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REFINEMENTS AND REVERSES OF YOUNG TYPE INEQUALITIES

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Abstract. Recently, some Young type inequalities have been promoted. The purpose of this paper is to give further refinements and reverses to them. Meanwhile, on the base of the scalars results, we obtain some corresponding operator inequalities and matrix versions including Hilbert-Schmidt norm, unitarily invariant norm, trace norm, which can be regarded as an application of the scalar inequalities.

1. Introduction

It is well known that the classical Young inequality for scalars says that if $a,b \geqslant 0$ and $v \in [0,1]$, then

$$a^v b^{1-v} \leqslant va + (1-v)b, \tag{1.1}$$

with equality if and only if $a = b$. This simple inequality is not only interesting in itself but also very useful. It is a particular case when $v = \frac{1}{2}$, by (1.1), we obtain the arithmetic-geometric mean inequality

$$\sqrt{ab} \leqslant \frac{a+b}{2}. \tag{1.2}$$

Some years later, Kittaneh and Manasrah [8] obtained a refinement of inequalities of (1.1), which can be stated in the following form

$$a^v b^{1-v} + r(\sqrt{a} - \sqrt{b})^2 \leqslant va + (1-v)b, \tag{1.3}$$

where $r = \min\{v, 1-v\}$.

Also, the inequality (1.3) was refined by Zhao and Wu [15] as follows:
If $0 \leqslant v \leqslant \frac{1}{2}$, then

$$a^{1-v} b^v + v(\sqrt{a} - \sqrt{b})^2 + r_0(\sqrt[4]{ab} - \sqrt{a})^2 \leqslant (1-v)a + vb. \tag{1.4}$$

If $\frac{1}{2} \leqslant v \leqslant 1$, then

$$a^{1-v} b^v + (1-v)(\sqrt{a} - \sqrt{b})^2 + r_0(\sqrt[4]{ab} - \sqrt{b})^2 \leqslant (1-v)a + vb, \tag{1.5}$$

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where $r_0 = \min\{2r, 1-2r\}$ for $r = \min\{v, 1-v\}$.

In a recent work, Hu [7] gave the following Young type inequalities:
If $0 \leqslant v \leqslant \frac{1}{2}$, then

$$v^2a^2 + (1-v)^2b^2 \geqslant (va)^{2v}b^{2-2v} + v^2(a-b)^2. \tag{1.6}$$

If $\frac{1}{2} \leqslant v \leqslant 1$, then

$$v^2a^2 + (1-v)^2b^2 \geqslant a^{2v}[(1-v)b]^{2-2v} + (1-v)^2(a-b)^2. \tag{1.7}$$

Later, on the basis of the result of Hu [7], Nasiri, Shakoori and Liao [12] have given some refinements of (1.6) and (1.7). These inequalities can be written as:
If $0 \leqslant v \leqslant \frac{1}{2}$, then

$$v^2a^2 + (1-v)^2b^2 \geqslant (va)^{2v}b^{2-2v} + v^2(a-b)^2 + v_0b(\sqrt{va} - \sqrt{b})^2, \tag{1.8}$$

where $v_0 = \min\{2v, 1-2v\}$.
If $\frac{1}{2} \leqslant v \leqslant 1$, then

$$v^2a^2 + (1-v)^2b^2 \geqslant a^{2v}[(1-v)b]^{2-2v} + (1-v)^2(a-b)^2 + v_1a(\sqrt{a} - \sqrt{(1-v)b})^2, \tag{1.9}$$

where $v_1 = \min\{2v-1, 2-2v\}$.

Here we remark that some other refinements and reserves of Young's inequality by the partitions of the weighted parameter $v \in [0,1]$ can be found in [3, 4, 13].

Meanwhile, based on Young type inequalities (1.8) and (1.9), Nasiri, Shakoori and Liao [12] have presented corresponding matrix versions for Hilbert-Schmidt norm, which can be stated that suppose $X \in M_n$ and $A, B \in M_n^+$, $v \in [0,1]$.
If $0 \leqslant v \leqslant \frac{1}{2}$, then

$$\begin{aligned} ||vAX(1-v)XB||_2^2 \geqslant & v^{2v}||A^vXB^{1-v}||_2^2 + v^2||AX-XB||_2^2 \\ & + v_0[v||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2 + ||XB||_2^2 - 2\sqrt{v}||A^{\frac{1}{4}}XB^{\frac{3}{4}}||_2^2] \\ & + 2v(1-v)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2, \end{aligned} \tag{1.10}$$

where $v_0 = \min\{2v, 1-2v\}$.
If $\frac{1}{2} \leqslant v \leqslant 1$, then

$$\begin{aligned} ||vAX + (1-v)XB||_2^2 \geqslant & (1-v)^{2-2v}||A^vXB^{1-v}||_2^2 + (1-v)^2||AX-XB||_2^2 \\ & + v_1[(1-v)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2 + ||AX||_2^2 - 2\sqrt{1-v}||A^{\frac{3}{4}}XB^{\frac{1}{4}}||_2^2] \\ & + 2v(1-v)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2, \end{aligned} \tag{1.11}$$

where $v_1 = \min\{2v-1, 2-2v\}$.

Furthermore, an interesting thing is that authors in [1] obtain the following reverse of inequalities (1.6) and (1.7)

$$\begin{cases} v^2a^2+(1-v)^2b^2 \leqslant (1-v)^2(a-b)^2+(1-v)^{2(1-v)}(a^vb^{1-v})^2, & 0\leqslant v\leqslant \frac{1}{2}, \\ v^2a^2+(1-v)^2b^2 \leqslant v^2(a-b)^2+v^{2v}(a^vb^{1-v})^2, & \frac{1}{2}\leqslant v\leqslant 1. \end{cases} \tag{1.12}$$

In this paper, we will present some another refinements of these inequalities. Throughout the paper, M_n denotes the space of all $n\times n$ complex matrices. M_n^+ denotes the set of all positive semidefinite matrices in M_n, $X\geqslant Y$ for $X,Y\in M_n$ means that X and Y are Hermitians and $X-Y\in M_n^+$. The set of all strictly positive definite matrices in M_n is denoted by M_n^{++}. The unitarily invariance of the $|||.|||$ on M_n means that $|||UAV|||=|||A|||$ for all $A\in M_n$ and for all the unitary matrices $U,V\in M_n$. For $A=[a_{ij}]\in M_n$, the Hilbert-Schmidt (or Frobenius) norm and the trace norm of A are defined by

$$||A||_2=\sqrt{\sum_{j=1}^{n}s_j^2(A)}, \qquad ||A||_1=\sum_{j=1}^{n}s_j(A),$$

respectively, where $s_1(A)\geqslant s_2(A)\geqslant ...\geqslant s_n(A)$ are the singular values of A, that is, the eigenvalues of the positive matrix $|A|=(A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. Moreover, it is well known that $||.||_2$ is unitarily invariant. For the notations adopted in this paper, the difinitions

$$A\nabla_vB=(1-v)A+vB, \quad v\in[0,1],$$

$$A\sharp_vB=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^vA^{\frac{1}{2}}, \quad v\in[0,1]$$

will be used for the matrix arithmetic and geometric means, when $A,B\in M_n^{++}$. In particular, denoted by $A\nabla B$ and $A\sharp B$ respectively when $v=\frac{1}{2}$.

Furthermore, let $B(H)$ denotes the C^*-algebra of all bounded linear operators on a complex Hilbert space H. In the case of dim $H=n$, we identify $B(H)$ with the matrix algebra of all $n\times n$ matrices with entries in the complex field. An operator $A\in B(H)$ is called positive, if

$$\langle Ax,x\rangle\geqslant 0$$

for all $x\in H$, and we write $A\geqslant 0$. The set of all positive operators on a complex Hilbert space H is denoted by $B^+(H)$. Also, the set of all positive invertible operators on a complex Hilbert space H is denoted with $B^{++}(H)$. If $A\in B^{++}(H)$, we write $A>0$.

The paper is organized in the following way: In section 2, we give Young type inequalities for scalar consisting of the reverse version of inequalities (1.12) and the refined version of (1.8), (1.9) and so on. In section 3, on the basis of our main scalar results, we obtained the related operator version, including the reverse and the refinement. In the last section, as an application, we establish some corresponding inequalities of matrix version for Hilbert-Schmidt norm, unitarily invariant norm, and related trace versions based on the result of part one.

2. Young type inequality for scalar

In this section, we mainly present some refined and reversed scalar Young type inequalities, including the reverse of (1.8), (1.9). Therefore, we start from the following improved reverse Young type inequalities.

THEOREM 2.1. *Suppose that* $a,b\geqslant 0$ *and* $v\in[0,1]$.

i) If $0\leqslant v\leqslant \frac{1}{2}$, *then*

$$v^2a^2+(1-v)^2b^2\leqslant a^{2v}[(1-v)b]^{2-2v}+(1-v)^2(a-b)^2-v_0a(\sqrt{a}-\sqrt{(1-v)b})^2, \tag{2.1}$$

where $v_0=\min\{2v,1-2v\}$.

ii) If $\frac{1}{2}\leqslant v\leqslant 1$, *then*

$$v^2a^2+(1-v)^2b^2\leqslant (va)^{2v}b^{2-2v}+v^2(a-b)^2-v_1b(\sqrt{va}-\sqrt{b})^2, \tag{2.2}$$

where $v_1=\min\{2v-1,2-2v\}$.

Proof.

i) If $0\leqslant v\leqslant \frac{1}{2}$, by (1.3), then we have

$$\begin{aligned}
&a^{2v}[(1-v)b]^{2-2v}+(1-v)^2(a-b)^2-v^2a^2-(1-v)^2b^2\\
=&2v[(1-v)ab]+(1-2v)a^2-2(1-v)ab+a^{2v}[(1-v)b]^{2-2v}\\
\geqslant&a^{2-2v}[(1-v)b]^{2v}+v_0(a-\sqrt{(1-v)ab})^2-2(1-v)ab+a^{2v}[(1-v)b]^{2-2v}\\
=&v_0a(\sqrt{a}-\sqrt{(1-v)b})^2+[(1-v)^va^{1-v}b^v-(1-v)^{1-v}a^vb^{1-v}]^2\\
\geqslant&v_0a(\sqrt{a}-\sqrt{(1-v)b})^2.
\end{aligned}$$

That is

$$v^2a^2+(1-v)^2b^2\leqslant a^{2v}[(1-v)b]^{2-2v}+(1-v)^2(a-b)^2-v_0a(\sqrt{a}-\sqrt{(1-v)b})^2.$$

So (2.1) holds.

ii) If $\frac{1}{2}\leqslant v\leqslant 1$, by (1.3) again, then we have

$$\begin{aligned}
&(va)^{2v}b^{2-2v}+v^2(a-b)^2-v^2a^2-(1-v)^2b^2\\
=&(2v-1)b^2+(2-2v)vab-2vab+(va)^{2v}b^{2-2v}\\
\geqslant&(va)^{2-2v}b^{2v}+v_1b(\sqrt{va}-\sqrt{b})^2-2vab+(va)^{2v}b^{2-2v}\\
=&v_1b(\sqrt{va}-\sqrt{b})^2+[(va)^{1-v}b^v-(va)^vb^{1-v}]^2\geqslant v_1b(\sqrt{va}-\sqrt{b})^2.
\end{aligned}$$

That is

$$v^2a^2+(1-v)^2b^2\leqslant (va)^{2v}b^{2-2v}+v^2(a-b)^2-v_1b(\sqrt{va}-\sqrt{b})^2.$$

This completes the proof.

REMARK 2.2. Obviously, inequalities (2.1) and (2.2) are reverses of inequalities (1.8), (1.9) and are refinements of inequalities (1.12).

Next, we give our second main result which is a refinements of (1.8), (1.9), (2.1) and (2.2).

THEOREM 2.4. *Let* $a,b \geqslant 0$ *and* $v \in [0,1]$.

i) If $0 \leqslant v \leqslant \frac{1}{4}$, *then*

$$\begin{aligned}
&(va)^{2v}b^{2-2v} + v^2(a-b)^2 + 2vb(\sqrt{va} - \sqrt{b})^2 + v_2 b(\sqrt[4]{vab} - \sqrt{b})^2 \\
\leqslant &v^2a^2 + (1-v)^2b^2 \\
\leqslant &a^{2v}[(1-v)b]^{2-2v} + (1-v)^2(a-b)^2 - 2va(\sqrt{a} - \sqrt{(1-v)b})^2 \\
&- v_2 a(\sqrt[4]{(1-v)ab} - \sqrt{a})^2,
\end{aligned} \tag{2.3}$$

where $v_2 = \min\{4v, 1-4v\}$.

ii) If $\frac{1}{4} \leqslant v \leqslant \frac{1}{2}$, *then*

$$\begin{aligned}
&(va)^{2v}b^{2-2v} + v^2(a-b)^2 + (1-2v)b(\sqrt{va} - \sqrt{b})^2 + v_3 b(\sqrt[4]{vab} - \sqrt{va})^2 \\
\leqslant &v^2a^2 + (1-v)^2b^2 \\
\leqslant &a^{2v}[(1-v)b]^{2-2v} + (1-v)^2(a-b)^2 - (1-2v)a(\sqrt{a} - \sqrt{(1-v)b})^2 \\
&- v_3 a(\sqrt[4]{(1-v)ab} - \sqrt{(1-v)b})^2,
\end{aligned} \tag{2.4}$$

where $v_3 = \min\{2-4v, 4v-1\}$.

iii) If $\frac{1}{2} \leqslant v \leqslant \frac{3}{4}$, *then*

$$\begin{aligned}
&a^{2v}[(1-v)b]^{2-2v} + (1-v)^2(a-b)^2 + (2v-1)a(\sqrt{a} - \sqrt{(1-v)b})^2 \\
&+ v_4 a(\sqrt[4]{(1-v)ab} - \sqrt{(1-v)b})^2 \\
\leqslant &v^2a^2 + (1-v)^2b^2 \\
\leqslant &(va)^{2v}b^{2-2v} + v^2(a-b)^2 - (2v-1)b(\sqrt{va} - \sqrt{b})^2 \\
&- v_4 b(\sqrt[4]{vab} - \sqrt{va})^2,
\end{aligned} \tag{2.5}$$

where $v_4 = \min\{4v-2, 3-4v\}$.

iv) If $\frac{3}{4} \leqslant v \leqslant 1$, *then*

$$\begin{aligned}
&a^{2v}[(1-v)b]^{2-2v} + (1-v)^2(a-b)^2 + (2-2v)a(\sqrt{a} - \sqrt{(1-v)b})^2 \\
&+ v_5 a(\sqrt[4]{(1-v)ab} - \sqrt{a})^2 \\
\leqslant &v^2a^2 + (1-v)^2b^2 \\
\leqslant &(va)^{2v}b^{2-2v} + v^2(a-b)^2 - (2-2v)b(\sqrt{va} - \sqrt{b})^2 \\
&- v_5 b(\sqrt[4]{vab} - \sqrt{b})^2,
\end{aligned} \tag{2.6}$$

where $v_5 = \min\{4-4v, 4v-3\}$.

Proof.

i) When $0 \leqslant v \leqslant \frac{1}{4}$, by simple calculation and (1.4), then we have

$$\begin{aligned}
& v^2a^2 + (1-v)^2b^2 - v^2(a-b)^2 \\
=& b[(1-2v)b + 2v(va)] \\
\geqslant& b[(va)^{2v}b^{1-2v} + 2v(\sqrt{va}-\sqrt{b})^2 + v_2(\sqrt[4]{vab}-\sqrt{b})^2] \\
=& (va)^{2v}b^{2-2v} + 2vb(\sqrt{va}-\sqrt{b})^2 + v_2b(\sqrt[4]{vab}-\sqrt{b})^2
\end{aligned}$$

and

$$\begin{aligned}
& (1-v)^2(a-b)^2 - v^2a^2 - (1-v)^2b^2 + a^{2v}[(1-v)b]^{2-2v} \\
=& 2v[(1-v)ab] + (1-2v)a^2 - 2(1-v)ab + a^{2v}[(1-v)b]^{2-2v} \\
\geqslant& a^{2-2v}[(1-v)b]^{2v} + 2va(\sqrt{a}-\sqrt{(1-v)b})^2 + v_2a(\sqrt[4]{(1-v)ab}-\sqrt{a})^2 \\
& - 2(1-v)ab + a^{2v}[(1-v)b]^{2-2v} \\
=& 2va(\sqrt{a}-\sqrt{(1-v)b})^2 + v_2a(\sqrt[4]{(1-v)ab}-\sqrt{a})^2 \\
& + [a^{1-v}((1-v)b)^v - a^v((1-v)b)^{1-v}]^2 \\
\geqslant& 2va(\sqrt{a}-\sqrt{(1-v)b})^2 + v_2a(\sqrt[4]{(1-v)ab}-\sqrt{a})^2.
\end{aligned}$$

So we completed the proof of (2.3).

ii) According to the inequality (1.5), the proof of (2.4) can be completed by an argument similar to that used in i).

iii) When $\frac{1}{2} \leqslant v \leqslant \frac{3}{4}$, by (1.4), then we have

$$\begin{aligned}
& v^2a^2 + (1-v)^2b^2 - (1-v)^2(a-b)^2 \\
=& a[(2v-1)a + (2-2v)((1-v)b)] \\
\geqslant& a\{a^{2v-1}[(1-v)b]^{2-2v} + (2v-1)(\sqrt{a}-\sqrt{(1-v)b})^2 \\
& + v_4(\sqrt[4]{(1-v)ab}-\sqrt{(1-v)b})^2\} \\
=& a^{2v}[(1-v)b]^{2-2v} + (2v-1)a(\sqrt{a}-\sqrt{(1-v)b})^2 \\
& + v_4a(\sqrt[4]{(1-v)ab}-\sqrt{(1-v)b})^2
\end{aligned}$$

and

$$\begin{aligned}
& v^2(a-b)^2 - v^2a^2 - (1-v)^2b^2 + (va)^{2v}b^{2-2v} \\
=& (2v-1)b^2 + (2-2v)vab - 2vab + (va)^{2v}b^{2-2v} \\
\geqslant& (va)^{2-2v}b^{2v} + (2v-1)b(\sqrt{va}-\sqrt{b})^2 + v_4b(\sqrt[4]{vab}-\sqrt{va})^2 - 2vab + (va)^{2v}b^{2-2v} \\
\geqslant& (2v-1)b(\sqrt{va}-\sqrt{b})^2 + v_4b(\sqrt[4]{vab}-\sqrt{va})^2.
\end{aligned}$$

So we completed the proof of (2.5).

iv) According to the inequality (1.5), the proof of (2.6) can be completed by an argument similar to that used in iii).

3. Inequalities for operator

In this section, we are going to present some Young type inequalities for an operator version by the monotonic property of operator functions. These operator inequalities are easily given by a standard functional calculus(or Kubo-Ando theory [11]). First, we give the following basic Lemma.

LEMMA 3.1. *([5]) Let $X \in B(H)$ be self-adjoint and let f and g be continuous real functions such that $f(t) \geqslant g(t)$ for all $t \in Sp(X)$ (the Spectrum of X). Then $f(X) \geqslant g(X)$.*

Now, according to the scalar result of (1.6), (1.7) which are presented by Hu, we obtain some of their operator versions as follows.

COROLLARY 3.2. *([7]) Let $A,B \in B^{++}(H)$ and $v \in [0,1]$.*

i) If $v \in [0,\frac{1}{2}]$, then

$$v^2A+(1-v)^2B \geqslant v^{2v}(A\sharp_{1-v}B)+v^2[A+B-2(A\sharp B)]. \tag{3.1}$$

ii) If $v \in [\frac{1}{2},1]$, then

$$v^2B+(1-v)^2A \geqslant (1-v)^{2-2v}(A\sharp_v B)+(1-v)^2[A+B-2(A\sharp B)]. \tag{3.2}$$

And then, by applying Lemma 3.1, the operator version of $(1.8),(1.9),(2,1)$ and (2.2) can be obtained as follows.

COROLLARY 3.3. *Let $A,B \in B^{++}(H)$ and $v \in [0,1]$.*

i) If $v \in [0,\frac{1}{2}]$, then

$$\begin{aligned} & v^{2v}(A\sharp_{1-v}B)+v^2[A+B-2(A\sharp B)]+v_0[B+v(A\sharp B)-2\sqrt{v}(A\sharp_{\frac{3}{4}}B)] \\ \leqslant & v^2A+(1-v)^2B \\ \leqslant & (1-v)^{2-2v}(A\sharp_{1-v}B)+(1-v)^2[A+B-2(A\sharp B)] \\ & -v_0[A+(1-v)(A\sharp B)-2\sqrt{1-v}(A\sharp_{\frac{1}{4}}B)], \end{aligned} \tag{3.5}$$

where $v_0=\min\{2v,1-2v\}$.

ii) If $v \in [\frac{1}{2},1]$, then

$$\begin{aligned} & (1-v)^{2-2v}(A\sharp_v B)+(1-v)^2[A+B-2(A\sharp B)] \\ & +v_1[B+(1-v)A-2\sqrt{1-v}(A\sharp_{\frac{3}{4}}B)] \\ \leqslant & v^2B+(1-v)^2A \\ \leqslant & v^{2v}(A\sharp_v B)+v^2[A+B-2(A\sharp B)]-v_1[A+v(A\sharp B)-2\sqrt{v}(A\sharp_{\frac{1}{4}}B)], \end{aligned} \tag{3.6}$$

where $v_1=\min\{2v-1,2-2v\}$.

Also, the operator version of Corollary 2.4 is deduced immediately as follows, which can be regarded as a generalization of Corollary 3.2 and Corollary 3.3.

COROLLARY 3.4. *Let* $A,B \in B^{++}(H)$ *and* $v \in [0,1]$.

i) If $v \in [0,\frac{1}{4}]$, *then*

$$\begin{aligned}
&v^{2v}(A\sharp_{1-v}B)+v^2[A+B-2(A\sharp B)]+2v[v(A\sharp B)+B-2\sqrt{v}(A\sharp_{\frac{3}{4}}B)]\\
&\quad+v_2[\sqrt{v}(A\sharp_{\frac{3}{4}}B)+B-2\sqrt[4]{v}(A\sharp_{\frac{7}{8}}B)]\\
&\leqslant v^2A+(1-v)^2B\\
&\leqslant (1-v)^{2-2v}(A\sharp_{1-v}B)+(1-v)^2[A+B-2(A\sharp B)]\\
&\quad-2v[A+(1-v)(A\sharp B)-2\sqrt{1-v}(A\sharp_{\frac{1}{4}}B)]\\
&\quad-v_2[\sqrt{1-v}(A\sharp_{\frac{1}{4}}B)+A-2\sqrt[4]{1-v}(A\sharp_{\frac{1}{8}}B)],
\end{aligned} \tag{3.3}$$

where $v_2=\min\{4v,1-4v\}$.

ii) If $v \in [\frac{1}{4},\frac{1}{2}]$, *then*

$$\begin{aligned}
&v^{2v}(A\sharp_{1-v}B)+v^2[A+B-2(A\sharp B)]+(1-2v)[v(A\sharp B)+B-2\sqrt{v}(A\sharp_{\frac{3}{4}}B)]\\
&\quad+v_3[\sqrt{v}(A\sharp_{\frac{3}{4}}B)+v(A\sharp B)-2\sqrt[4]{v^3}(A\sharp_{\frac{5}{8}}B)]\\
&\leqslant v^2A+(1-v)^2B\\
&\leqslant (1-v)^{2-2v}(A\sharp_{1-v}B)+(1-v)^2[A+B-2(A\sharp B)]\\
&\quad-(1-2v)[A+(1-v)(A\sharp B)-2\sqrt{1-v}(A\sharp_{\frac{1}{4}}B)]\\
&\quad-v_3[\sqrt{1-v}(A\sharp_{\frac{1}{4}}B)+(1-v)(A\sharp B)-2\sqrt[4]{(1-v)^3}(A\sharp_{\frac{3}{8}}B)],
\end{aligned} \tag{3.4}$$

where $v_3=\min\{2-4v,4v-1\}$.

iii) If $v \in [\frac{1}{2},\frac{3}{4}]$, *then*

$$\begin{aligned}
&(1-v)^{2-2v}(A\sharp_{v}B)+(1-v)^2[A+B-2(A\sharp B)]\\
&\quad+(2v-1)[(1-v)(A\sharp B)+B-2\sqrt{1-v}(A\sharp_{\frac{3}{4}}B)]\\
&\quad+v_4[\sqrt{1-v}(A\sharp_{\frac{3}{4}}B)+(1-v)(A\sharp B)-\sqrt[4]{(1-v)^3}(A\sharp_{\frac{5}{8}}B)]\\
&\leqslant v^2B+(1-v)^2A\\
&\leqslant v^{2v}(A\sharp_{v}B)+v^2[A+B-2(A\sharp B)]-(2v-1)[v(A\sharp B)+A-2\sqrt{v}(A\sharp_{\frac{1}{4}}B)]\\
&\quad-v_4[\sqrt{v}(A\sharp_{\frac{1}{4}}B)+v(A\sharp B)-\sqrt[4]{v^3}(A\sharp_{\frac{3}{8}}B)],
\end{aligned} \tag{3.5}$$

where $v_4=\min\{4v-2,3-4v\}$.

iv) If $v \in [\frac{3}{4},1]$, *then*

$$(1-v)^{2-2v}(A\sharp_{v}B)+(1-v)^2[A+B-2(A\sharp B)]$$

$$
\begin{aligned}
&+(2-2v)[(1-v)(A\sharp B)+B-2\sqrt{1-v}(A\sharp_{\frac{3}{4}}B)]\\
&+v_5[\sqrt{1-v}(A\sharp_{\frac{3}{4}}B)+B-2\sqrt[4]{1-v}(A\sharp_{\frac{7}{8}}B)]\\
\leqslant& v^2B+(1-v)^2A\\
\leqslant& v^{2v}(A\sharp_v B)+v^2[A+B-2(A\sharp B)]-(2-2v)[v(A\sharp B)+A-2\sqrt{v}(A\sharp_{\frac{1}{4}}B)]\\
&-v_5[\sqrt{v}(A\sharp_{\frac{1}{4}}B)+A-2\sqrt[4]{v}(A\sharp_{\frac{1}{8}}B)],
\end{aligned}
\tag{3.6}
$$

where $v_5=\min\{4-4v,4v-3\}$.

Proof.

i) Let $a=1, b^2=t>0$ in (2.3), then the inequalities become

$$
\begin{aligned}
&v^{2v}t^{1-v}+v^2(1+t-2t^{\frac{1}{2}})+2v(vt^{\frac{1}{2}}+t-2\sqrt{v}t^{\frac{3}{4}})+v_2(\sqrt{v}t^{\frac{3}{4}}+t-2\sqrt[4]{v}t^{\frac{7}{8}})\\
\leqslant& v^2+(1-v)^2t\\
\leqslant&(1-v)^{2-2v}t^{1-v}+(1-v)^2(1+t-2t^{\frac{1}{2}})-2v[1+(1-v)t^{\frac{1}{2}}-2\sqrt{1-v}t^{\frac{1}{4}}]\\
&-v_2[\sqrt{1-v}t^{\frac{1}{4}}+1-2\sqrt[4]{1-v}t^{\frac{1}{8}}].
\end{aligned}
\tag{3.7}
$$

For the operator $X=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ has a positive spectrum, I be the identity operator. By Lemma 3.1 and replace t with X in inequalities (3.7), then we have

$$
\begin{aligned}
&v^{2v}X^{1-v}+v^2(I+X-2X^{\frac{1}{2}})+2v(vX^{\frac{1}{2}}+X-2\sqrt{v}X^{\frac{3}{4}})\\
&+v_2(\sqrt{v}X^{\frac{3}{4}}+X-2\sqrt[4]{v}X^{\frac{7}{8}})\\
\leqslant& v^2I+(1-v)^2X\\
\leqslant&(1-v)^{2-2v}X^{1-v}+(1-v)^2(I+X-2X^{\frac{1}{2}})-2v[I+(1-v)X^{\frac{1}{2}}-2\sqrt{1-v}X^{\frac{1}{4}}]\\
&-v_2[\sqrt{1-v}X^{\frac{1}{4}}+I-2\sqrt[4]{1-v}X^{\frac{1}{8}}].
\end{aligned}
\tag{3.8}
$$

Then multiplying (3.8) by $A^{\frac{1}{2}}$ on both sides, we get the required inequalities (3.3).

ii) The line of proof ii) is similar to the one presented in i) by applying the inequalities (2.4), thus we omit it.

iii) Let $b=1, a^2=t>0$, in (2.5), then the inequalities become

$$
\begin{aligned}
&(1-v)^{2-2v}t^v+(1-v)^2(1+t-2t^{\frac{1}{2}})+(2v-1)[(1-v)t^{\frac{1}{2}}+t-2\sqrt{1-v}t^{\frac{3}{4}}]\\
&+v_4(\sqrt{1-v}t^{\frac{3}{4}}+(1-v)t^{\frac{1}{2}}-2\sqrt[4]{(1-v)^3}t^{\frac{5}{8}})\\
\leqslant& v^2t+(1-v)^2\\
\leqslant& v^{2v}t^v+v^2(1+t-2t^{\frac{1}{2}})-(2v-1)[vt^{\frac{1}{2}}+1-2\sqrt{v}t^{\frac{1}{4}}]
\end{aligned}
$$

$$-v_4[\sqrt{v}t^{\frac{1}{4}}+vt^{\frac{1}{2}}-2\sqrt[4]{v^3}t^{\frac{3}{8}}]. \tag{3.9}$$

For $X=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ has a positive spectrum, I be the identity operator. By Lemma 3.1 and inserting X in inequalities (3.9), we have

$$\begin{aligned}
&(1-v)^{2-2v}X^v+(1-v)^2(I+X-2X^{\frac{1}{2}})\\
&+(2v-1)[(1-v)X^{\frac{1}{2}}+X-2\sqrt{1-v}X^{\frac{3}{4}}]\\
&+v_4[\sqrt{1-v}X^{\frac{3}{4}}+(1-v)X^{\frac{1}{2}}-2\sqrt[4]{(1-v)^3}X^{\frac{5}{8}}]\\
\leqslant& v^2X+(1-v)^2I\\
\leqslant& v^{2v}X^v+v^2(I+X-2X^{\frac{1}{2}})-(2v-1)[vX^{\frac{1}{2}}+I-2\sqrt{v}X^{\frac{1}{4}}]\\
&-v_4[\sqrt{v}X^{\frac{1}{4}}+vX^{\frac{1}{2}}-2\sqrt[4]{v^3}X^{\frac{3}{8}}].
\end{aligned} \tag{3.10}$$

Then multiplying both sides of the inequality (3.10) by $A^{\frac{1}{2}}$, we get the desired inequalities (3.5).

iv) The line of proof is similar to the one presented in iii) by applying the inequality (2.6), thus we omit it.

4. Inequalities for matrix

It is known that some matrix norm inequalities can be obtained by the point-wise order (scalar inequality). (See, for example, [6, 10, 13]).

In this section, we only present some interesting matrix versions of Theorem 2.4 for Hilbert-Schmidt norm, unitarily invariant norm, trace norm and trace. To do these, we need the following Lemmas. However, it is worth mentioning that the first Lemma is a Heinz-Kato type inequality for unitarily invariant norms.

LEMMA 4.1. *([9]) Suppose $A,B\in M_n^+$ and $X\in M_n$. If $0\leqslant v\leqslant 1$, then*

$$|||A^vXB^{1-v}|||\leqslant |||AX|||^v|||XB|||^{1-v}.$$

In particular,

$$tr|A^vB^{1-v}|\leqslant (trA)^v(trB)^{1-v}.$$

LEMMA 4.2. *([2]) Let $A,B\in M_n$, then*

$$\sum_{j=1}^n s_j(AB)\leqslant \sum_{j=1}^n s_j(A)s_j(B).$$

LEMMA 4.3. *Let $A,B \in M_n^+$ and $v \in [0,1]$, then*

$$\|A^v B^{1-v}\|_2^2 \leqslant \sum_{j=1}^{n} \left[s_j^v(A) s_j^{1-v}(B)\right]^2.$$

Proof. For any $C \in M_n$, we have the following inequality by [14],

$$|trC| \leqslant \sum_{j=1}^{n} s_j(C).$$

So by Lemma 4.2,

$$\begin{aligned}\sum_{j=1}^{n} \left[s_j^v(A) s_j^{1-v}(B)\right]^2 &\geqslant \sum_{j=1}^{n} s_j(A^{2v}B^{2(1-v)}) \geqslant tr(A^{2v}B^{2(1-v)}) = tr(B^{1-v}A^{2v}B^{1-v}) \\ &= \sum_{j=1}^{n} s_j^2(A^v B^{1-v}) = \|A^v B^{1-v}\|_2^2.\end{aligned}$$

Now, we first establish matrix versions of Theorem 2.4 for Hilbert-Schmidt norm, whose proof is based on the spectral theorem.

THEOREM 4.4. *Suppose $A,B,X \in M_n$ such that $A,B \in M_n^+$ and satisfy $v \in [0,1]$.*

i) If $v \in [0,\frac{1}{4}]$, then

$$\begin{aligned}& v^{2v}||A^v X B^{1-v}||_2^2 + 2v||A^{\frac{1}{2}} X B^{\frac{1}{2}}||_2^2 + v^2||AX - XB||_2^2 \\ & + (v_2 + 2v)||XB||_2^2 + (v_2 - 4v)\sqrt{v}||A^{\frac{1}{4}} X B^{\frac{3}{4}}||_2^2 - 2v_2\sqrt[4]{v}||A^{\frac{1}{8}} X B^{\frac{7}{8}}||_2^2 \\ \leqslant & ||vAX + (1-v)XB||_2^2 \\ \leqslant & (1-v)^{2-2v}||A^v X B^{1-v}||_2^2 + (1-v)^2||AX - XB||_2^2 - (v_2 + 2v)||AX||_2^2 \\ & - (v_2 - 4v)\sqrt{1-v}||A^{\frac{3}{4}} X B^{\frac{1}{4}}||_2^2 + 2v_2\sqrt[4]{1-v}||A^{\frac{7}{8}} X B^{\frac{1}{8}}||_2^2, \end{aligned} \tag{4.1}$$

where $v_2 = \min\{4v, 1-4v\}$.

ii) If $v \in [\frac{1}{4},\frac{1}{2}]$, then

$$\begin{aligned}& v^{2v}||A^v X B^{1-v}||_2^2 + v(v_3 + 3 - 4v)||A^{\frac{1}{2}} X B^{\frac{1}{2}}||_2^2 + v^2||AX - XB||_2^2 \\ & + (1-2v)||XB||_2^2 + (v_3 - 2 + 4v)\sqrt{v}||A^{\frac{1}{4}} X B^{\frac{3}{4}}||_2^2 - 2v_3\sqrt[4]{v^3}||A^{\frac{3}{8}} X B^{\frac{5}{8}}||_2^2 \\ \leqslant & ||vAX + (1-v)XB||_2^2 \\ \leqslant & (1-v)^{2-2v}||A^v X B^{1-v}||_2^2 + (1-v)(4v - 1 - v_3)||A^{\frac{1}{2}} X B^{\frac{1}{2}}||_2^2 \\ & + (1-v)^2||AX - XB||_2^2 - (1-2v)||AX||_2^2 \\ & - (v_3 + 4v - 2)\sqrt{1-v}||A^{\frac{3}{4}} X B^{\frac{1}{4}}||_2^2 + 2v_3\sqrt[4]{(1-v)^3}||A^{\frac{5}{8}} X B^{\frac{3}{8}}||_2^2, \end{aligned} \tag{4.2}$$

where $v_3 = \min\{2-4v, 4v-1\}$.

iii) *If* $v \in [\frac{1}{2}, \frac{3}{4}]$, *then*

$$\begin{aligned}
&(1-v)^{2-2v}||A^vXB^{1-v}||_2^2 + (1-v)(v_4+4v-1)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2 \\
&+(1-v)^2||AX-XB||_2^2 + (2v-1)||AX||_2^2 \\
&+(v_4+2-4v)\sqrt{1-v}||A^{\frac{3}{4}}XB^{\frac{1}{4}}||_2^2 - 2v_4\sqrt[4]{(1-v)^3}||A^{\frac{5}{8}}XB^{\frac{3}{8}}||_2^2 \\
\leqslant &||vAX+(1-v)XB||_2^2 \\
\leqslant &v^{2v}||A^vXB^{1-v}||_2^2 + v(3-4v-v_4)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2 \\
&+v^2||AX-XB||_2^2 - (2v-1)||XB||_2^2 \\
&-(v_4+2-4v)\sqrt{v}||A^{\frac{1}{4}}XB^{\frac{3}{4}}||_2^2 + 2v_4\sqrt[4]{v^3}||A^{\frac{3}{8}}XB^{\frac{5}{8}}||_2^2,
\end{aligned} \tag{4.3}$$

where $v_4 = \min\{4v-2, 3-4v\}$.

iv) *If* $v \in [\frac{3}{4}, 1]$, *then*

$$\begin{aligned}
&(1-v)^{2-2v}||A^vXB^{1-v}||_2^2 + 2(1-v)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2 \\
&+(1-v)^2||AX-XB||_2^2 + (v_5+2-2v)||AX||_2^2 \\
&+(v_5-4+4v)\sqrt{1-v}||A^{\frac{3}{4}}XB^{\frac{1}{4}}||_2^2 - 2v_5\sqrt[4]{1-v}||A^{\frac{7}{8}}XB^{\frac{1}{8}}||_2^2 \\
\leqslant &||vAX+(1-v)XB||_2^2 \\
\leqslant &v^{2v}||A^vXB^{1-v}||_2^2 + v^2||AX-XB||_2^2 - (v_5+2-2v)||XB||_2^2 \\
&-(v_5-4+4v)\sqrt{v}||A^{\frac{1}{4}}XB^{\frac{3}{4}}||_2^2 + 2v_5\sqrt[4]{v}||A^{\frac{1}{8}}XB^{\frac{7}{8}}||_2^2,
\end{aligned} \tag{4.4}$$

where $v_5 = \min\{4-4v, 4v-3\}$.

Proof. Since A and B are positive semidefinite, it follows by spectral theorem that there exist unitary matrices $U,V \in M_n$, such that $A = U\Lambda_1U^*, B = V\Lambda_2V^*$, where $\Lambda_1 = diag(\lambda_1, \lambda_2, \cdots, \lambda_n), \Lambda_2 = diag(\mu_1, \mu_2, \cdots, \mu_n), \lambda_i, \mu_j \geqslant 0, i,j = 1,2,\cdots,n$. For our computations, let $Y = U^*XV = [y_{ij}]$. Then we have

$$vAX+(1-v)XB = U[(v\lambda_i+(1-v)\mu_j)y_{ij}]V^*, AX-XB = U[(\lambda_i-\mu_j)y_{ij}]V^*,$$

$$A^vXB^{1-v} = U[(\lambda_i^v\mu_j^{1-v})y_{ij}]V^*, A^{\frac{1}{2}}XB^{\frac{1}{2}} = U[(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}})y_{ij}]V^*, AX = U[\lambda_iy_{ij}]V^*,$$

and $XB = U[\mu_jy_{ij}]V^*, A^{\frac{1}{4}}XB^{\frac{3}{4}} = U[(\lambda_i^{\frac{1}{4}}\mu_j^{\frac{3}{4}})y_{ij}]V^*, A^{\frac{1}{8}}XB^{\frac{7}{8}} = U[(\lambda_i^{\frac{1}{8}}\mu_j^{\frac{7}{8}})y_{ij}]V^*$.

i) If $v \in [0, \frac{1}{4}]$, now, by (2.3) and the unitarily invariant property of the Hilbert-Schmidt norm, then we have

$$\begin{aligned}
&||vAX+(1-v)XB||_2^2 \\
=&\sum_{i,j=1}^n (v\lambda_i+(1-v)\mu_j)^2|y_{ij}|^2
\end{aligned}$$

$$\begin{aligned}
\geqslant & v^{2v}\sum_{i,j=1}^{n}(\lambda_i^{v}\mu_j^{1-v})^2|y_{ij}|^2+2v(1-v)\sum_{i,j=1}^{n}(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}})^2|y_{ij}|^2+v^2\sum_{i,j=1}^{n}(\lambda_i-\mu_j)^2|y_{ij}|^2\\
&+2v\sum_{i,j=1}^{n}\left[v\lambda_i\mu_j+\mu_j^2-2\sqrt{v}\lambda_i^{\frac{1}{2}}\mu_j^{\frac{3}{2}}\right]|y_{ij}|^2\\
&+v_2\sum_{i,j=1}^{n}\left[\sqrt{v}\lambda_i^{\frac{1}{2}}\mu_j^{\frac{3}{2}}+\mu_j^2-2\sqrt[4]{v}\lambda_i^{\frac{1}{4}}\mu_j^{\frac{7}{4}}\right]|y_{ij}|^2\\
= & v^{2v}||A^vXB^{1-v}||_2^2+2v(1-v)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2+v^2||AX-XB||_2^2\\
&+2v\left[v||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2+||XB||_2^2-2\sqrt{v}||A^{\frac{1}{4}}XB^{\frac{3}{4}}||_2^2\right]\\
&+v_2\left[\sqrt{v}||A^{\frac{1}{4}}XB^{\frac{3}{4}}||_2^2+||XB||_2^2-2\sqrt[4]{v}||A^{\frac{1}{8}}XB^{\frac{7}{8}}||_2^2\right]
\end{aligned}$$

and

$$\begin{aligned}
&||vAX+(1-v)XB||_2^2\\
= & \sum_{i,j=1}^{n}(v\lambda_i+(1-v)\mu_j)^2|y_{ij}|^2\\
\leqslant & (1-v)^{2-2v}\sum_{i,j=1}^{n}(\lambda_i^{v}\mu_j^{1-v})^2|y_{ij}|^2+2v(1-v)\sum_{i,j=1}^{n}(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}})^2|y_{ij}|^2\\
&+(1-v)^2\sum_{i,j=1}^{n}(\lambda_i-\mu_j)^2|y_{ij}|^2-2v\sum_{i,j=1}^{n}\left[(1-v)\lambda_i\mu_j+\lambda_i^2-2\sqrt{1-v}\lambda_i^{\frac{3}{2}}\mu_j^{\frac{1}{2}}\right]|y_{ij}|^2\\
&-v_2\sum_{i,j=1}^{n}\left[\sqrt{1-v}\lambda_i^{\frac{3}{2}}\mu_j^{\frac{1}{2}}+\lambda_i^2-2\sqrt[4]{1-v}\lambda_i^{\frac{7}{4}}\mu_j^{\frac{1}{4}}\right]|y_{ij}|^2\\
= & (1-v)^{2-2v}||A^vXB^{1-v}||_2^2+2v(1-v)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2+(1-v)^2||AX-XB||_2^2\\
&-2v\left[(1-v)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2+||AX||_2^2-2\sqrt{1-v}||A^{\frac{3}{4}}XB^{\frac{1}{4}}||_2^2\right]\\
&-v_2\left[\sqrt{1-v}||A^{\frac{3}{4}}XB^{\frac{1}{4}}||_2^2+||AX||_2^2-2\sqrt[4]{(1-v)}||A^{\frac{7}{8}}XB^{\frac{1}{8}}||_2^2\right].
\end{aligned}$$

So we completed the proof of (4.1).

Using the same technique in (2.4), (2.5) and (2.6), respectively, we can get (4.2), (4.3) and (4.4). Here we completed the proof.

REMARK 4.5. It is clear that the first inequality of (4.1), (4.2), (4.3) and (4.4) are refinements of inequalities (1.10) and (1.11).

The following results for unitarily invariant norm and trace are established by Lemma 4.1.

THEOREM 4.6. *Suppose $A,B,X\in M_n$ such that $A,B\in M_n^{++}$ and satisfy $v\in[0,1]$, for any unitarily invariant norm $|||.|||$.*

i) If $v\in[0,\frac{1}{4}]$, then

$$\left[v|||AX|||+(1-v)|||XB|||\right]^2$$

$$
\begin{aligned}
&\geqslant v^{2v}|||A^v XB^{1-v}|||^2 + 2v(1-v)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||^2 + v^2\big(|||AX||| - |||XB|||\big)^2 \\
&\quad + 2v|||XB|||\big(\sqrt{v|||AX|||} - \sqrt{|||XB|||}\big)^2 \\
&\quad + v_2|||XB|||\big(\sqrt[4]{v|||AX|||\;|||XB|||} - \sqrt{|||XB|||}\big)^2, \qquad (4.5)
\end{aligned}
$$

where $v_2 = \min\{4v, 1-4v\}$.

ii) *If* $v \in [\frac{1}{4}, \frac{1}{2}]$, *then*

$$
\begin{aligned}
&\big[v|||AX||| + (1-v)|||XB|||\big]^2 \\
&\geqslant v^{2v}|||A^v XB^{1-v}|||^2 + 2v(1-v)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||^2 + v^2\big(|||AX||| - |||XB|||\big)^2 \\
&\quad + (1-2v)|||XB|||\big(\sqrt{v|||AX|||} - \sqrt{|||XB|||}\big)^2 \\
&\quad + v_3|||XB|||\big(\sqrt[4]{v|||AX|||\;|||XB|||} - \sqrt{v|||AX|||}\big)^2, \qquad (4.6)
\end{aligned}
$$

where $v_3 = \min\{2-4v, 4v-1\}$.

iii) *If* $v \in [\frac{1}{2}, \frac{3}{4}]$, *then*

$$
\begin{aligned}
&\big[v|||AX||| + (1-v)|||XB|||\big]^2 \\
&\geqslant (1-v)^{2-2v}|||A^v XB^{1-v}|||^2 + 2v(1-v)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||^2 \\
&\quad + (1-v)^2\big(|||AX||| - |||XB|||\big)^2 \\
&\quad + (2v-1)|||AX|||\big(\sqrt{|||AX|||} - \sqrt{(1-v)|||XB|||}\big)^2 \\
&\quad + v_4|||AX|||\big(\sqrt[4]{(1-v)|||AX|||\;|||XB|||} - \sqrt{(1-v)|||XB|||}\big)^2, \qquad (4.7)
\end{aligned}
$$

where $v_4 = \min\{4v-2, 3-4v\}$.

iv) *If* $v \in [\frac{3}{4}, 1]$, *then*

$$
\begin{aligned}
&\big[v|||AX||| + (1-v)|||XB|||\big]^2 \\
&\geqslant (1-v)^{2-2v}|||A^v XB^{1-v}|||^2 + 2v(1-v)|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||^2 \\
&\quad + (1-v)^2\big(|||AX||| - |||XB|||\big)^2 \\
&\quad + (2-2v)|||AX|||\big(\sqrt{|||AX|||} - \sqrt{(1-v)|||XB|||}\big)^2 \\
&\quad + v_5|||AX|||\big(\sqrt[4]{(1-v)|||AX|||\;|||XB|||} - \sqrt{|||AX|||}\big)^2, \qquad (4.8)
\end{aligned}
$$

where $v_5 = \min\{4-4v, 4v-3\}$.

Proof.

i) When $v \in [0, \frac{1}{4}]$, by Lemma 4.1 and the left inequality of (2.3), then we have

$$
\big[v|||AX||| + (1-v)|||XB|||\big]^2
$$

$$
\begin{aligned}
&\geqslant v^{2v}\left[|||AX|||^{v}|||XB|||^{1-v}\right]^2+2v(1-v)|||AX|||\,|||XB|||\\
&\quad+v^2\big(|||AX|||-|||XB|||\big)^2+2v|||XB|||\big(\sqrt{v|||AX|||}-\sqrt{|||XB|||}\big)^2\\
&\quad+v_2|||XB|||\big(\sqrt[4]{v|||AX|||\,|||XB|||}-\sqrt{|||XB|||}\big)^2\\
&\geqslant v^{2v}|||A^vXB^{1-v}|||^2+2v(1-v)|||A^{\frac12}XB^{\frac12}|||^2\\
&\quad+v^2\big(|||AX|||-|||XB|||\big)^2+2v|||XB|||\big(\sqrt{v|||AX|||}-\sqrt{|||XB|||}\big)^2\\
&\quad+v_2|||XB|||\big(\sqrt[4]{v|||AX|||\,|||XB|||}-\sqrt{|||XB|||}\big)^2.
\end{aligned}
$$

So we completed the proof of (4.5).

The proof of the line ii), iii), iv) are similar to the one presented in i) by using the Lemma 4.1 and the left inequalities of (2.4), (2.5) and (2.6), respectively, thus we omit them.

THEOREM 4.7. *Suppose $A,B,X\in M_n$ such that $A,B\in M_n^{++}$ and satisfy $v\in[0,1]$.*

i) If $v\in[0,\frac14]$, then

$$
\begin{aligned}
\left[tr(A\nabla_{1-v}B)\right]^2\geqslant& v^{2v}\left[tr\,|\,A^vB^{1-v}\,|\,\right]^2+v^2\big(trA-trB\big)^2\\
&+2v(1-v)\big(tr\,|\,A^{\frac12}B^{\frac12}\,|\,\big)^2+2vtrB\big(\sqrt{vtrA}-\sqrt{trB}\big)^2\\
&+v_2trB\big(\sqrt[4]{vtrAtrB}-\sqrt{trB}\big)^2, \qquad (4.9)
\end{aligned}
$$

where $v_2=\min\{4v,1-4v\}$.

ii) If $v\in[\frac14,\frac12]$, then

$$
\begin{aligned}
\left[tr(A\nabla_{1-v}B)\right]^2\geqslant& v^{2v}\left[tr|A^vB^{1-v}|\right]^2+v^2\big(trA-trB\big)^2\\
&+2v(1-v)\big(tr\,|\,A^{\frac12}B^{\frac12}\,|\,\big)^2+(1-2v)trB\big(\sqrt{vtrA}-\sqrt{trB}\big)^2\\
&+v_3trB\big(\sqrt[4]{vtrAtrB}-\sqrt{vtrA}\big)^2, \qquad (4.10)
\end{aligned}
$$

where $v_3=\min\{2-4v,4v-1\}$.

iii) If $v\in[\frac12,\frac34]$, then

$$
\begin{aligned}
\left[tr(A\nabla_{1-v}B)\right]^2\geqslant&(1-v)^{2-2v}\left[tr|A^vB^{1-v}|\right]^2+(1-v)^2\big(trA-trB\big)^2\\
&+2v(1-v)\big(tr\,|\,A^{\frac12}B^{\frac12}\,|\,\big)^2+(2v-1)trA\big(\sqrt{trA}-\sqrt{(1-v)trB}\big)^2\\
&+v_4trA\big(\sqrt[4]{(1-v)trAtrB}-\sqrt{(1-v)trB}\big)^2, \qquad (4.11)
\end{aligned}
$$

where $v_4=\min\{4v-2,3-4v\}$.

iv) If $v\in[\frac34,1]$, then

$$
\left[tr(A\nabla_{1-v}B)\right]^2\geqslant(1-v)^{2-2v}\left[tr|A^vB^{1-v}|\right]^2+(1-v)^2\big(trA-trB\big)^2
$$

$$+2v(1-v)\left(tr\mid A^{\frac{1}{2}}B^{\frac{1}{2}}\mid\right)^2+(2-2v)trA\left(\sqrt{trA}-\sqrt{(1-v)trB}\right)^2$$
$$+v_5trA\left(\sqrt[4]{(1-v)trAtrB}-\sqrt{trA}\right)^2, \tag{4.12}$$

where $v_5=\min\{4-4v,4v-3\}$.

Proof.

i) When $v\in[0,\frac{1}{4}]$, by Lemma 4.1 and the left inequality of (2.3), then we have

$$\begin{aligned}
&[tr(A\nabla_{1-v}B)]^2\\
=&\left[vtrA+(1-v)trB\right]^2\\
\geqslant&v^{2v}\left[(trA)^v(trB)^{1-v}\right]^2+v^2\left(trA-trB\right)^2+2v(1-v)trAtrB\\
&+2vtrB\left(\sqrt{vtrA}-\sqrt{trB}\right)^2+v_2trB\left(\sqrt[4]{vtrA\,trB}-\sqrt{trB}\right)^2\\
\geqslant&v^{2v}\left[tr\mid A^vB^{1-v}\mid\right]^2+v^2\left(trA-trB\right)^2+2v(1-v)\left(tr\mid A^{\frac{1}{2}}B^{\frac{1}{2}}\mid\right)^2\\
&+2vtrB\left(\sqrt{vtrA}-\sqrt{trB}\right)^2+v_2trB\left(\sqrt[4]{vtrAtrB}-\sqrt{trB}\right)^2.
\end{aligned}$$

So (4.9) holds.

The proof of the line ii), iii), iv) are similar to the one presented in i) by using the Lemma 4.1 and the left inequalities of $(2.4)-(2.6)$, respectively, thus we omit them.

Also, we get the following result which gives refinements of the trace norm version of Young type inequality.

THEOREM 4.8. *Let* $A,B\in M_n$ *such that* $A,B\in M_n^+$ *and satisfy* $v\in[0,1]$.

i) If $v\in[0,\frac{1}{4}]$, *then*

$$\begin{aligned}
&v^2||A||_2^2+(1-v)^2||B||_2^2=tr(v^2A^2+(1-v)^2B^2)\\
\geqslant&v^{2v}||A^vB^{1-v}||_2^2+v^2\left(||A||_2-||B||_2\right)^2\\
&+2v\left[v||AB||_1+||B||_2^2-2\sqrt{v}\sqrt{||A||_1\,||B^3||_1}\right]\\
&+v_2\left[\sqrt{v}||A^{\frac{1}{2}}B^{\frac{3}{2}}||_1+||B||_2^2-2\sqrt[4]{v}\sqrt{||A^{\frac{1}{2}}||_1\,||B^{\frac{7}{2}}||_1}\right],
\end{aligned}\tag{4.13}$$

where $v_2=\min\{4v,1-4v\}$.

ii) If $v\in[\frac{1}{4},\frac{1}{2}]$, *then*

$$\begin{aligned}
&v^2||A||_2^2+(1-v)^2||B||_2^2=tr\left(v^2A^2+(1-v)^2B^2\right)\\
\geqslant&v^{2v}||A^vB^{1-v}||_2^2+v^2\left(||A||_2-||B||_2\right)^2\\
&+(1-2v)\left[v||AB||_1+||B||_2^2-2\sqrt{v}\sqrt{||A||_1\,||B^3||_1}\right]\\
&+v_3\left[\sqrt{v}||A^{\frac{1}{2}}B^{\frac{3}{2}}||_1+v||AB||_1-2\sqrt[4]{v^3}\sqrt{||A^{\frac{3}{2}}||_1\,||B^{\frac{5}{2}}||_1}\right],
\end{aligned}\tag{4.14}$$

where $v_3=\min\{2-4v,4v-1\}$.

iii) If $v \in [\frac{1}{2}, \frac{3}{4}]$*, then*

$$\begin{aligned} &v^2||A||_2^2 + (1-v)^2||B||_2^2 = tr(v^2A^2 + (1-v)^2B^2) \\ \geqslant &(1-2v)^{2-2v}||A^vB^{1-v}||_2^2 + (1-v)^2(||A||_2 - ||B||_2)^2 \\ &+ (2v-1)\big[(1-v)||AB||_1 + ||A||_2^2 - 2\sqrt{1-v}\sqrt{||A^3||_1\ ||B||_1}\big] \\ &+ v_4\big[\sqrt{1-v}||A^{\frac{3}{2}}B^{\frac{1}{2}}||_1 + (1-v)||AB||_1 - 2\sqrt[4]{(1-v)^3}\sqrt{||A^{\frac{5}{2}}||_1\ ||B^{\frac{3}{2}}||_1}\big], \end{aligned} \tag{4.15}$$

where $v_4 = \min\{4v-2, 3-4v\}$.

iv) If $v \in [\frac{3}{4}, 1]$*, then*

$$\begin{aligned} &v^2||A||_2^2 + (1-v)^2||B||_2^2 = tr(v^2A^2 + (1-v)^2B^2) \\ \geqslant &(1-2v)^{2-2v}||A^vB^{1-v}||_2^2 + (1-v)^2(||A||_2 - ||B||_2)^2 \\ &+ (2-2v)\big[(1-v)||AB||_1 + ||A||_2^2 - 2\sqrt{1-v}\sqrt{||A^3||_1\ ||B||_1}\big] \\ &+ v_5\big[\sqrt{1-v}||A^{\frac{3}{2}}B^{\frac{1}{2}}||_1 + ||A||_2^2 - 2\sqrt[4]{(1-v)}\sqrt{||A^{\frac{7}{2}}||_1\ ||B^{\frac{1}{2}}||_1}\big], \end{aligned} \tag{4.16}$$

where $v_5 = \min\{4-4v, 4v-3\}$.

Proof. We shall prove the first inequality and leave the others to the reader because the proof is similar to each other.

i) When $v \in [0, \frac{1}{4}]$, then using Lemma 4.2, Lemma 4.3, Cauchy-Schwarz inequality and the left inequality of (2.3), we have

$$\begin{aligned} &tr(v^2A^2 + (1-v)^2B^2) \\ = &\sum_{j=1}^{n}[v^2s_j^2(A) + (1-v)^2s_j^2(B)] \\ \geqslant &\sum_{j=1}^{n}\big[(vs_j(A))^{2v}s_j^{2-2v}(B) + v^2(s_j(A) - s_j(B))^2 \\ &+ 2vs_j(B)\big(\sqrt{vs_j(A)} - \sqrt{s_j(B)}\big)^2 + v_2s_j(B)\big(\sqrt[4]{vs_j(A)s_j(B)} - \sqrt{s_j(B)}\big)^2\big] \\ = &v^{2v}\sum_{j=1}^{n}\big[s_j^v(A)s_j^{1-v}(B)\big]^2 + v^2\big[\sum_{j=1}^{n}s_j^2(A) + \sum_{j=1}^{n}s_j^2(B) - 2\sum_{j=1}^{n}s_j(A)s_j(B)\big] \\ &+ 2v\big[v\sum_{j=1}^{n}s_j(A)s_j(B) + \sum_{j=1}^{n}s_j^2(B) - 2\sqrt{v}\sum_{j=1}^{n}s_j^{\frac{1}{2}}(A)s_j^{\frac{3}{2}}(B)\big] \\ &+ v_2\big[\sqrt{v}\sum_{j=1}^{n}s_j^{\frac{1}{2}}(A)s_j^{\frac{3}{2}}(B) + \sum_{j=1}^{n}s_j^2(B) - 2\sqrt[4]{v}\sum_{j=1}^{n}s_j^{\frac{1}{4}}(A)s_j^{\frac{7}{4}}(B)\big] \\ \geqslant &v^{2v}\sum_{j=1}^{n}\big[s_j^v(A)s_j^{1-v}(B)\big]^2 + v^2\big[\sum_{j=1}^{n}s_j^2(A) + \sum_{j=1}^{n}s_j^2(B) - 2\big(\sum_{j=1}^{n}s_j^2(A)\big)^{\frac{1}{2}}\big(\sum_{j=1}^{n}s_j^2(B)\big)^{\frac{1}{2}}\big] \end{aligned}$$

$$+2v\big[v\sum_{j=1}^{n}s_j(AB)+\sum_{j=1}^{n}s_j^2(B)-2\sqrt{v}\big(\sum_{j=1}^{n}s_j(A)\big)^{\frac{1}{2}}\big(\sum_{j=1}^{n}s_j^3(B)\big)^{\frac{1}{2}}\big]$$
$$+v_2\big[\sqrt{v}\sum_{j=1}^{n}s_j(A^{\frac{1}{2}}B^{\frac{3}{2}})+\sum_{j=1}^{n}s_j^2(B)-2\sqrt[4]{v}\big(\sum_{j=1}^{n}s_j^{\frac{1}{2}}(A)\big)^{\frac{1}{2}}\big(\sum_{j=1}^{n}s_j^{\frac{7}{2}}(B)\big)^{\frac{1}{2}}\big]$$
$$\geqslant v^{2v}||A^vB^{1-v}||_2^2+v^2(||A||_2-||B||_2)^2$$
$$+2v\big[v||AB||_1+||B||_2^2-2\sqrt{v}\sqrt{||A||_1\ ||B^3||_1}\big]$$
$$+v_2\big[\sqrt{v}||A^{\frac{1}{2}}B^{\frac{3}{2}}||_1+||B||_2^2-2\sqrt[4]{v}\sqrt{||A^{\frac{1}{2}}||_1\ ||B^{\frac{7}{2}}||_1}\big].$$

For another, since

$$tr(v^2A^2+(1-v)^2B^2)=v^2trA^2+(1-v)^2trB^2=v^2\sum_{j=1}^{n}s_j^2(A)+(1-v)^2\sum_{j=1}^{n}s_j^2(B),$$

that is

$$tr(v^2A^2+(1-v)^2B^2)=v^2||A||_2^2+(1-v)^2||B||_2^2.$$

This completed the proof.

REMARK 4.9. It is obvious that inequalities (4.13) – (4.16) are refinements of the well-known results in [12].

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