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IMPROVEMENT AND GENERALIZATION OF SOME JENSEN–MERCER–TYPE INEQUALITIES

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Abstract. The present paper is devoted to the study of Jensen-Mercer-type inequalities. Our results generalize and improve some earlier results in the literature.

1. Introduction

The well-known Jensen inequality for the convex functions states that if f is a convex function on the interval $[m,M]$, then

$$f\left(\sum_{i=1}^{n} w_i a_i\right) \leqslant \sum_{i=1}^{n} w_i f(a_i) \tag{1.1}$$

for all $a_i \in [m,M]$ and $w_i \in [0,1]$ $(i=1,\ldots,n)$ with $\sum_{i=1}^{n} w_i = 1$. Various inequalities improving and extending (1.1) have been studied in [1, 6, 7].

Mercer [5] proved that if f is a convex function on $[m,M]$, then

$$f\left(M+m-\sum_{i=1}^{n} w_i a_i\right) \leqslant f(M)+f(m)-\sum_{i=1}^{n} w_i f(a_i) \tag{1.2}$$

for all $a_i \in [m,M]$ and $w_i \in [0,1]$ $(i=1,\ldots,n)$ with $\sum_{i=1}^{n} w_i = 1$. Several refinements and generalizations of the inequality (1.2) have been given in [4, 8].

In [3, Theorem 2.1] it has been shown that if f is a convex function on the interval $[m,M]$, then

$$f\left(M+m-\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(M+m-u)\,du \leqslant f(M)+f(m)-\frac{f(a)+f(b)}{2} \tag{1.3}$$

for all $a,b \in [m,M]$.

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In this paper we prove the following general result:

$$f(M+m-\overline{a}) \leqslant \sum_{i=1}^{n} \frac{w_i}{\overline{a}-a_i} \int_{M+m-\overline{a}}^{M+m-a_i} f(t)\,dt \leqslant f(M)+f(m)-\sum_{i=1}^{n} w_i f(a_i) \tag{1.4}$$

where $\overline{a} := \sum_{i=1}^{n} w_i a_i$. After that, we show a refinement of inequality (1.4) in the following form

$$f(M+m-\overline{a}) \leqslant \sum_{i=1}^{n} w_i f\left(M+m-\frac{\overline{a}+a_i}{2}\right) \leqslant \sum_{i=1}^{n} \frac{w_i}{\overline{a}-a_i} \int_{M+m-\overline{a}}^{M+m-a_i} f(t)\,dt$$
$$\leqslant f(M)+f(m)-\sum_{i=1}^{n} w_i f(a_i).$$

Though we confine our discussion to scalars, by changing the convex function assumption with the operator convex, the inequalities we obtain in this paper can be extended in a natural way to Hilbert space operators.

2. Main results

The following lemma is well-known in [5, Lemma 1.3], but we prove it for the reader convenience.

LEMMA 2.1. *Let f be a convex function on $[m,M]$, then*

$$f(M+m-a_i) \leqslant f(M)+f(m)-f(a_i), \ (m \leqslant a_i \leqslant M, \ i=1,\ldots,n).$$

Proof. If $f:[m,M] \to \mathbb{R}$ is a convex function, then for any $x,y \in [m,M]$ and $t \in [0,1]$, we have

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y). \tag{2.1}$$

It can be verified that if $m \leqslant a_i \leqslant M\,(i=1,\ldots,n)$, then $\frac{M-a_i}{M-m}, \frac{a_i-m}{M-m} \leqslant 1$ and $\frac{M-a_i}{M-m} + \frac{a_i-m}{M-m} = 1$. Thanks to (2.1), we have

$$f(a_i) \leqslant \frac{M-a_i}{M-m} f(m) + \frac{a_i-m}{M-m} f(M). \tag{2.2}$$

One the other hand, $m \leqslant a_i \leqslant M\,(i=1,\ldots,n)$ implies $m \leqslant M+m-a_i \leqslant M\,(i=1,\ldots,n)$. Thus, from (2.2) we infer

$$f(M+m-a_i) \leqslant \frac{a_i-m}{M-m} f(m) + \frac{M-a_i}{M-m} f(M). \tag{2.3}$$

Summing up (2.2) and (2.3), we get the desired result.

Based on this, our first result can be stated as follows:

THEOREM 2.1. *Let f be a convex function on $[m,M]$ and $t \in [0,1]$. Then*

$$f(M+m-\overline{a}) \leqslant \sum_{i=1}^{n} w_i f(M+m-((1-t)\overline{a}+ta_i)) \leqslant f(M)+f(m)-\sum_{i=1}^{n} w_i f(a_i) \tag{2.4}$$

for all $a_i \in [m,M]$ and $w_i \in [0,1]$ $(i=1,\dots,n)$ with $\sum_{i=1}^{n} w_i = 1$, where $\overline{a} := \sum_{i=1}^{n} w_i a_i$. Moreover, the function $F : [0,1] \to \mathbb{R}$ defined by

$$F(t) = \sum_{i=1}^{n} w_i f(M+m-((1-t)\overline{a}+ta_i))$$

is monotonically nondecreasing and convex on $[0,1]$.

Proof. Firstly, we have

$$\begin{aligned}\sum_{i=1}^{n} w_i f(M+m-((1-t)\overline{a}+ta_i)) &\geqslant f\left(\sum_{i=1}^{n} w_i (M+m-((1-t)\overline{a}+ta_i))\right) \\ &= f(M+m-\overline{a}).\end{aligned} \tag{2.5}$$

On the other hand,

$$\begin{aligned}&\sum_{i=1}^{n} w_i f(M+m-((1-t)\overline{a}+ta_i)) \\ =&\sum_{i=1}^{n} w_i f((1-t)(M+m-\overline{a})+t(M+m-a_i)) \\ \leqslant&\sum_{i=1}^{n} w_i ((1-t) f(M+m-\overline{a})+tf(M+m-a_i)) \\ \leqslant&\sum_{i=1}^{n} w_i \left((1-t)\left(f(M)+f(m)-\sum_{j=1}^{n} w_j f(a_j)\right)+t(f(M)+f(m)-f(a_i))\right) \\ =&f(M)+f(m)-\sum_{i=1}^{n} w_i f(a_i).\end{aligned}$$

For the convexity of F, we have

$$\begin{aligned}&F\left(\frac{t+s}{2}\right) \\ =&\sum_{i=1}^{n} w_i f\left(M+m-\left(\left(1-\frac{t+s}{2}\right)\overline{a}+\frac{t+s}{2}a_i\right)\right) \\ =&\sum_{i=1}^{n} w_i f\left(M+m-\left(\frac{(1-t)\overline{a}+ta_i+(1-s)\overline{a}+sa_i}{2}\right)\right) \\ =&\sum_{i=1}^{n} w_i f\left(\frac{M+m-((1-t)\overline{a}+ta_i)+M+m-((1-s)\overline{a}+sa_i)}{2}\right)\end{aligned}$$

$$\leqslant \frac{1}{2}\left[\sum_{i=1}^{n} w_i f(M+m-((1-t)\overline{a}+ta_i))+\sum_{i=1}^{n} w_i f(M+m-((1-s)\overline{a}+sa_i))\right]$$
$$=\frac{F(t)+F(s)}{2}.$$

Now, if $0<s<t<1$, then $s=\frac{t-s}{t}\cdot 0+\frac{s}{t}\cdot t$ and hence the convexity of F implies

$$F(s)=F\left(\frac{t-s}{t}\cdot 0+\frac{s}{t}\cdot t\right)\leqslant \frac{t-s}{t}F(0)+\frac{s}{t}F(t)\leqslant \frac{t-s}{t}F(t)+\frac{s}{t}F(t)=F(t).$$

We remark that the second inequality in the above follows from (2.5) and the fact

$$F(0)=\sum_{i=1}^{n} w_i f(M+m-\overline{a})=f(M+m-\overline{a}).$$

Therefore F is monotonically nondecreasing on $[0,1]$.

COROLLARY 2.1. *Let all the assumptions of Theorem 2.1 hold, then*

$$f(M+m-\overline{a})\leqslant \sum_{i=1}^{n}\frac{w_i}{\overline{a}-a_i}\int_{M+m-\overline{a}}^{M+m-a_i} f(t)\,dt\leqslant f(M)+f(m)-\sum_{i=1}^{n} w_i f(a_i). \tag{2.6}$$

Proof. Integrating the inequality (2.4) over $t\in[0,1]$, we get the inequalities (2.6). Here we used the fact

$$\begin{aligned}\int_0^1 f(M+m-((1-t)\overline{a}+ta_i))\,dt &= \int_0^1 f((1-t)(M+m-\overline{a})+t(M+m-a_i))\,dt\\ &= \int_0^1 f(t(M+m-\overline{a})+(1-t)(M+m-a_i))\,dt\\ &= \frac{1}{\overline{a}-a_i}\int_{M+m-\overline{a}}^{M+m-a_i} f(t)\,dt.\end{aligned}$$

REMARK 2.1. Put $n=2$, $w_1=w_2=1/2$, $a_1=a$, and $a_2=b$ in Corollary 2.1, then

$$f\left(M+m-\frac{a+b}{2}\right)\leqslant \frac{1}{b-a}\int_a^b f(M+m-u)\,du\leqslant f(M)+f(m)-\frac{f(a)+f(b)}{2}$$

which shows that our inequalities (2.6) generalize inequalities (1.3).

We give a more precise estimate in the next theorem.

THEOREM 2.2. *Let f be a convex function on $[m,M]$. Then*

$$\begin{aligned}f(M+m-\overline{a})\leqslant \sum_{i=1}^{n} w_i f\left(M+m-\frac{\overline{a}+a_i}{2}\right)&\leqslant \sum_{i=1}^{n}\frac{w_i}{\overline{a}-a_i}\int_{M+m-\overline{a}}^{M+m-a_i} f(t)\,dt\\ &\leqslant f(M)+f(m)-\sum_{i=1}^{n} w_i f(a_i)\end{aligned} \tag{2.7}$$

for all $a_i \in [m,M]$ *and* $w_i \in [0,1]$ $(i=1,\dots,n)$ *with* $\sum_{i=1}^{n} w_i = 1$.

Proof. If $f:[m,M]\to\mathbb{R}$ is a convex function, then we have for any $a,b\in[m,M]$

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b+tb+(1-t)a}{2}\right) \\ &\leqslant \frac{f(ta+(1-t)b)+f(tb+(1-t)a)}{2} \\ &\leqslant \frac{f(a)+f(b)}{2}. \end{aligned}$$

Replacing a and b by $M+m-a$ and $M+m-b$, respectively, we get

$$\begin{aligned} f\left(M+m-\frac{a+b}{2}\right) &\leqslant \frac{f(M+m-(ta+(1-t)b))+f(M+m-(tb+(1-t)a))}{2} \\ &\leqslant \frac{f(M+m-a)+f(M+m-b)}{2}. \end{aligned}$$

Integrating the inequality over $t\in[0,1]$, and using the fact

$$\int_0^1 f(tx+(1-t)y)\,dt = \int_0^1 f(ty+(1-t)x)\,dt,$$

we infer that

$$\begin{aligned} f\left(M+m-\frac{a+b}{2}\right) &\leqslant \int_0^1 f(M+m-(ta+(1-t)b))\,dt \\ &\leqslant \frac{f(M+m-a)+f(M+m-b)}{2}. \end{aligned}$$

Since $a_i, \overline{a}\in[m,M]$, we can write

$$f\left(M+m-\frac{\overline{a}+a_i}{2}\right) \leqslant \frac{1}{\overline{a}-a_i}\int_{M+m-\overline{a}}^{M+m-a_i} f(t)\,dt \leqslant \frac{f(M+m-\overline{a})+f(M+m-a_i)}{2},$$

due to

$$\int_0^1 f(M+m-(t\overline{a}+(1-t)a_i))\,dt = \frac{1}{\overline{a}-a_i}\int_{M+m-\overline{a}}^{M+m-a_i} f(t)\,dt.$$

Multiplying by $w_i>0$ $(i=1,\dots,n)$ and summing over i from 1 to n we may deduce

$$\begin{aligned} \sum_{i=1}^{n} w_i f\left(M+m-\frac{\overline{a}+a_i}{2}\right) &\leqslant \sum_{i=1}^{n} \frac{w_i}{\overline{a}-a_i}\int_{M+m-\overline{a}}^{M+m-a_i} f(t)\,dt \\ &\leqslant \frac{f(M+m-\overline{a})+\sum_{i=1}^{n} w_i f(M+m-a_i)}{2}. \end{aligned} \tag{2.8}$$

On the other hand, by (1.1)

$$f(M+m-\overline{a}) = f\left(\sum_{i=1}^{n} w_i\left(M+m-\frac{\overline{a}+a_i}{2}\right)\right) \leqslant \sum_{i=1}^{n} w_i f\left(M+m-\frac{\overline{a}+a_i}{2}\right) \tag{2.9}$$

and by Lemma 2.1

$$\begin{aligned}
&\frac{f(M+m-\overline{a})+\sum_{i=1}^{n} w_i f(M+m-a_i)}{2} \\
\leqslant&\frac{f(M)+f(m)-\sum_{j=1}^{n} w_j f(a_j)+f(M)+f(m)-\sum_{i=1}^{n} w_i f(a_i)}{2} \qquad (2.10)\\
=&f(M)+f(m)-\sum_{i=1}^{n} w_i f(a_i).
\end{aligned}$$

Combining (2.8), (2.9), and (2.10), we get (2.7).

COROLLARY 2.2. *Let* $a_i \in [m,M]$ *and* $w_i \in [0,1]$ $(i=1,\ldots,n)$ *with* $\sum_{i=1}^{n} w_i = 1$. *Then*

$$\begin{aligned}
\frac{Mm}{\prod_{i=1}^{n} a_i^{w_i}} &\leqslant \exp\left[\sum_{i=1}^{n} \frac{w_i}{\overline{a}-a_i}\int_{M+m-\overline{a}}^{M+m-a_i} \log t\,dt\right] \leqslant \prod_{i=1}^{n}\left(M+m-\frac{\overline{a}+a_i}{2}\right)^{w_i} \\
&\leqslant M+m-\sum_{i=1}^{n} w_i a_i.
\end{aligned}$$

Proof. Put $f(t) = -\log t$, $(0 < t \leqslant 1)$ in Theorem 2.2.

REMARK 2.2. If we set $n=2$, $a_1 = m, a_2 = M$ and $w_1 = w_2 = 1/2$ in Corollary 2.2, then we have

$$\sqrt{Mm} \leqslant \frac{M^{\frac{M}{M-m}}}{em^{\frac{m}{M-m}}} \leqslant \frac{1}{4}\sqrt{(M+3m)(m+3M)} \leqslant \frac{1}{2}(M+m).$$

One can obtain the inequalities for the weighted parameter in means of two variables m and M by elementary calculations. We leave it to the interested readers.

REMARK 2.3. Let all the assumptions of Theorem 2.2 hold, then

$$\begin{aligned}
f(\overline{a}) &\leqslant \sum_{i=1}^{n} w_i f\left(\frac{\overline{a}+a_i}{2}\right) \leqslant \sum_{i=1}^{n} \frac{w_i}{a_i-\overline{a}}\int_{\overline{a}}^{a_i} f(t)\,dt \leqslant \frac{f(\overline{a})+\sum_{i=1}^{n} w_i f(a_i)}{2} \\
&\leqslant \sum_{i=1}^{n} w_i f(a_i).
\end{aligned}$$

The proof is in the same spirit as that of Theorem 2.2 (see also [2, Corollary 3]).

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