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EXTENSIONS OF HÖLDER'S INEQUALITY AND ITS APPLICATIONS IN OSTROWSKI INEQUALITY

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Abstract. In this paper, we present several new extensions and refinements of Hölder's inequality and some related inequalities. These Hölder's inequalities are presented via function $f(t)$. My results generalize and extend the results of Tian. Furthermore, we apply our results to Ostrowski inequality to obtain some other interesting inequalities as special cases.

1. Introduction

Let $a_{ij} > 0$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $p_j > 1$, and $\sum_{j=1}^{n} p_j^{-1} = 1$. Then it is well known that the following Hölder's inequality holds :

$$\sum_{i=1}^{m}\prod_{j=1}^{n} a_{ij} \leqslant \prod_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}^{p_j}\right)^{\frac{1}{p_j}}. \tag{1}$$

In addition, we can get the opposite result according to $0 < p_n < 1, p_j < 0, j = 1,2,\dots, n-1$, $a_{ij} > 0$ and $\sum_{j=1}^{n} \frac{1}{p_j} = 1$.

$$\sum_{i=1}^{m}\prod_{j=1}^{n} a_{ij} \geqslant \prod_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}^{p_j}\right)^{\frac{1}{p_j}}. \tag{2}$$

Similarly, we can get the integral form of the Hölder's inequality,

$$\int_a^b \left(\prod_{j=1}^{n} f_j(x)\right) dx \leqslant \prod_{j=1}^{n}\left(\int_a^b f_j^{p_j}(x)dx\right)^{1/p_j}, \tag{3}$$

where $p_j > 1$, $f_j(x) > 0, j = 1,2,\dots,n$ and $\sum_{j=1}^{n} \frac{1}{p_j} = 1$.

$$\int_a^b \left(\prod_{j=1}^{n} f_j(x)\right) dx \geqslant \prod_{j=1}^{n}\left(\int_a^b f_j^{p_j}(x)dx\right)^{1/p_j}, \tag{4}$$

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where $0 < p_n < 1, p_j < 0, j = 1,2,\ldots,n-1,\ f_j(x) > 0$ and $\sum_{j=1}^{n} \frac{1}{p_j} = 1$.

Hölder's inequality occupies a very crucial position in the field of mathematical analysis. Through this study, we present some new versions of Hölder's inequalities and give their application to some special inequalities, such as Ostrowski inequality. In addition to the above considerations, this paper also introduces the application of Hölder's inequality on the time scales, and gives new forms of inequality based on the existing lemmas.

Yang's[25, 26] conclusions are of great help to this paper, which makes further research possible. The applications of integral inequality in Qi's[6, 7] gives profound inspiration to this paper and prompts many theorems on integral Hölder's inequality to be obtained. Tian[10, 11, 12, 13, 14, 16], Tian and Ha[17, 18], Tian et al.[19, 21] give many meaningful generalizations and applications of Hölder's inequality. For other recent Hölder's inequalities, please see the references.

Firstly, we learned the inspiration from the article in Yang[26] to get the functions $h(t)$ and $g(t)$

$$h(\tau) = \prod_{k=1}^{n} \left[\sum_{i=1}^{m} \left(\prod_{j=1}^{n} a_{ij} \right)^{1-\tau} \left(a_{ik}^{p_k} \right)^{\tau} \right]^{1/p_k}, \tag{5}$$

where $h : (-\infty, +\infty) \to (0, +\infty)$.

$$g(\tau) = \prod_{k=1}^{n} \left[\int_{\lambda_1}^{\lambda_2} \left(\prod_{j=1}^{n} f_j(\theta) \right)^{1-\tau} f_k^{p_k \tau}(\theta) d\theta \right]^{1/p_k}, t \in \mathbb{R}, \tag{6}$$

where $f_k(\theta) > 0,\ x \in [\lambda_1, \lambda_2], k = 1,2,\ldots,n$, and $f_k \in L^{p_k}[\lambda_1, \lambda_2]$.

In addition, we take into account more possibilities on this basis, thus extending it and getting a corresponding series of results (listed below).

Secondly, we also introduce Ostrowski inequalities in this paper:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \leqslant \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]$$

and

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \leqslant \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \parallel f' \parallel_\infty,$$

where $\parallel f' \parallel_\infty := sup_{t \in (a,b)} \mid f'(t) \mid < \infty$ and $x \in [a,b]$.

Moreover, we will consider applying Hölder's inequality to Ostrowski inequality in this paper, so as to extend Ostrowski inequality.

In addition to taking into account the above situation, we have added two additional inequalities on the time scales in the article, and we are particularly grateful for the inspiration provided by Chen[2].

2. Main results

In order to prove the main results, we need the following lemmas.

LEMMA 2.1. [25] *If* ξ_1 *and* ξ_2 *are positive numbers, then*

$$\left(ln\xi_1 - ln\xi_2\right)\left(\xi_1^\tau - \xi_2^\tau\right)\begin{cases}\geqslant 0, \text{ if } \tau \geqslant 0,\\ \leqslant 0, \text{ if } \tau \leqslant 0,\end{cases}$$

and the equal sign is established only when $(\xi_1 - \xi_2)\tau = 0$.

LEMMA 2.2. [25] *Let* $rp_k > 1$, $\sum_{k=1}^n \frac{1}{p_k} = r$, $a_{ij} > 0$, $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$, $b_i = \left(\prod_{j=1}^n a_{ij}\right)^{\frac{1}{r}}$, $d_{ik} = \frac{a_{ik}^{rp_k}}{\prod_{j=1}^n a_{ij}}$. *Then*

$$\sum_{k=1}^n \frac{1}{p_k}\left(\sum_{i=1}^m b_i ln d_{ik}\right) = 0.$$

LEMMA 2.3. [25] *Let* $rp_k > 1$, $\sum_{k=1}^n \frac{1}{p_k} = r$, $f_k(\theta) > 0$, $F(\theta) = \left(\prod_{j=1}^n f_j(\theta)\right)^{\frac{1}{r}}$, $g_k(\theta) = f_k^{rp_k}/F^r(\theta)$, $\theta \in [a,b] \to \mathbb{R}$, $k = 1,2,\ldots,n$. *Then*

$$\sum_{k=1}^n p_k^{-1} ln g_k(\theta) = 0, \theta \in [a,b].$$

Next, we can get the Lemma 2.4 from Kwon and Bae[5].

LEMMA 2.4. [5] *Let* ξ, ζ *be real numbers. Then*

$$\left(\xi - \zeta\right)\left(e^{t\xi} - e^{t\zeta}\right) \geqslant 0 \text{ if } t \geqslant 0,$$

and

$$\left(\xi - \zeta\right)\left(e^{t\xi} - e^{t\zeta}\right) \leqslant 0 \text{ if } t \leqslant 0.$$

LEMMA 2.5. [1] *Let* $f : [\xi, \delta] \to \mathbb{R}$ *be a twice differentiable mapping on* (ξ, δ), *then this equality holds*

$$\begin{aligned}\int_\xi^\delta f(t)dt =& (\delta - \xi)(1-h)f(\theta) - (\delta - \xi)(1-h)\left(\theta - \frac{\xi + \delta}{2}\right)f'(\theta)\\ &+ h\frac{\delta - \xi}{2}\left(f(\xi) + f(\delta)\right) - \frac{h^2(\delta - \xi)^2}{8}\left(f'(\delta) - f'(\xi)\right)\\ &+ \int_\xi^\delta K(\theta, t)f''(t)dt,\end{aligned}$$

for all $\theta \in \left[\xi + h\frac{\delta-\xi}{2}, \delta - h\frac{\delta-\xi}{2}\right]$ *and* $h \in [0,1]$. *Here* $K : [\xi, \delta]^2 \to \mathbb{R}$

$$K(\theta,t) = \begin{cases} \frac{1}{2}\left[t - \left(\xi + h\frac{\delta-\xi}{2}\right)\right]^2, \text{ if } t \in [\xi,\theta], \\ \frac{1}{2}\left[t - \left(\delta - h\frac{\delta-\xi}{2}\right)\right]^2, \text{ if } t \in (\theta,\delta]. \end{cases}$$

Analytic inequality([15], [22], [28], [29], [30], [31]), especially Hölder's inequality occupies a very high position in mathematics, Tian and Hu [20], Zhao and Cheung[32] have conducted in-depth research on Hölder's inequality. We refer to Qi's([8], [9]) researches on some characteristics of inequalities and give some theorems as follows. At this point, we use the discrete form of $h_r(\theta)$ derived above to give some theorems and corollaries.

The following theorems and corollaries are the results of this study.

THEOREM 2.6. *Let* $S_{ij} > 0$, $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$, $rp_k > 1$, $\sum_{k=1}^n 1/p_k = r$. *Define a positive function* $h_r(\theta) : \mathbb{R} \to \mathbb{R}^+$ *by*

$$h_r(\theta) = \prod_{k=1}^{n}\left[\sum_{i=1}^{m}\left(\prod_{j=1}^{n} S_{ij}\right)^{1-\theta}\left(S_{ik}^{rp_k}\right)^{\theta}\right]^{1/p_k}, \theta \in \mathbb{R}. \tag{7}$$

Then

$$h_r'(\theta) \geqslant 0 \qquad \text{if } \theta \geqslant 0, \text{and } h_r'(\theta) \leqslant 0 \qquad \text{if } \theta \leqslant 0. \tag{8}$$

Here, $h_r'(\theta) = 0$ *if and only if* $\theta = 0$ *or*

$$\frac{S_{ik}^{rp_k}}{\prod_{j=1}^n S_{ij}} = \frac{S_{jk}^{rp_k}}{\prod_{j=1}^n S_{jl}}, 1 \leqslant i,l \leqslant m, k = 1,2,\ldots,n. \tag{9}$$

Thus, we have

$$\left(\sum_{i=1}^{m}\prod_{j=1}^{n} S_{ij}\right)^r = \prod_{k=1}^{n}\left(\sum_{i=1}^{m}\prod_{j=1}^{n} S_{ij}\right)^{\frac{1}{p_k}} = h_r(0) \leqslant h_r(1) = \prod_{k=1}^{n}\left(\sum_{i=1}^{n} S_{ik}^{rp_k}\right)^{\frac{1}{p_k}}. \tag{10}$$

If $r = 1$, *then it is a refinement of* (1)

$$\sum_{i=1}^{m}\prod_{j=1}^{n} S_{ij} = h(0) \leqslant h(1) = \prod_{k=1}^{n}\left(\sum_{i=1}^{m} S_{ik}^{p_k}\right)^{\frac{1}{p_k}}.$$

Proof. According to the Yan[24], we can get

$$h_r'(\theta) = \begin{cases} \geqslant 0, \ \theta \geqslant 0, \\ \leqslant 0, \ \theta \leqslant 0, \end{cases}$$

where the conditions for $h_r'(\theta) = 0$ are $\theta = 0$ or $\frac{S_{ik}^{rp_k}}{\prod_{j=1}^n S_{ij}} = \frac{S_{jk}^{rp_k}}{\prod_{j=1}^n S_{jl}}, 1 \leqslant i,l \leqslant m, k = 1,2,\ldots,n$.

At the same time, we also can get $h_r''(\theta) \geqslant 0$ and $h_r'(\theta)\theta > 0$. So, for $h_r(0)$ and $h_r(1)$, we can get the following conclusion

$$\left(\sum_{i=1}^{m}\prod_{j=1}^{n}S_{ij}\right)^{r} = \prod_{k=1}^{n}\left(\sum_{i=1}^{m}\prod_{j=1}^{n}S_{ij}\right)^{\frac{1}{p_k}} = h_r(0) \leqslant h_r(1) = \prod_{k=1}^{n}\left(\sum_{i=1}^{n}S_{ik}^{rp_k}\right)^{\frac{1}{p_k}}.$$

If $r = 1$, then it is a refinement of (1)

$$\sum_{i=1}^{m}\prod_{j=1}^{n}S_{ij} = h(0) \leqslant h(1) = \prod_{k=1}^{n}\left(\sum_{i=1}^{m}S_{ik}^{p_k}\right)^{\frac{1}{p_k}}.$$

Thus, the proof of Theorem 2.6 is completed.

COROLLARY 2.7. *If (9) holds, then $h_r(\theta) = const$. Otherwise, $h_r''(\theta) > 0, t \in R$ and $h_r'(\theta)\theta > 0$ for $\theta \neq 0$, especially for $0 = \theta_1 < \theta_2 < \cdots < \theta_N = 1$, we have*

$$h_r(0) = h_r(\theta_1) < h_r(\theta_2) < \cdots < h_r(\theta_N) = h_r(1).$$

COROLLARY 2.8. *Let $R(a) = \prod_{k=1}^{n}\left(\sum_{i=1}^{n}S_{ik}^{rp_k}\right)^{\frac{1}{p_k}} - \left(\sum_{i=1}^{m}\prod_{j=1}^{n}S_{ij}\right)^{r}$, then $R(a) \geqslant 0$, and $R(a) = 0$ if and only if*

$$\frac{S_{ik}^{rp_k}}{\prod_{j=1}^{n}S_{ij}} = \frac{S_{jk}^{rp_k}}{\prod_{j=1}^{n}S_{jl}}, 1 \leqslant i,l \leqslant m, k = 1,2,\ldots,n.$$

COROLLARY 2.9. *Let $n = 2, p > 1, q = p/(p-1), r = 1$, then we have*

$$\sum_{i=1}^{m}S_{i1}S_{i2} = \left(\sum_{i=1}^{m}S_{i1}^{p}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{m}S_{i2}^{q}\right)^{\frac{1}{q}} - R(a).$$

Let $S_k = S_{i1}, T_k = S_{i2}$, then

$$\sum_{k=1}^{m}S_kT_k = \left(\sum_{k=1}^{m}S_k^{p}\right)^{\frac{1}{p}}\left(\sum_{k=1}^{m}T_k^{q}\right)^{\frac{1}{q}} - R(a),$$

where $R(a) \geqslant 0$ and $R(a) = 0$ if and only if

$$\frac{S_k^p}{\sum_{k=1}^{m}S_k^p} = \frac{T_k^q}{\sum_{k=1}^{m}T_k^q}.$$

THEOREM 2.10. *Let $f_k(x) > 0$, $x \in [a,b]$, $f_k \in L^{p_k}[a,b]$ and let $p_k, k = 1,2,\ldots,n$, be positive real numbers satisfying $\sum_{k=1}^{n}\frac{1}{p_k} = r$. Define a positive function $g_r(\theta)$: $\mathbb{R} \to \mathbb{R}^+$ by*

$$g_r(\theta) = \prod_{k=1}^{n}\left[\int_a^b \left(\prod_{j=1}^{n}f_j(x)\right)^{1-\theta} f_k^{rp_k\theta}(x)dx\right]^{\frac{1}{p_k}}. \tag{11}$$

Then

$$g_r'(\theta) \geqslant 0 \text{ if } \theta \geqslant 0, \text{ and } g_r'(\theta) \leqslant 0 \text{ if } \theta \leqslant 0.$$

Here $g_r'(\theta) = 0$ *if and only if* $\theta = 0$ *or*

$$\frac{f_k^{rp_k}(x)}{\prod_{j=1}^n f_j(x)} = c_k = const. \tag{12}$$

The function $g_r(\theta)$ *defined by (14) is concave, that is* $g_r''(\theta) \geqslant 0$, *for all* $\theta \in \mathbb{R}$, *and the equal sign is established only when*

$$\frac{f_k^{rp_k}(x)}{\prod_{j=1}^n f_j(x)} = c_k = const.$$

Thus, we can have $g_r(0) \leqslant g_r(1)$, *and this reduces to (2) because*

$$\left[\int_a^b \left(\prod_{j=1}^n f_j(x)\right) dx\right]^r = \prod_{k=1}^n \left[\int_a^b \left(\prod_{j=1}^n f_j(x)\right) dx\right]^{\frac{1}{p_k}} = g_r(0)$$
$$\leqslant g_r(1) = \prod_{k=1}^n \left(\int_a^b f_k^{rp_k}(x) dx\right)^{\frac{1}{p_k}}.$$

If $r = 1$, *then it is a refinement of (2)*

$$\int_a^b \left(\prod_{j=1}^n f_j(x)\right) dx = g_r(0) \leqslant g_r(1) = \prod_{k=1}^n \left(\int_a^b f_k^{p_k}(x) dx\right)^{\frac{1}{p_k}}.$$

Proof. According to the Yan[24], we can get

$$g_r'(\theta) = \begin{cases} \geqslant 0, \ \theta \geqslant 0, \\ \leqslant 0, \ \theta \leqslant 0, \end{cases}$$

where the conditions for $g_r'(\theta) = 0$ are $\theta = 0$ or $\frac{f_k^{rp_k}(x)}{\prod_{j=1}^n f_j(x)} = c_k = const$.
At the same time, we also can get $g_r''(\theta) \geqslant 0$ and $g_r'(\theta)\theta > 0$. So, for $g_r(0)$ and $g_r(1)$, we can get the following conclusion

$$\left[\int_a^b \left(\prod_{j=1}^n f_j(x)\right) dx\right]^r = \prod_{k=1}^n \left[\int_a^b \left(\prod_{j=1}^n f_j(x)\right) dx\right]^{\frac{1}{p_k}} = g_r(0)$$
$$\leqslant g_r(1) = \prod_{k=1}^n \left(\int_a^b f_k^{rp_k}(x) dx\right)^{\frac{1}{p_k}}.$$

If $r = 1$, then it is a refinement of (2)

$$\int_a^b \left(\prod_{j=1}^n f_j(x)\right) dx = g_r(0) \leqslant g_r(1) = \prod_{k=1}^n \left(\int_a^b f_k^{p_k}(x) dx\right)^{\frac{1}{p_k}}.$$

Thus, the proof of Theorem 2.10 is completed.

COROLLARY 2.11. *If* $g_k(x) = const$, *then* $g_r(\theta) = const$. *Otherwise,* $g_r''(\theta) > 0, \theta \in \mathbb{R}$ *and* $g_r'(\theta)\theta > 0$ *for* $\theta \neq 0$, *especially for* $0 = \theta_1 < \theta_2 < \cdots < \theta_N = 1$, *we have*

$$g_r(0) = g_r(\theta_1) < g_r(\theta_2) < \cdots < g_r(\theta_N) = g_r(1),$$
$$g_r(0) = \prod_{k=1}^{n}\left(\int_a^b F^r(x)dx\right)^{\frac{1}{p_k}} \leqslant g_r(1) = \prod_{k=1}^{n}\left(\int_a^b F^r(x)g_k(x)dx\right)^{\frac{1}{p_k}}.$$

COROLLARY 2.12. *Let* $p_k > 1, \sum_{k=1}^{n}\frac{1}{p_k} = r, f_j(x) > 0, 1 \leqslant j \leqslant n$. *Then*

$$g_r(0) \leqslant g_r(\frac{1}{2}) \leqslant g_r(1),$$

that is

$$\prod_{k=1}^{n}\left(\int_a^b \prod_{j=1}^{n} f_j(x)dx\right)^{\frac{1}{p_k}} \leqslant \prod_{k=1}^{n}\left(\int_a^b (\prod_{j=1}^{n} f_j(x))^{\frac{1}{2}} f_k^{\frac{rp_k}{2}} dx\right)^{\frac{1}{p_k}} \leqslant \prod_{k=1}^{n}\left(\int_a^b f_k^{rp_k}(x)dx\right)^{\frac{1}{p_k}}.$$

The equalities hold if and only if (12) *hold*

$$\frac{f_k^{rp_k}(x)}{\prod_{j=1}^{n} f_j(x)} = c_k = const.$$

THEOREM 2.13. *Let* $f : [\theta_1, \theta_2] \to \mathbb{R}$ *be continuous, differentiable on* (θ_1, θ_2) *and* $| f'' |$ *is log-convex. Thus, for some* $p > 1$, *we have the following inequality:*

$$\begin{aligned}
&\left|\int_{\theta_1}^{\theta_2} f(t)dt - \left(\theta_2 - \theta_1\right)\left(1-h\right)f(x) + \left(\theta_2 - \theta_1\right)\left(1-h\right)\left(x - \frac{\theta_1+\theta_2}{2}\right)f'(x)\right.\\
&\left. - h\frac{\theta_2-\theta_1}{2}\left(f(\theta_1)+f(\theta_2)\right) + \frac{h^2\left(\theta_2-\theta_1\right)^2}{8}\left(f'(\theta_2)-f'(\theta_1)\right)\right|\\
\leqslant &\frac{1}{4}\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left[\left(x-\theta_1-h\frac{\theta_2-\theta_1}{2}\right)^{2q+1} + 2\left(h\frac{\theta_2-\theta_1}{2}\right)^{2q+1}\right.\\
&\left. + \left(\theta_2 - x - h\frac{\theta_2-\theta_1}{2}\right)^{2q+1}\right]^{\frac{1}{q}}\left(\int_{\theta_1}^{\theta_2} | f''(t) |^p \, dt\right)^{\frac{1}{p}},
\end{aligned}$$

where $q = \frac{p}{p-1}$, $h \in [0,1]$ *and* $\theta_1 + \frac{\theta_2-\theta_1}{2}h \leqslant x \leqslant \theta_2 - \frac{\theta_2-\theta_1}{2}h$.

Proof. By Lemma 2.5, we can get

$$\begin{aligned}
&\left|\int_{\theta_1}^{\theta_2} f(t)dt - (\theta_2-\theta_1)(1-h)f(x) + (\theta_2-\theta_1)(1-h)\left(x - \frac{\theta_1+\theta_2}{2}\right)f'(x)\right.\\
&\left. - h\frac{\theta_2-\theta_1}{2}\left(f(\theta_1)+f(\theta_2)\right) + \frac{h^2(\theta_2-\theta_1)^2}{8}\left(f'(\theta_2)-f'(\theta_1)\right)\right|
\end{aligned}$$

$$=\left|\int_{\theta_1}^{\theta_2} f''(t)K(x,t)dt\right| \leqslant \int_{\theta_1}^{\theta_2} \mid f''(t) \mid\mid K(x,t) \mid dt$$
$$\leqslant\left(\int_{\theta_1}^{\theta_2} \mid f''(t) \mid^p dt\right)^{\frac{1}{p}}\left(\int_{\theta_1}^{\theta_2} \mid K(x,t) \mid^q dt\right)^{\frac{1}{q}},$$

where $q=\frac{p}{p-1}$ and

$$K(x,t)=\begin{cases}\frac{1}{2}\left[t-\left(\theta_1+h\frac{\theta_2-\theta_1}{2}\right)\right]^2, \text{ if } t\in[\theta_1,x],\\ \frac{1}{2}\left[t-\left(\theta_2-h\frac{\theta_2-\theta_1}{2}\right)\right]^2, \text{ if } t\in(x,\theta_2].\end{cases}$$

Since for $\zeta_1 \leqslant \xi \leqslant \zeta_2$,

$$\int_{\zeta_1}^{\zeta_2} \mid \tau-\xi \mid^q dt=\int_{\zeta_1}^{\xi} \mid \tau-\xi \mid^q dt+\int_{\xi}^{\zeta_2} \mid \tau-\xi \mid^q dt$$
$$=\frac{1}{q+1}\left((\xi-\zeta_1)^{q+1}+(\zeta_2-\xi)^{q+1}\right).$$

We have

$$\left(\int_{\theta_1}^{\theta_2} \mid K(x,t) \mid^q dt\right)^{\frac{1}{q}}$$
$$=\left(\int_{\theta_1}^{x} \mid K(x,t) \mid^q dt+\int_{x}^{\theta_2} \mid K(x,t) \mid^q dt\right)^{\frac{1}{q}}$$
$$=\left[\int_{\theta_1}^{x}\left(\frac{1}{2}\left(t-\left(\theta_1+h\frac{\theta_2-\theta_1}{2}\right)\right)\right)^{2q}dt+\int_{x}^{\theta_2}\left(\frac{1}{2}\left(t-\left(\theta_2-h\frac{\theta_2-\theta_1}{2}\right)\right)\right)^{2q}dt\right]^{\frac{1}{q}}$$
$$=\frac{1}{4}\left[\int_{\theta_1}^{x}\left(t-\left(\theta_1+h\frac{\theta_2-\theta_1}{2}\right)\right)^{2q}dt+\int_{x}^{\theta_2}\left(t-\left(\theta_2-h\frac{\theta_2-\theta_1}{2}\right)\right)^{2q}dt\right]^{\frac{1}{q}}$$
$$=\frac{1}{4}\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left[\left(x-\theta_1-h\frac{\theta_2-\theta_1}{2}\right)^{2q+1}-\left(-h\frac{\theta_2-\theta_1}{2}\right)^{2q+1}\right.$$
$$\left.+\left(h\frac{\theta_2-\theta_1}{2}\right)^{2q+1}-\left(x-\theta_2+h\frac{\theta_2-\theta_1}{2}\right)^{2q+1}\right]^{\frac{1}{q}}$$
$$=\frac{1}{4}\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left[\left(x-\theta_1-h\frac{\theta_2-\theta_1}{2}\right)^{2q+1}+2\left(h\frac{\theta_2-\theta_1}{2}\right)^{2q+1}\right.$$
$$\left.+\left(\theta_2-x-h\frac{\theta_2-\theta_1}{2}\right)^{2q+1}\right]^{\frac{1}{q}}.$$

Thus, the proof of Theorem 2.13 is completed.

COROLLARY 2.14. *Let $f(t)=t^{\alpha}, \alpha \in (0,1), [\theta_1,\theta_2] \to [0,1]$. Then the inequality in theorem 2.13 can be changed into*

$$\left|\frac{1}{\alpha+1}-(1-h)x^{\alpha}+(1-h)\left(x-\frac{1}{2}\right)\alpha x^{\alpha-1}-\frac{h}{2}+\frac{\alpha h^2}{8}\right|$$
$$\leqslant \frac{1}{4}\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left(2\left(\frac{h}{2}\right)^{2q+1}+\left(x-\frac{h}{2}\right)^{2q+1}+\left(1-x-\frac{h}{2}\right)^{2q+1}\right)^{\frac{1}{q}}\frac{\alpha(1-\alpha)}{(1+(\alpha-2)p)^{\frac{1}{p}}},$$

where $x \in [\frac{h}{2}, 1-\frac{h}{2}]$, $h \in [0,1]$ and $p>1, q=\frac{p}{p-1}$.
In particular, for $h=0$, we have

$$\left|\frac{1}{\alpha+1}-x^{\alpha}+\left(x-\frac{1}{2}\right)\alpha x^{\alpha-1}\right|$$
$$\leqslant \frac{1}{4}\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left(x^{2q+1}+(1-x)^{2q+1}\right)^{\frac{1}{q}}\frac{\alpha(1-\alpha)}{(1+(\alpha-2)p)^{\frac{1}{p}}},$$

where $x \in [0,1]$.

COROLLARY 2.15. *Let $h=\frac{1}{2}$, this inequality in Theorem 2.13 reduces to*

$$\left|\int_{\theta_1}^{\theta_2} f(t)dt-\frac{\theta_2-\theta_1}{2}f(x)+\frac{\theta_2-\theta_1}{2}\left(x-\frac{\theta_1+\theta_2}{2}\right)f'(x)\right.$$
$$\left.-\frac{\theta_2-\theta_1}{4}\left(f(\theta_1)+f(\theta_2)\right)+\frac{(\theta_2-\theta_1)^2}{32}\left(f'(\theta_2)-f'(\theta_1)\right)\right|$$
$$\leqslant \frac{1}{4}\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left[\left(x-\theta_1-\frac{\theta_2-\theta_1}{4}\right)^{2q+1}\right.$$
$$\left.+2\left(\frac{\theta_2-\theta_1}{4}\right)^{2q+1}+\left(\theta_2-x-\frac{\theta_2-\theta_1}{4}\right)^{2q+1}\right]^{\frac{1}{q}}\left(\int_{\theta_1}^{\theta_2}|f''(t)|^p\,dt\right)^{\frac{1}{p}}.$$

Let $x=\frac{\theta_1+\theta_2}{2}$, this inequality reduces to

$$\left|\int_{\theta_1}^{\theta_2} f(t)dt-\frac{\theta_2-\theta_1}{2}f\left(\frac{\theta_1+\theta_2}{2}\right)-\frac{\theta_2-\theta_1}{4}\left(f(\theta_1)+f(\theta_2)\right)\right.$$
$$\left.+\frac{(\theta_2-\theta_1)^2}{32}\left(f'(\theta_2)-f'(\theta_1)\right)\right|$$
$$\leqslant \frac{1}{4}\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left[2\left(\frac{\theta_2-\theta_1}{2}\right)^{2q+1}\right]^{\frac{1}{q}}\left(\int_{\theta_1}^{\theta_2}|f''(t)|^p\,dt\right)^{\frac{1}{p}}.$$

COROLLARY 2.16. *Under the assumptions of Theorem 2.13, the following inequality holds:*

$$\left|\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2} f(t)dt-\frac{f(\theta_1)+f(\theta_2)}{2}+\frac{\theta_2-\theta_1}{8}\left(f'(\theta_2)-f'(\theta_1)\right)\right|$$

$$\leqslant \frac{1}{2(\theta_2-\theta_1)}\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left[\left(x-\frac{\theta_1+\theta_2}{2}\right)^{2q+1}+\left(\frac{\theta_2-\theta_1}{2}\right)^{2q+1}\right]^{\frac{1}{q}}\left(\int_{\theta_1}^{\theta_2}|f''(t)|^p\,dt\right)^{\frac{1}{p}}.$$

If we choose $x=\frac{\theta_1+\theta_2}{2}$, *we can get*

$$\left|\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}f(t)dt-\frac{f(\theta_1)+f(\theta_2)}{2}+\frac{\theta_2-\theta_1}{8}\left(f'(\theta_2)-f'(\theta_1)\right)\right|$$
$$\leqslant \frac{1}{2(\theta_2-\theta_1)}\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left(\frac{\theta_2-\theta_1}{2}\right)^{2+1/q}\left(\int_{\theta_1}^{\theta_2}|f''(t)|^p\,dt\right)^{\frac{1}{p}}.$$

Next we apply the Hölder's inequality to the Ostrowski inequality and will further extend the application of the inequality proved by Yang[27].

THEOREM 2.17. *Let* $f:[\xi,\delta]\to\mathbb{R}$ *be a twice differentiable mapping on* (ξ,δ) *and* $f'\in L^p[\xi,\delta]$ *it for some* $p>1$. *Then*

$$\left|\int_\xi^\delta f(t)\mathrm{d}t-\left[f(x)(1-h)+\frac{f(\xi)+f(\delta)}{2}h\right](\delta-\xi)\right|$$
$$\leqslant\left(\frac{|f'(\xi)|^x}{|f'(x)|^\xi}\right)^{\frac{1}{x-\xi}}\left(\frac{T^{qx}-T^{q\xi}}{q\ln T}\right)^{\frac{1}{q}}\times\left[\frac{\left(x-\left(\xi+h\frac{\delta-\xi}{2}\right)\right)^{p+1}-\left(h\frac{\delta-\xi}{2}\right)^{p+1}}{p+1}\right]^{\frac{1}{p}}$$
$$+\left(\frac{|f'(x)|^\delta}{|f'(\delta)|^x}\right)^{\frac{1}{\delta-x}}\left(\frac{M^{q\delta}-M^{qx}}{q\ln M}\right)^{\frac{1}{q}}\times\left[\frac{\left(h\frac{\delta-\xi}{2}\right)^{p+1}-\left(x-\left(\delta-h\frac{\delta-\xi}{2}\right)\right)^{p+1}}{p+1}\right]^{\frac{1}{p}},$$

where $q=p/(p-1), h\in[0,1]$ *and* $\xi+h(\delta-\xi)/2\leqslant x\leqslant\delta-h(\delta-\xi)/2$. *Here* $T=\left(\frac{|f'(x)|}{|f'(\xi)|}\right)^{\frac{1}{x-\xi}}$ *and* $M=\left(\frac{|f'(\delta)|}{|f'(x)|}\right)^{\frac{1}{\delta-x}}$. *And also* $T,M\neq 1$.

Proof. Let $r:[\xi,\delta]^2\to\mathbb{R}$ be given by

$$r(x,t)=\begin{cases} t-\left[\xi+h\frac{(\delta-\xi)}{2}\right], & t\in[\xi,x],\\ t-\left[\delta-h\frac{(\delta-\xi)}{2}\right], & t\in[x,\delta].\end{cases}$$

From Yang[27], we can get the following conclusions:

$$\int_\xi^\delta r(x,t)f'(t)\mathrm{d}t=\int_\xi^x\left[t-\left(\xi+h\frac{(\delta-\xi)}{2}\right)\right]f'(t)\mathrm{d}t$$
$$+\int_x^\delta\left[t-\left(\delta-h\frac{(\delta-\xi)}{2}\right)\right]f'(t)\mathrm{d}t$$
$$=(\delta-\xi)h\frac{f(\xi)+f(\delta)}{2}+(\delta-\xi)(1-h)f(x)-\int_\xi^\delta f(t)\mathrm{d}t.$$

Obviously, we can get

$$
\begin{aligned}
&\left|\int_\xi^\delta r(x,t) f'(t)\mathrm{d}t\right| \\
\leqslant &\int_\xi^\delta |r(x,t)|\left|f'(t)\right|\mathrm{d}t \\
\leqslant &\left(\int_\xi^\delta |r(x,t)|^p \mathrm{d}t\right)^{1/p}\left(\int_\xi^\delta \left|f'(t)\right|^q \mathrm{d}t\right)^{1/q} \\
\leqslant &\left(\int_\xi^x \left|t-\left(\xi+h\frac{\delta-\xi}{2}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_\xi^x \left|f'\left(\frac{t-\xi}{x-\xi}x+\frac{x-t}{x-\xi}\xi\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_x^\delta \left|t-\left(\delta-h\frac{\delta-\xi}{2}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_x^\delta \left|f'\left(\frac{t-x}{\delta-x}\delta+\frac{\delta-t}{\delta-x}x\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leqslant &\left(\int_\xi^x \left|t-\left(\xi+h\frac{\delta-\xi}{2}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_\xi^x \left|f'(x)\right|^{\left(\frac{t-\xi}{x-\xi}\right)q}\left|f'(\xi)\right|^{\left(\frac{x-t}{x-\xi}\right)q} dt\right)^{\frac{1}{q}} \\
&+\left(\int_x^\delta \left|t-\left(\delta-h\frac{\delta-\xi}{2}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_x^\delta \left|f'(\delta)\right|^{\left(\frac{t-x}{\delta-x}\right)q}\left|f'(x)\right|^{\left(\frac{\delta-t}{\delta-x}\right)q} dt\right)^{\frac{1}{q}} \\
\leqslant &\left(\frac{|f'(\xi)|^x}{|f'(x)|^\xi}\right)^{\frac{1}{x-\xi}}\left(\frac{T^{qx}-T^{q\xi}}{q\ln T}\right)^{\frac{1}{q}}\times\left[\frac{\left(x-\left(\xi+h\frac{\delta-\xi}{2}\right)\right)^{p+1}-\left(h\frac{\delta-\xi}{2}\right)^{p+1}}{p+1}\right]^{\frac{1}{p}} \\
&+\left(\frac{|f'(x)|^\delta}{|f'(\delta)|^x}\right)^{\frac{1}{\delta-x}}\left(\frac{M^{q\delta}-M^{qx}}{q\ln M}\right)^{\frac{1}{q}}\times\left[\frac{\left(h\frac{\delta-\xi}{2}\right)^{p+1}-\left(x-\left(\delta-h\frac{\delta-\xi}{2}\right)\right)^{p+1}}{p+1}\right]^{\frac{1}{p}}.
\end{aligned}
$$

Thus, the proof of Theorem 2.17 is completed.

THEOREM 2.18. *Let $f:[\xi,\delta]\to\mathbb{R}$ be a twice differentiable mapping on (ξ,δ) and its derivative $f':(\xi,\delta)\to\mathbb{R}$ is bounded in (ξ,δ). Then for any $x\in[\xi,\delta]$ and $p>1$, we have*

$$
\begin{aligned}
&\left|\frac{(\delta-x)f(\delta)+(x-\xi)f(\xi)}{\delta-\xi}-\frac{1}{\delta-\xi}\int_\xi^\delta f(t)dt\right| \\
\leqslant &\frac{1}{\delta-\xi}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left((x-\xi)^{q+1}+(\delta-x)^{q+1}\right)^{\frac{1}{q}}\left(\int_\xi^\delta |\ f'(t)\ |^p\ dt\right)^{\frac{1}{p}}.
\end{aligned}
$$

Proof. After simple calculation, we can get

$$\int_\xi^\delta (t-x)f'(t)dt=(\delta-x)f(\delta)-(\xi-x)f(\xi)-\int_\xi^\delta f(x)dx.$$

This implies that

$$
\begin{aligned}
&\left|\frac{(\delta-x)f(\delta)+(x-\xi)f(\xi)}{\delta-\xi}-\frac{1}{\delta-\xi}\int_{\xi}^{\delta}f(t)dt\right| \\
=&\left|\frac{1}{\delta-\xi}\int_{\xi}^{\delta}(t-x)f'(t)dt\right| \\
\leqslant&\frac{1}{\delta-\xi}\int_{\xi}^{\delta}\mid t-x\mid\mid f'(t)\mid dt \\
\leqslant&\frac{1}{\delta-\xi}\left(\int_{\xi}^{\delta}\mid t-x\mid^{q}dt\right)^{\frac{1}{q}}\left(\int_{\xi}^{\delta}\mid f'(x)\mid^{p}dx\right)^{\frac{1}{p}} \\
\leqslant&\frac{1}{\delta-\xi}\left(\int_{\xi}^{x}\mid x-t\mid^{q}dt+\int_{x}^{\delta}\mid t-x\mid^{q}dt\right)^{\frac{1}{q}}\left(\int_{\xi}^{\delta}\mid f'(x)\mid^{p}dx\right)^{\frac{1}{p}} \\
\leqslant&\frac{1}{\delta-\xi}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left((x-\xi)^{q+1}+(\delta-x)^{q+1}\right)^{\frac{1}{q}}\left(\int_{\xi}^{\delta}\mid f'(t)\mid^{p}dt\right)^{\frac{1}{p}}.
\end{aligned}
$$

Thus, the proof of Theorem 2.18 is completed.

In 1998, Hilger[3, 4] presented the theory of time scales. Tian [16] and Tian et al.[23] gave some new time scales versions of Hölder's inequality and Minkowski's inequality via the Diamond-Alpha integral. Now we give two new inequalities on time scales based on Chen[2].

THEOREM 2.19. *Let* $\mathbb{T}$ *be a time scale* $\theta_1,\theta_2\in\mathbb{T}$ *with* $\theta_1<\theta_2$ *and* $\alpha_{kj}\in\mathbb{R}, j=1,2,\ldots,n, k=1,2,\ldots,n$, $\sum_{k=1}^{s}\frac{1}{p_k}=r$, $\sum_{k=1}^{s}\alpha_{kj}=0$. *If* $f_j(x)>0$, *and* $f_j(j=1,2,\ldots,n)$ *is a continuous real-valued function on* $[\theta_1,\theta_2]_{\mathbb{T}}$, *then:*

(1) *for* $rp_k>1$, *we have the following inequality:*

$$\int_{\theta_1}^{\theta_2}\prod_{j=1}^{n}f_j(x)\diamond_{\alpha}x\leqslant\prod_{k=1}^{n}\left[\int_{\theta_1}^{\theta_2}\left(\prod_{j=1}^{n}f_j(x)\right)^{1-t}\left(f_k^{rp_k}(x)\right)^{t}\diamond_{\alpha}x\right]^{\frac{1}{rp_k}},$$

(2) *for* $0<rp_s<1$, $rp_k<0, k=1,2,\ldots,s-1$, *we have the following reverse inequality:*

$$\int_{\theta_1}^{\theta_2}\prod_{j=1}^{n}f_j(x)\diamond_{\alpha}x\geqslant\prod_{k=1}^{n}\left[\int_{\theta_1}^{\theta_2}\left(\prod_{j=1}^{n}f_j(x)\right)^{1-t}\left(f_k^{rp_k}(x)\right)^{t}\diamond_{\alpha}x\right]^{\frac{1}{rp_k}}.$$

Proof. Firstly, we can get the following inequality from Chen[2]

$$\int_{\theta_1}^{\theta_2}\prod_{j=1}^{n}f_j(x)\diamond_{\alpha}x\leqslant\prod_{k=1}^{s}\left(\int_{\theta_1}^{\theta_2}\prod_{j=1}^{n}f_j^{1+rp_k\alpha_{kj}}(x)\diamond_{\alpha}x\right)^{\frac{1}{rp_k}},$$

where $rp_k > 1$. Therefore, we can let $s = n, \alpha_{kj} = -t/rp_k$ for $k \neq j$ and $\alpha_{jj} = t(1-\frac{1}{rp_k})$ with $t \in \mathbb{R}$, then

$$\int_{\theta_1}^{\theta_2} \prod_{j=1}^{n} f_j(x) \diamond_\alpha x \leqslant \prod_{k=1}^{n} \left[\int_{\theta_1}^{\theta_2} \left(\prod_{j=1}^{n} f_j(x) \right)^{1-t} \left(f_k^{rp_k}(x) \right)^t \diamond_\alpha x \right]^{\frac{1}{rp_k}}.$$

Secondly, we can get the following conclusion from Chen[2]

$$\int_{\theta_1}^{\theta_2} \prod_{j=1}^{n} f_j(x) \diamond_\alpha x \geqslant \prod_{k=1}^{s} \left(\int_{\theta_1}^{\theta_2} \prod_{j=1}^{n} f_j^{1+rp_k\alpha_{kj}}(x) \diamond_\alpha x \right)^{\frac{1}{rp_k}},$$

where $0 < rp_s < 1, rp_k < 0 (k = 1, 2, \ldots, s-1)$. Therefore, we can let $s = n, \alpha_{kj} = -t/rp_k$ for $k \neq j$ and $\alpha_{jj} = t(1-\frac{1}{rp_k})$ with $t \in \mathbb{R}$, then

$$\int_{\theta_1}^{\theta_2} \prod_{j=1}^{n} f_j(x) \diamond_\alpha x \geqslant \prod_{k=1}^{n} \left[\int_{\theta_1}^{\theta_2} \left(\prod_{j=1}^{n} f_j(x) \right)^{1-t} \left(f_k^{rp_k}(x) \right)^t \diamond_\alpha x \right]^{\frac{1}{rp_k}}.$$

Thus, successful proof of Theorem 2.19.

3. Conclusions

Through this study, we present some new versions of the Hölder's inequality and give some of their applications. At the same time, we combine Ostrowski inequality with Hölder's inequality to obtain a generalized form of Ostrowski inequality. Furthermore, we obtain two new inequalities according to the further extension of $h_r(t)$ and $g_r(t)$ and consider its applications on time scales in order to obtain new inequalities based on time scales. In future research, we hope to further extend the theorems drawn in this paper and get some new results.

4. Competing interests

The authors declare that they have no competing interests.

5. Authors' contributions

The authors read and approved the final manuscript.

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