1).\end{aligned}$$

Hence, we have for $\phi\equiv 1$ on $B_{R,T}$,

$$\begin{aligned}F(x,T_1)\leqslant{}&n\alpha^2(T_1)\left[\frac{C}{R^2}\left(1+\sqrt{K}R\right)+CK\right]+\frac{n^2C}{R^2\gamma(T_1)}\\&+n^{\frac{3}{2}}\alpha^2(T_1)K+na\alpha^2(T_1).\end{aligned}$$

If γ is nondecreasing which satisfies the system

$$\begin{cases}\dfrac{\gamma'}{\gamma}-(\dfrac{2\varphi}{n}-\alpha')\dfrac{1}{\alpha}\leqslant 0,\\ \dfrac{\gamma}{\alpha-1}\leqslant C_2.\end{cases}$$

Recall that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1<T_1$. Hence, we have

$$\phi G(x,T_1)\leqslant(\phi G)(x_1,t_1)$$

$$\leqslant n\gamma(T_1)\alpha^2(T_1)\left[\frac{C}{R^2}\Big(1+\sqrt{K}R\Big)+\frac{Cn\alpha^4}{R^2}+CK\right]$$
$$+n^{\frac{3}{2}}\gamma(T_1)\alpha^2(T_1)K+na\alpha^2(T_1)\gamma(T_1).$$

Hence, we have for $\phi\equiv 1$ on $B_{R,T}$,

$$F(x,T_1)\leqslant n\alpha^2(T_1)\left[\frac{C}{R^2}\Big(1+\sqrt{K}R\Big)+K\right]+\frac{n^2C\alpha^4}{R^2\gamma(T_1)}$$
$$+n^{\frac{3}{2}}\alpha^2(T_1)K+na\alpha^2(T_1).$$

Because T_1 is arbitrary in $0<T_1<T$, the conclusion is valid.

Case 2 $a\geqslant 0$. It is not difficult to find $\Delta f\leqslant -\frac{F}{\alpha}$ form (3.14). Then, we have from (3.10)

$$(\Delta-\partial_t)G\geqslant \gamma|f_{ij}+\frac{\varphi}{n}\delta_{ij}|^2+\Big[(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}-\frac{\gamma'}{\gamma}\Big]G$$
$$-\gamma\alpha^2n^2K^22\nabla f\nabla G-aG.$$

Using (3.13) and (3.13), we infer

$$\begin{aligned}
0&\geqslant(\Delta-\partial_t)(\phi G)\\
&=G\Big(\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}\Big)+\phi(\Delta-\partial_t)G-\gamma G\phi_t\\
&\geqslant G\Big(\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}\Big)+\frac{\phi\gamma}{\alpha^2 n}\Big[\frac{G}{\gamma}+(\alpha-1)|\nabla f|^2\Big]^2\\
&\quad+\Big[(\frac{2\varphi}{2}-\alpha')\frac{1}{\alpha}-\frac{\gamma'}{\gamma}\Big]\phi G-\gamma\phi\alpha^2n^2K^2-2\phi\nabla f\nabla G\\
&\quad-a\phi G-G\sqrt{C}K.
\end{aligned}$$

Multiply ϕ, and we have

$$\begin{aligned}
0&\geqslant \phi G\Big[\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}+(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}-\frac{\gamma'}{\gamma}\Big]+\frac{\phi^2\gamma}{\alpha^2 n}\Big[\frac{G}{\gamma}+(\alpha-1)|\nabla f|^2\Big]^2\\
&\quad-\gamma\phi^2\alpha^2n^2K^2-2\phi^2\nabla f\nabla G-a\phi^2G-\phi G\sqrt{C}K\\
&\geqslant \phi G\Big[\Delta\phi-2\frac{|\nabla\phi|^2}{\phi}+(\frac{2\varphi}{n}-\alpha')\frac{1}{\alpha}-\frac{\gamma'}{\gamma}\Big]+\frac{\phi^2G^2}{\alpha^2n\gamma}+\frac{2\phi^2(\alpha-1)}{n\alpha^2}G|\nabla f|^2\\
&\quad-\gamma\phi^2\alpha^2n^2K^2+2\phi G\nabla\phi\nabla f-a\phi^2G-\phi G\sqrt{C}K,
\end{aligned}\tag{3.17}$$

where we dropped the term $\frac{\phi^2(\alpha-1)^2\gamma}{n\alpha^2}|\nabla f|^4$. We use the fact

$$\frac{2\phi^2(\alpha-1)}{n\alpha^2}G|\nabla f|^2+2\phi G\nabla\phi\nabla f\geqslant -\frac{n\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi}\phi G,$$

to (3.17), we deduce that

$$
\begin{aligned}
0 &\geqslant \phi G\Big[\Delta\phi - 2\frac{|\nabla\phi|^2}{\phi} + (\frac{2\varphi}{n} - \alpha')\frac{1}{\alpha} - \frac{\gamma'}{\gamma} - \frac{n\alpha^2}{2(\alpha-1)}\frac{|\nabla\phi|^2}{\phi} - a\phi - \sqrt{C}K\Big] \\
&\quad + \frac{\phi^2 G^2}{\alpha^2 n\gamma} - \gamma\phi^2\alpha^2 n^2 K^2 \\
&\geqslant \Big[-\frac{C}{R^2}(1+\sqrt{k}R) - \frac{2C}{R^2} + (\frac{2\varphi}{n} - \alpha')\frac{1}{\alpha} - \frac{\gamma'}{\gamma} - \frac{n\alpha^2}{2(\alpha-1)}\frac{C}{R^2} - \sqrt{C}K\Big]\phi G \\
&\quad + \frac{\phi^2 G^2}{\alpha^2 n\gamma} - \gamma\phi^2\alpha^2 n^2 K^2.
\end{aligned}
$$

For the inequality $Ax^2 - 2Bx \leqslant C$, one has $x \leqslant \frac{2B}{A} + \left(\frac{C}{A}\right)^{\frac{1}{2}}$, where $A, B, C > 0$.

$$
\begin{aligned}
\phi G(x, T_1) &\leqslant (\phi G)(x_1, t_1) \\
&\leqslant \Big\{n\gamma\alpha^2\Big[\frac{C}{R^2}(1+\sqrt{K}R) + \frac{n\alpha^2}{2(\alpha-1)}\frac{C}{R^2} + a\phi + \sqrt{C}K\Big] \\
&\quad + n\gamma\alpha^2\Big[\frac{\gamma'}{\gamma} - (\frac{2\varphi}{n} - \frac{\alpha'}{\alpha})\frac{1}{\alpha}\Big] + n^{\frac{3}{2}}\gamma\alpha^2\phi K\Big\}(x_1, t_1).
\end{aligned}
$$

If γ is nondecreasing which satisfies the system

$$
\begin{cases}
\dfrac{\gamma'}{\gamma} - (\dfrac{2\varphi}{n} - \alpha')\dfrac{1}{\alpha} \leqslant 0, \\
\dfrac{\gamma\alpha^4}{\alpha-1} \leqslant C_1.
\end{cases}
$$

Recall that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1 < T_1$. Hence, we have

$$
\begin{aligned}
\phi G(x, T_1) &\leqslant (\phi G)(x_1, t_1) \\
&\leqslant n\gamma(T_1)\alpha^2(T_1)\Big[\frac{C}{R^2}\Big(1+\sqrt{K}R\Big) + a\phi + K\Big] + \frac{n^2 C}{R^2} \\
&\quad + n^{\frac{3}{2}}\gamma(T_1)\alpha^2(T_1)K.
\end{aligned}
$$

Hence, we have for $\phi \equiv 1$ on $B_{R,T}$,

$$
\begin{aligned}
\sup_{B_R} F(x, T_1) &\leqslant n\alpha^2(T_1)\Big[\frac{C}{R^2}\Big(1+\sqrt{K}R\Big) + a + CK\Big] + \frac{n^2 C}{R^2\gamma(T_1)} \\
&\quad + n^{\frac{3}{2}}\alpha^2(T_1)K.
\end{aligned}
$$

If γ is nondecreasing which satisfies the system

$$
\begin{cases}
\dfrac{\gamma'}{\gamma} - (\dfrac{2\varphi}{n} - \alpha')\dfrac{1}{\alpha} \leqslant 0, \\
\dfrac{\gamma}{\alpha-1} \leqslant C_2.
\end{cases}
$$

Recall that $\alpha(t)$ and $\gamma(t)$ are non-decreasing and $t_1 < T_1$. Hence, we have

$$\begin{aligned}\phi G(x,T_1) &\leqslant (\phi G)(x_1,t_1)\\ &\leqslant n\gamma(T_1)\alpha^2(T_1)\left[\frac{C}{R^2}\left(1+\sqrt{K}R\right)+\frac{Cn\alpha^4}{R^2}+a\phi+CK\right]\\ &\quad +n^{\frac{3}{2}}\gamma(T_1)\alpha^2(T_1)K.\end{aligned}$$

Hence, we have for $\phi \equiv 1$ on $B_{R,T}$,

$$\begin{aligned}F(x,T_1) \leqslant n\alpha^2(T_1)\left[\frac{C}{R^2}\left(1+\sqrt{K}R\right)+a+K\right]+\frac{n^2C\alpha^4}{R^2\gamma(T_1)}\\ +n^{\frac{3}{2}}\alpha^2(T_1)K.\end{aligned}$$

Because T_1 is arbitrary in $0 < T_1 < T$, the conclusion is valid. This proof is complete.

4. Harnack inequalities

In this section, as application of main theorems, some Harnack inequalities are derived.

THEOREM 4.1. *Let $(M^n, g(x,t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.7). Suppose that $|\mathrm{Ric}| \leqslant K$ for some $K > 0$, and all $(x,t) \in M^n \times [0,T]$. Assume that $u(x,t)$ is a positive solution to (1.6). Let $h(x,t)$ be a function defined on $M^n \times [0,T]$ which is C^1 in t and C^2 in x, satisfying $|\nabla h|^2 \leqslant \delta_2 h$ and $\Delta h \geqslant -\delta_3$ on $M^n \times [0,T]$ for some positive constants δ_2 and δ_3. Then for all $(x_1,t_1) \in M^n \times (0,T)$ and $(x_2,t_2) \in M^n \times (0,T)$ such that $t_1 < t_2$, we have*

$$u(x_2,t_2) \leqslant \begin{cases} u(x_1,t_1)\times \exp\left(\Gamma(t_1,t_2,\delta_1,\delta_2,\delta_3,\overline{u}_1)\right), & l \leqslant 1,\\ u(x_1,t_1)\times \exp\left(\Lambda(t_1,t_2,\delta_1,\delta_2,\delta_3,\overline{u}_1)\right), & l > 1,\end{cases}$$

where $\gamma(s)$ is a smooth curve connecting x_1 and x_2 with $\gamma(1) = x_1$ and $\gamma(0) = x_2$, and

$$\begin{aligned}&\Gamma(t_1,t_2,\delta_1,\delta_2,\delta_3,\overline{u}_1)\\ &=\int_0^1 \frac{|\gamma'(s)|^4}{2(t_2-t_1)^2}ds+\int_{t_1}^{t_2}\frac{\alpha^2(t)}{32}dt+\int_{t_1}^{t_2}[\varphi+C\alpha^2(K+\mu_1)+\delta_1\overline{u}_1]dt,\\ &\Lambda(t_1,t_2,\delta_1,\delta_2,\delta_3,\overline{u}_1)\\ &=\int_0^1 \frac{|\gamma'(s)|^4}{2(t_2-t_1)^2}ds+\int_{t_1}^{t_2}\frac{\alpha^2(t)}{32}dt+\int_{t_1}^{t_2}[\varphi+C\alpha^2(K+\mu_2)]dt\\ &\quad+\int_{t_1}^{t_2}\left[\delta\sqrt{\frac{l\alpha-1}{l-1}}\sqrt{\overline{u}_2\delta_2}+\alpha^{\frac{3}{2}}\sqrt{n(l-1)\varphi\delta_1}\right]dt.\end{aligned}$$

Proof. Firstly, the estimate in Corollary 2.2 can be written as

$$\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha(t)\frac{u_t(x,t)}{u(x,t)} + \alpha(t)h(x,t)u^{l-1}(x,t)$$
$$\leqslant \begin{cases} \alpha\varphi + C\alpha^2(K+\mu_1), & l \leqslant 1, \\ \alpha\varphi + C\alpha^2(K+\mu_2) + \delta\sqrt{\frac{l\alpha-1}{l-1}}\sqrt{\overline{u}_2\delta_2} & \\ \qquad + \alpha^{\frac{3}{2}}\sqrt{n(l-1)\varphi\delta_1}, & l > 1, \end{cases} \tag{4.1}$$

where $\mu_1 = \sqrt{\overline{u}_1\delta_3} + \frac{\alpha(3-2l)-1}{\alpha-1}\overline{u}_1\delta_1 + \sqrt{(2-l)\overline{u}_1\delta_2}$ and $\mu_2 = (l-1)\overline{u}_2\delta_1 + \sqrt{\overline{u}_2\delta_3}$.

Now we only prove the conclusion for $l \leqslant 1$.

Define $l(s) = \log u(\gamma(s),(1-s)t_2 + st_1)$. Obviously, we infer that $l(0) = \log u(x_2,t_2)$ and $l(1) = \log u(x_1,t_1)$. Direct calculation shows

$$\begin{aligned} \frac{\partial l(s)}{\partial s} &= (t_2-t_1)\left(\frac{\nabla u}{u}\frac{\gamma'(s)}{t_2-t_1} - \frac{u_t}{u}\right) \\ &\leqslant (t_2-t_1)\left[\frac{\nabla u}{u}\frac{\gamma'(s)}{t_2-t_1} - \frac{1}{\alpha(t)}\frac{|\nabla u|^2}{u^2} - h(x,t)u^{l-1} + \varphi + C\alpha(K+\mu_1)\right] \\ &\leqslant \frac{\alpha(t)}{4}\frac{|\gamma'(s)|^2}{t_2-t_1} + (t_2-t_1)[\varphi + C\alpha(K+\mu) + \delta_1\overline{u}_1]. \end{aligned}$$

Integrating the above inequality over $\gamma(s)$, we obtain

$$\begin{aligned} \log\frac{u(x_1,t_1)}{u(x_2,t_2)} &= \int_0^1 \frac{\partial l(s)}{\partial s}ds \\ &\leqslant \int_0^1 \left[\frac{\alpha(t)}{4}\frac{|\gamma'(s)|^2}{t_2-t_1} + (t_2-t_1)[\varphi + C\alpha(K+\mu_1) + \delta_1\overline{u}_1]\right]ds \\ &\leqslant \int_0^1 \frac{|\gamma'(s)|^4}{2(t_2-t_1)^2}ds + \int_{t_1}^{t_2}\frac{\alpha^2(t)}{32}dt \\ &\quad + \int_{t_1}^{t_2}[\varphi + C\alpha(K+\mu) + \delta_1\overline{u}_1]dt. \end{aligned}$$

The proof is complete.

We also derive an Harnack inequality for the equation (1.6). The proof is similar to Theorem 4.1, so we omit it.

THEOREM 4.2. *Let $(M^n, g(x,t))_{t\in[0,T]}$ be a complete solution to the Ricci flow* (1.7). *Suppose that $|\mathrm{Ric}| \leqslant K$ for some $K > 0$, and all $(x,t) \in M^n \times [0,T]$. Assume that $u(x,t)$ is a positive solution to* (1.6). *Then for all $(x_1,t_1) \in M^n \times (0,T)$ and $(x_2,t_2) \in M^n \times (0,T)$ such that $t_1 < t_2$, we have*

$$\begin{aligned} u(x_2,t_2) \leqslant & u(x_1,t_1) \\ & \times \exp\left(\int_0^1 \frac{|\gamma'(s)|^4}{2(t_2-t_1)^2}ds + \int_{t_1}^{t_2}\frac{\alpha^2(t)}{32}dt + \int_{t_1}^{t_2}[\varphi + C\alpha(K+|a|) + |a\log N|]dt\right), \end{aligned}$$

where $N = \max_{M^n\times[0,T]} u$.

5. Application to heat equation

According to Theorem 2.1 and Theorem 3.1, we derive corresponding gradient estimates and Harnack inequalities to the heat equation along the Ricci flow.

THEOREM 5.1. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.7). Assume that $|\mathrm{Ric}(x,t)| \leqslant K$ for some $K > 0$ and all $t \in [0,T]$. Suppose that there exist three functions $\alpha(t)$, $\varphi(t)$ and $\gamma(t)$ satisfy the conditions (C1), (C2), (C3) and (C4).*

Given $x_0 \in M$ and $R > 0$, let $u(x,t)$ be a positive solution of the heat equation

$$u_t = \Delta u, \tag{5.1}$$

in the cube $B_{2R,T} := \{(x,t)|d(x,x_0,t) \leqslant 2R, 0 \leqslant t \leqslant T\}$, where c is a constant.

If $\frac{\gamma\alpha^4}{\alpha-1} \leqslant C_1$ for some constant C_1, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + K\right) + \frac{Cn^2}{R^2\gamma} + \alpha\varphi.$$

Here $C = C(n, C_2)$ is a constant.

If $\frac{\gamma}{\alpha-1} \leqslant C_2$ for some constant C_2, then

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leqslant C\alpha^2\left(\frac{1}{R^2} + \frac{\sqrt{K}}{R} + K\right) + \frac{Cn^2\alpha^4}{R^2\gamma} + \alpha\varphi.$$

Here $C = C(n, C_2)$ is a constant.

COROLLARY 5.1. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.7). Assume that $|\mathrm{Ric}(x,t)| \leqslant K$ for some $K > 0$ and all $t \in [0,T]$. Given $x_0 \in M$ and $R > 0$, let $u(x,t)$ be a positive solution of the heat equation (5.1) in the cube $B_{2R,T} := \{(x,t)|d(x,x_0,t) \leqslant 2R, 0 \leqslant t \leqslant T\}$. Then the following special estimates are valid.*

1. *Li-Yau type:*

$$\alpha(t) = constant, \quad \varphi(t) = \frac{n}{t} + \frac{nK\alpha^2}{\alpha-1}, \gamma(t) = t^\theta \quad with \quad 0 < \theta \leqslant 2.$$

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \leqslant C\alpha^2\left[\frac{1}{R^2}(1+\sqrt{K}R) + \frac{\alpha^2}{\alpha-1}\frac{1}{R^2} + K\right] + \alpha\varphi + n^{\frac{3}{2}}\alpha^2 K.$$

2. *Hamilton type:*

$$\alpha(t) = e^{2Kt}, \quad \varphi(t) = \frac{n}{t}e^{4Kt}, \quad \gamma(t) = te^{2Kt}.$$

$$\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}\leqslant C\alpha^2\left[\frac{1}{R^2}(1+\sqrt{K}R)+K\right]+\frac{C\alpha^4}{R^2te^{2Kt}}$$
$$+\alpha\varphi+n^{\frac{3}{2}}\alpha^2K.$$

3. Li-Xu type:

$$\alpha(t)=1+\frac{\sinh(Kt)\cosh(Kt)-Kt}{\sinh^2(Kt)},\quad \varphi(t)=2nK[1+\coth(Kt)],$$
$$\gamma(t)=\tanh(Kt).$$

$$\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}\leqslant C\alpha^2\left[\frac{1}{R^2}(1+\sqrt{K}R)+K\right]+\frac{C}{R^2\tanh(Kt)}$$
$$+\alpha\varphi+n^{\frac{3}{2}}\alpha^2K.$$

4. Linear Li-Xu type:

$$\alpha(t)=1+2Kt,\quad \varphi(t)=\frac{n}{t}+nK(1+2Kt+\mu Kt),\gamma(t)=Kt\quad \textit{with}\quad \mu\geqslant\frac{1}{4}.$$

$$\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}\leqslant C\alpha^2\left[\frac{1}{R^2}(1+\sqrt{K}R)+K\right]+\frac{C\alpha^4}{R^2Kt}$$
$$+\alpha\varphi+n^{\frac{3}{2}}\alpha^2K.$$

Let $R\to\infty$, a global estimate is derived.

COROLLARY 5.2. *Let $(M^n,g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.7). Assume that $|\mathrm{Ric}(x,t)|\leqslant K$ for some $K>0$ and all $t\in[0,T]$. Suppose that there exist three functions $\alpha(t)$, $\varphi(t)$ and $\gamma(t)$ which satisfy the following conditions (C1), (C2), (C3) and (C4).*

Given $x_0\in M$ and $R>0$, let $u(x,t)$ be a positive solution of the heat equation (5.2) in the cube $M^n\times[0,T]$. Then

$$\frac{|\nabla u|^2}{u^2}-\alpha\frac{u_t}{u}\leqslant C\alpha^2K+\alpha\varphi,$$

where $C=C(n,C_1,C_2)$ is a constant.

Using Theorem 4.1, we derive a Harnack inequality.

COROLLARY 5.3. *(Harnack Inequality) Let $(M^n,g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.7). Suppose that $|\mathrm{Ric}|\leqslant K$ for some $K>0$, and all $(x,t)\in M^n\times[0,T]$. Assume that $u(x,t)$ is a positive solution to (5.1). Then for all $(x_1,t_1)\in M^n\times(0,T)$ and $(x_2,t_2)\in M^n\times(0,T)$ such that $t_1<t_2$, we have*

$$u(x_2,t_2)\leqslant u(x_1,t_1)$$

$$\times \exp\left(\int_0^1 \frac{|\gamma'(s)|^4}{2(t_2-t_1)^2}ds + \int_{t_1}^{t_2} \frac{\alpha^2(t)}{32}dt + \int_{t_1}^{t_2}[\varphi + C\alpha K]dt\right),$$

where $C = C(n)$ and $\gamma(s)$ is a smooth curve connecting x_1 and x_2 with $\gamma(1) = x_1$ and $\gamma(0) = x_2$.

6. Appendix

We will check some special functions $\alpha(t) > 1$, $\varphi(t) > 0$ and $\gamma(t) > 0$ satisfy the following two systems

$$\begin{cases} \dfrac{2\varphi}{n} - 2\alpha K \geqslant (\dfrac{2\varphi}{n} - \alpha')\dfrac{1}{\alpha}, \\ \dfrac{2\varphi}{n} - \alpha' > 0, \\ \dfrac{\varphi^2}{n} + \alpha\varphi' \geqslant 0, \end{cases} \tag{6.1}$$

and

$$\begin{cases} \dfrac{\gamma'}{\gamma} - (\dfrac{2\varphi}{n} - \alpha')\dfrac{1}{\alpha} \leqslant 0, \\ \dfrac{\gamma\alpha^4}{\alpha - 1} \leqslant C, \ or \ \dfrac{\gamma}{\alpha - 1} \leqslant C. \end{cases} \tag{6.2}$$

Besides, $\alpha(t)$ and $\gamma(t)$ are non-decreasing.

(1) Let $\alpha(t) = 1 + 2Kt$, $\varphi(t) = \frac{n}{t} + nK(1 + 2Kt + \mu Kt)$ $(\mu \geqslant \frac{1}{4})$ and $\gamma(t) = Kt$.

One can has

(i) $$\frac{2\varphi}{n} - \alpha' = \frac{2}{t} + 2K(1 + 2Kt + \mu Kt) - 2K > 0,$$

(ii) $$\begin{aligned} \frac{\varphi^2}{n} + \alpha\varphi' &= \frac{n}{t^2} + nK^2(1 + 2Kt + \mu Kt)^2 + \frac{2nK}{t}(1 + 2Kt + \mu Kt) \\ &\quad + (1 + 2Kt)(-\frac{n}{t^2} + 2nK^2 + n\mu K^2) \\ &= nK^2(1 + 2Kt + \mu Kt)^2 + \frac{2nK}{t}(2Kt + \mu Kt) \\ &\quad + (1 + 2Kt)(2nK^2 + n\mu K^2) > 0, \end{aligned}$$

(iii) $$\begin{aligned} &\frac{2\varphi}{n} - 2\alpha K - (\frac{2\varphi}{n} - \alpha')\frac{1}{\alpha} \\ =&\frac{2}{t} + 2K(1 + 2Kt + \mu Kt) - 2K(1 + 2Kt) \\ &- \left[\frac{2}{t} + 2K(1 + 2Kt + \mu Kt) - 2K\right] \cdot \frac{1}{1 + 2Kt} \\ =&\frac{4Kt(\mu K^2t^2 - Kt + 1)}{t(1 + 2Kt)} \geqslant 0, \quad for \quad \mu \geqslant \frac{1}{4}. \end{aligned}$$

Hence, $\alpha(t) = 1+2Kt$, $\varphi(t) = \frac{n}{t} + nK(1+2Kt+\mu Kt)$ $(0 < \mu \leqslant \frac{1}{4})$ satisfy system (6.1).

On the other hand, one has

$$\begin{aligned}
&\frac{\gamma'}{\gamma} - (\frac{2\varphi}{n} - \alpha')\frac{1}{\alpha} \\
&= \frac{1}{t} - \left(\frac{2}{t} + 2K(1+2Kt+\mu Kt) - 2K\right)\frac{1}{1+2Kt} \\
&= \frac{1}{t(1+2Kt)}\left[-(4K^2+2K\mu)t^2 + 2Kt - 1\right] \\
&= \frac{1}{t(1+2Kt)}\left[-(3K^2+2K\mu)t^2 - (Kt-1)^2\right] \\
&\leqslant 0, \quad for \quad t \geqslant 0,
\end{aligned}$$

and $\frac{\gamma}{\alpha-1} = \frac{1}{2}$. So, (6.2) is also satisfied.

(2) $\alpha(t) = e^{2Kt}$, $\varphi(t) = \frac{n}{t}e^{4Kt}$ and $\gamma(t) = te^{2Kt}$, where $(0 < Kt \leqslant 1)$. Direct calculation gives

$$\text{(i)} \quad \frac{2\varphi}{n} - \alpha' = \frac{2}{t}e^{2Kt}(e^{2Kt} - Kt) > 0,$$

$$\text{(ii)} \quad \frac{\varphi^2}{n} + \alpha\varphi' = \frac{n}{t^2}e^{6Kt}(e^{2Kt} - 1 + 4Kt) > 0,$$

$$\begin{aligned}
\text{(iii)} \quad &\frac{2\varphi}{n} - 2\alpha K - (\frac{2\varphi}{n} - \alpha')\frac{1}{\alpha} = \frac{2}{t}e^{4Kt} - 2Ke^{2Kt} - \frac{2}{t}e^{2Kt} + 2K \\
&= (e^{2Kt} - 1)(\frac{2}{t}e^{2Kt} - 2K) \geqslant 0.
\end{aligned}$$

Hence, $\alpha(t) = e^{2Kt}$ and $\varphi(t) = \frac{n}{t}e^{4Kt}$ satisfy system (6.1).

Besides, we have

$$\begin{aligned}
&\frac{\gamma'}{\gamma} - (\frac{2\varphi}{n} - \alpha')\frac{1}{\alpha} \\
&= \frac{1+2Kt}{t} - \left(\frac{2}{t}e^{2Kt} - 2K\right) \\
&= \frac{1}{t}(1 + 4Kt - 2e^{2Kt}) \\
&\leqslant 0, \quad for \quad t \geqslant 0,
\end{aligned}$$

and as $t \to 0^+$, $\frac{\gamma}{\alpha-1} = \frac{te^{2Kt}}{e^{2Kt}-1} \to \frac{1}{2K}$. This implies $\frac{\gamma}{\alpha-1} \leqslant C$. So, (6.2) is also satisfied.

(3) $\alpha(t) = 1 + \frac{\sinh(Kt)\cosh(Kt) - Kt}{\sinh^2(Kt)}$, $\varphi(t) = 2nK[1+\coth(Kt)]$ and $\gamma(t) = \tanh(Kt)$. Direct calculation gives

$$\text{(i)} \quad \frac{2\varphi}{n} - \alpha' = 4K[1+\coth(Kt)] - 2K + 2K\coth^2(Kt) - \frac{2K^2t}{\sinh^2(Kt)}\coth(Kt),$$

$$= 2K + 2K(1+\alpha)\coth(Kt) > 0,$$

(ii) $$\alpha(\frac{2\varphi}{n} - 2\alpha K) - (\frac{2\varphi}{n} - \alpha')$$
$$= 4K\alpha[1+\coth(Kt)] - 2K\alpha^2 - [2K + 2K(1+\alpha)\coth(Kt)]$$
$$= 2K\alpha\Big[1+\coth(Kt) + \frac{Kt}{\sinh^2(Kt)}\Big] - [2K + 2K(1+\alpha)\coth(Kt)]$$
$$= 2K(\alpha-1)\frac{Kt}{\sinh^2(Kt)} > 0,$$

(iii) $$\frac{\varphi^2}{n} + \alpha\varphi' = \frac{2nK^2}{\sinh^2(Kt)}\Big[2(1+\coth(Kt))^2\sinh^2(Kt) - \alpha\Big]$$
$$= \frac{2nK^2}{\sinh^2(Kt)}\Big[2e^{2Kt} - 1 - \frac{e^{4Kt} - 1 - 4Kte^{2Kt}}{(e^{2Kt}-1)^2}\Big]$$
$$= \frac{4nK^2e^{2Kt}}{(e^{2Kt}-1)^2\sinh^2(Kt)}\Big[e^{4Kt} - 3e^{2Kt} + 2 + 4Kt\Big].$$

Let $f(x) = e^{4x} - 3e^{2x} + 2 + 4x$ with $x \leqslant 0$. Obviously, $f(0) = 0$ and

$$f'(x) = 4e^{4x} - 6e^{2x} + 4 > 0.$$

Then we get $f(x) > 0$ for $x > 0$. Hence, we have

$$(\frac{2\varphi}{n} - \alpha')\varphi + \alpha\varphi' + \alpha'\varphi - \frac{\varphi^2}{n}$$
$$= \frac{4nK^2e^{2Kt}}{(e^{2Kt}-1)^2\sinh^2(Kt)}\Big[e^{4Kt} - 3e^{2Kt} + 2 + 4Kt\Big] > 0.$$

Hence, $\alpha(t) = 1 + \frac{\sinh(Kt)\cosh(Kt) - Kt}{\sinh^2(Kt)}$ and $\varphi(t) = 2nK[1+\coth(Kt)]$ satisfy system (6.1).

On the other hand, as $t \to 0$, we have $\frac{\gamma\alpha^4}{\alpha-1} \to 2$; $\frac{\gamma\alpha^4}{\alpha-1} \to 1$ for $t \to \infty$. These imply $\frac{\gamma\alpha^4}{\alpha-1} \leqslant C$, here C is a universal constant.
Besides, we have

$$\frac{\gamma'}{\gamma} - (\frac{2\varphi}{n} - \alpha')\frac{1}{\alpha}$$
$$= \frac{1}{\alpha}\left[\frac{K\alpha}{\sinh(Kt)\cosh(Kt)} - 2K - 2K(1+\alpha)\coth(Kt)\right]$$
$$= \frac{1}{\alpha}\left[\frac{K}{\sinh(Kt)\cosh(Kt)}[\alpha - 2(1+\alpha)\cosh^2(Kt)] - 2K\right]$$
$$= \frac{1}{\alpha}\left[\frac{K}{\sinh(Kt)}[\alpha(1-2\cosh(Kt)) - 2\cosh(Kt)] - 2K\right]$$
$$\leqslant 0, \quad for \quad t \geqslant 0.$$

So, (6.2) is also satisfied.

(4) $\alpha(t) = constant$, $\varphi(t) = \frac{\alpha n}{t} + \frac{nK\alpha^2}{\alpha-1}$ and $\gamma(t) = t^\theta$ with $0 < \theta \leqslant 2$. Direct calculation gives

$$\text{(i)}\quad \frac{2\varphi}{n} - \alpha' = \frac{2}{n}\Big[\frac{\alpha n}{t} + \frac{nK\alpha^2}{\alpha-1}\Big] > 0,$$

$$\text{(ii)}\quad \frac{\varphi^2}{n} + \alpha\varphi' = \frac{n\alpha^2}{t^2} + \frac{n^2K^2\alpha^4}{n(\alpha-1)^2} + \frac{2nK\alpha^2}{(\alpha-1)t} - \frac{n\alpha^2}{t^2} > 0,$$

$$\begin{aligned}\text{(iii)}\quad &(\frac{2\varphi}{n} - 2\alpha K) - (\frac{2\varphi}{n} - \alpha')\frac{1}{\alpha} \\ =&\frac{2\varphi}{n\alpha}(\alpha-1) - 2K\alpha \\ \geqslant&\frac{2}{n\alpha}(\alpha-1)\frac{nK\alpha^2}{\alpha-1} - 2K\alpha = 0.\end{aligned}$$

Hence, $\alpha(t) = constant$, and $\varphi(t) = \frac{\alpha n}{t} + \frac{nK\alpha^2}{\alpha-1}$ which satisfy system (6.1).

On the other hand, we have

$$\begin{aligned}\frac{\gamma'}{\gamma} - (\frac{2\varphi}{n} - \alpha')\frac{1}{\alpha} &= \frac{\theta}{t} - \frac{2}{t} - \frac{2K\alpha}{\alpha-1} \\ &\leqslant 0, \quad for \quad t \geqslant 0 \quad and \quad 0 < \theta \leqslant 2.\end{aligned}$$

So, (6.2) is also satisfied.

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