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ULAM STABILITY OF AN ADDITIVE–QUADRATIC FUNCTIONAL EQUATION IN BANACH SPACES

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Abstract. Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the following additive-quadratic functional equation

$$f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(x,w)=0. \tag{0.1}$$

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [25] in 1940, concerning the stability of group homomorphisms. Let $(G_1,.)$ be a group and let $(G_2,*)$ be a metric group with the metric $d(.,.)$. Given $\varepsilon>0$, does there exist a $\delta 0$, such that if a mapping $h: G_1 \to G_2$ satisfies the inequality $d(h(x.y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H: G_1\to G_2$ with $d(h(x),H(x))<\varepsilon$ for all $x\in G_1$? In the other words, Under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [15] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f: E\to E'$ be a mapping between Banach spaces such that

$$\|f(x+y)-f(x)-f(y)\|\leqslant\delta$$

for all $x,y\in E$, and for some $\delta>0$. Then there exists a unique additive mapping $T: E\to E'$ such that

$$\|f(x)-T(x)\|\leqslant\delta$$

for all $x\in E$. In 1978, Rassias [22] proved the following theorem.

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THEOREM 1.1. [22] *Let* $f : E \to E'$ *be a mapping from a normed vector space* E *into a Banach space* E' *subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, *where* ε *and* p *are constants with* $\varepsilon > 0$ *and* $p < 1$. *Then there exists a unique additive mapping* $T : E \to E'$ *such that*

$$\|f(x) - T(x)\| \leqslant \frac{2\varepsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. *If* $p < 0$ *then inequality* (1.1) *holds for all* $x, y \neq 0$, *and* (1.2) *for* $x \neq 0$. *Also, if the function* $t \mapsto f(tx)$ *from* $\mathbb{R}$ *into* E' *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in E$, *then* T *is* $\mathbb{R}$*-linear.*

A generalization of the Rassias' theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function.

THEOREM 1.2. [12] *Suppose* $(G, +)$ *is an abelian group,* E *is a Banach space, and that the so-called admissible control function* $\varphi : G \times G \to \mathbb{R}$ *satisfies*

$$\tilde{\varphi}(x,y) := 2^{-1}\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in G$. *If* $f : G \to E$ *is a mapping with*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varphi(x,y)$$

for all $x, y \in G$, *then there exists a unique mapping* $T : G \to E$ *such that* $T(x+y) = T(x) + T(y)$ *and* $\|f(x) - T(x)\| \leqslant \tilde{\varphi}(x,x)$ *for all* $x, y \in G$.

Gilányi [13] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leqslant \|f(x+y)\| \tag{1.3}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [23]. Fechner [11] and Gilányi [14] proved the Hyers-Ulam stability of the functional inequality (1.3). Park [18, 19] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [6, 7, 9, 10, 18, 24]).

We recall a fundamental result in fixed point theory.

THEOREM 1.3. [2, 5] *Let* (X,d) *be a complete generalized metric space and let* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $\alpha < 1$. *Then for each given element* $x \in X$, *either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n *or there exists a positive integer* n_0 *such that*

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geqslant n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;

(4) $d(y,y^*) \leqslant \frac{1}{1-\alpha} d(y,Jy)$ *for all* $y \in Y$.

In 1996, Isac and Rassias [16] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [3, 4, 8, 20, 21]).

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (0.1) in Banach spaces by using the direct method. In Section 3, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (0.1) in Banach spaces by using the fixed point method.

Throughout this paper, let X be a complex normed space and Y be a complex Banach space.

2. Hyers-Ulam stability of the additive-quadratic functional equation (0.1): direct method

We investigate the additive-quadratic functional equation (0.1) in complex normed spaces.

LEMMA 2.1. *If a mapping* $f : X^2 \to Y$ *satisfies* $f(0,z) = f(x,0) = 0$ *and*

$$f(x+y,z+w) + f(x-y,z-w) - 2f(x,z) - 2f(x,w) = 0 \tag{2.1}$$

for all $x,y,z,w \in X$, *then* $f : X^2 \to Y$ *is additive in the first variable and quadratic in the second variable.*

Proof. If $w = 0$, then $f(x+y,z) + f(x-y,z) - 2f(x,z) = 0$ for all $x,y,z \in X$. So f is additive in the first variable.

If $y = 0$, then $f(x,z+w) + f(x,z-w) - 2f(x,z) - 2f(x,w) = 0$ for all $x,z,w \in X$. So f is quadratic in the second variable.

Note that if $f : X \to Y$ satisfies (2.1), then the mapping $f : X \to Y$ is called an *additive-quadratic* mapping.

Now we prove the Hyers-Ulam stability of the additive-quadratic functional inequality (0.1) in complex Banach spaces.

THEOREM 2.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying*

$$\Phi(x,y) := \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.2}$$

for all $x,y \in X$ and $f : X^2 \to Y$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and

$$\|f(x+y,z+w) + f(x-y,z-w) - 2f(x,z) - 2f(x,w)\| \leqslant \varphi(x,y)\varphi(z,w) \tag{2.3}$$

for all $x,y,z,w \in X$. Then there exists a unique additive-quadratic mapping $F : X^2 \to Y$ such that

$$\|f(x,z) - F(x,z)\| \leqslant \min\left\{\frac{1}{2}\Psi(x,x)\varphi(z,0), \frac{1}{4}\varphi(x,0)\Phi(z,z)\right\}$$

for all $x,z \in X$, where

$$\Psi(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)$$

for all $x,y \in X$.

Proof. Letting $y = x$ and $w = 0$ in (2.3), we get

$$\|f(2x,z) - 2f(x,z)\| \leqslant \varphi(x,x)\varphi(z,0) \tag{2.4}$$

and so

$$\left\|f(x,z) - 2f\left(\frac{x}{2}, z\right)\right\| \leqslant \varphi\left(\frac{x}{2}, \frac{x}{2}\right)\varphi(z,0)$$

for all $x,z \in X$. Hence

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}, z\right) - 2^m f\left(\frac{x}{2^m}, z\right)\right\| &\leqslant \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}, z\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}, z\right)\right\| \\ &\leqslant \frac{1}{2}\sum_{j=l+1}^{m} 2^j \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}\right)\varphi(z,0)\end{aligned} \tag{2.5}$$

for all nonnegative integers m and l with $m > l$ and all $x,z \in X$. It follows from (2.5) that the sequence $\{2^k f(\frac{x}{2^k}, z)\}$ is Cauchy for all $x,z \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k}, z)\}$ converges. So one can define the mapping $P : X^2 \to Y$ by

$$P(x,z) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}, z\right)$$

for all $x,z \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get

$$\|f(x,z) - P(x,z)\| \leqslant \frac{1}{2}\Psi(x,x)\varphi(z,0) \tag{2.6}$$

for all $x,z \in X$.

It follows from (2.2) and (2.3) that

$$\begin{aligned}
&\|P(x+y,z+w)+P(x-y,z-w)-2P(x,z)-2P(x,w)\| \\
&= \lim_{n\to\infty}\left\|2^n\left(f\left(\frac{x+y}{2^n},z+w\right)+f\left(\frac{x-y}{2^n},z-w\right)-2f\left(\frac{x}{2^n},z\right)-2f\left(\frac{x}{2^n},w\right)\right)\right\| \\
&\leqslant \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)\varphi(z,w)=0
\end{aligned}$$

for all $x,y,z,w \in X$. So

$$P(x+y,z+w)+P(x-y,z-w)-2P(x,z)-2P(x,w)=0$$

for all $x,y,z,w \in X$. By Lemma 2.1, the mapping $P: X^2 \to Y$ is additive in the first variable and quadratic in second variable.

Now, let $T: X^2 \to Y$ be another additive-quadratic mapping satisfying (2.6). Then we have

$$\begin{aligned}
\|P(x,z)-T(x,z)\| &= \left\|2^qP\left(\frac{x}{2^q},z\right)-2^qT\left(\frac{x}{2^q},z\right)\right\| \\
&\leqslant \left\|2^qP\left(\frac{x}{2^q},z\right)-2^qf\left(\frac{x}{2^q},z\right)\right\|+\left\|2^qT\left(\frac{x}{2^q},z\right)-2^qf\left(\frac{x}{2^q},z\right)\right\| \\
&\leqslant 2^q\Psi\left(\frac{x}{2^q},\frac{x}{2^q}\right)\varphi(z,0),
\end{aligned}$$

which tends to zero as $q\to\infty$ for all $x,z \in X$. So we can conclude that $P(x,z)=T(x,z)$ for all $x,z\in X$. This proves the uniqueness of P.

On the other hand, letting $y=0$ and $w=z$ in (2.3), we get

$$\|f(x,2z)-4f(x,z)\| \leqslant \varphi(x,0)\varphi(z,z) \tag{2.7}$$

and so

$$\left\|f(x,z)-4f\left(x,\frac{z}{2}\right)\right\| \leqslant \varphi(x,0)\varphi\left(\frac{z}{2},\frac{z}{2}\right)$$

for all $x,z\in X$. Hence

$$\begin{aligned}
\left\|4^lf\left(x,\frac{z}{2^l}\right)-4^mf\left(x,\frac{z}{2^m}\right)\right\| &\leqslant \sum_{j=l}^{m-1}\left\|4^jf\left(x,\frac{z}{2^j}\right)-4^{j+1}f\left(x,\frac{z}{2^{j+1}}\right)\right\| \\
&\leqslant \frac{1}{4}\sum_{j=l+1}^{m}4^j\varphi(x,0)\varphi\left(\frac{z}{2^j},\frac{z}{2^j}\right)
\end{aligned} \tag{2.8}$$

for all nonnegative integers m and l with $m>l$ and all $x,z\in X$. It follows from (2.8) that the sequence $\{4^kf(x,\frac{z}{2^k})\}$ is Cauchy for all $x,z\in X$. Since Y is a Banach space, the sequence $\{4^kf(x,\frac{z}{2^k}\}$ converges. So one can define the mapping $Q: X^2\to Y$ by

$$Q(x,z):=\lim_{k\to\infty}4^kf\left(x,\frac{z}{2^k}\right)$$

for all $x,z \in X$. Moreover, letting $l=0$ and passing the limit $m \to \infty$ in (2.8), we get

$$\|f(x,z)-Q(x,z)\| \leqslant \frac{1}{4}\varphi(x,0)\Phi(z,z) \tag{2.9}$$

for all $x,z \in X$.

It follows from (2.2) and (2.3) that

$$\begin{aligned}
&\|Q(x+y,z+w)+Q(x-y,z-w)-2Q(x,z)-2Q(x,w)\| \\
&= \lim_{n\to\infty}\left\|4^n\left(f\left(x+y,\frac{z+w}{2^n}\right)+f\left(x-y,\frac{z-w}{2^n}\right)-2f\left(x,\frac{z}{2^n}\right)-2f\left(x,\frac{w}{2^n}\right)\right)\right\| \\
&\leqslant \lim_{n\to\infty} 4^n\varphi(x,y)\varphi\left(\frac{z}{2^n},\frac{w}{2^n}\right)=0
\end{aligned}$$

for all $x,y,z,w \in X$. So

$$Q(x+y,z+w)+Q(x-y,z-w)-2Q(x,z)-2Q(x,w)=0$$

for all $x,y,z,w \in X$. By Lemma 2.1, the mapping $Q: X^2 \to Y$ is additive in the first variable and quadratic in second variable.

Now, let $T: X^2 \to Y$ be another additive-quadratic mapping satisfying (2.9). Then we have

$$\begin{aligned}
\|Q(x,z)-T(x,z)\| &= \left\|4^qQ\left(x,\frac{z}{2^q}\right)-4^qT\left(x,\frac{z}{2^q}\right)\right\| \\
&\leqslant \left\|4^qQ\left(x,\frac{z}{2^q}\right)-4^qf\left(x,\frac{z}{2^q}\right)\right\|+\left\|4^qT\left(x,\frac{z}{2^q}\right)-4^qf\left(x,\frac{z}{2^q}\right)\right\| \\
&\leqslant \frac{4^q}{2}\varphi(x,0)\Phi\left(\frac{z}{2^q},\frac{z}{2^q}\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x,z \in X$. So we can conclude that $Q(x,z)=T(x,z)$ for all $x,z \in X$. This proves the uniqueness of Q.

It follows from (2.9) that

$$2^n\left\|f\left(\frac{x}{2^n},z\right)-Q\left(\frac{x}{2^n},z\right)\right\| \leqslant \frac{2^n}{4}\varphi\left(\frac{x}{2^n},0\right)\Phi(z,z),$$

which tends to zero as $n \to \infty$ for all $x,z \in X$. Since $Q: X^2 \to Y$ is additive in the first variable, we get $\|P(x,z)-Q(x,z)\|=0$, i.e., $F(x,z) := P(x,z) = Q(x,z)$ for all $x,z \in X$. Thus there is an additive-quadratic mapping $F: X^2 \to Y$ such that

$$\|f(x,z)-F(x,z)\| \leqslant \min\left\{\frac{1}{2}\Psi(x,x)\varphi(z,0), \frac{1}{4}\varphi(x,0)\Phi(z,z)\right\}$$

for all $x,z \in X$.

COROLLARY 2.3. *Let $r>2$ and θ be nonnegative real numbers and $f: X^2 \to Y$ be a mapping satisfying $f(x,0)=f(0,z)=0$ and*

$$\|f(x+y,z+w)+f(x-y,z-w)-2f(x,z)-2f(x,w)\|$$

$$\leqslant \theta(\|x\|^r+\|y\|^r)(\|z\|^r+\|w\|^r) \tag{2.10}$$

for all $x,y,z,w\in X$. *Then there exists a unique additive-quadratic mapping* $F:X^2\to Y$ *such that*

$$\|f(x,z)-F(x,z)\|\leqslant \frac{2\theta}{2^r-2}\|x\|^r\|z\|^r$$

for all $x,z\in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y)=\sqrt{\theta}(\|x\|^r+\|y\|^r)$ for all $x,y\in X$, since $\min\{\frac{2\theta}{2^r-2}\|x\|^r\|z\|^r,\frac{2\theta}{2^r-4}\|x\|^r\|z\|^r\}=\frac{2\theta}{2^r-2}\|x\|^r\|z\|^r$ for all $x,z\in X$.

THEOREM 2.4. *Let* $\varphi:X^2\to[0,\infty)$ *be a function satisfying*

$$\Psi(x,y):=\sum_{j=0}^{\infty}\frac{1}{2^j}\varphi\left(2^jx,2^jz\right)<\infty \tag{2.11}$$

for all $x,y\in X$ *and let* $f:X^2\to Y$ *be a mapping satisfying* $f(x,0)=f(0,z)=0$ *and* (2.3) *for all* $x,z\in X$. *Then there exists a unique additive-quadratic mapping* $F:X^2\to Y$ *such that*

$$\|f(x,z)-F(x,z)\|\leqslant \min\left\{\frac{1}{2}\Psi(x,x)\varphi(z,0),\frac{1}{4}\varphi(x,0)\Phi(z,z)\right\} \tag{2.12}$$

for all $x,z\in X$, *where*

$$\Phi(x,y):=\sum_{j=0}^{\infty}\frac{1}{4^j}\varphi\left(2^jx,2^jy\right)$$

for all $x,y\in X$.

Proof. It follows from (2.4) that

$$\left\|f(x,z)-\frac{1}{2}f(2x,z)\right\|\leqslant\frac{1}{2}\varphi(x,x)\varphi(z,0)$$

for all $x,z\in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f\left(2^lx,z\right)-\frac{1}{2^m}f(2^mx,z)\right\|&\leqslant\sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^jx,z\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x,z\right)\right\| \\ &\leqslant\frac{1}{2}\sum_{j=l}^{m-1}\frac{1}{2^j}\varphi\left(2^jx,2^jx\right)\varphi(z,0)\end{aligned} \tag{2.13}$$

for all nonnegative integers m and l with $m>l$ and all $x,z\in X$. It follows from (2.13) that the sequence $\{\frac{1}{2^k}f(2^kx,z)\}$ is Cauchy for all $x,z\in X$. Since Y is a Banach space, the sequence $\{\frac{1}{2^k}f(2^kx,z)\}$ converges. So one can define the mapping $P:X^2\to Y$ by

$$P(x,z):=\lim_{k\to\infty}\frac{1}{2^k}f\left(2^kx,z\right)$$

for all $x,z\in X$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (2.13), we get

$$\|f(x,z)-P(x,z)\|\leqslant\frac{1}{2}\Psi(x,x)\varphi(z,0) \tag{2.14}$$

for all $x,z\in X$.

It follows from (2.3) and (2.11) that

$$\begin{aligned}&\|P(x+y,z+w)+P(x-y,z-w)-2P(x,z)-2P(x,w)\|\\ =&\lim_{n\to\infty}\left\|\frac{1}{2^n}\left(f\left(2^n(x+y),z+w\right)+f\left(2^n(x-y),z-w\right)-2f\left(2^nx,z\right)-2f\left(2^nx,w\right)\right)\right\|\\ \leqslant&\lim_{n\to\infty}\frac{1}{2^n}\varphi\left(2^nx,2^ny\right)\varphi(z,w)=0\end{aligned}$$

for all $x,y,z,w\in X$. So

$$P(x+y,z+w)+P(x-y,z-w)-2P(x,z)-2P(x,w)=0$$

for all $x,y,z,w\in X$. By Lemma 2.1, the mapping $P:X^2\to Y$ is additive in the first variable and quadratic in second variable.

Now, let $T:X^2\to Y$ be another additive-quadratic mapping satisfying (2.14). Then we have

$$\begin{aligned}\|P(x,z)-T(x,z)\|&=\left\|\frac{1}{2^q}P\left(2^qx,z\right)-\frac{1}{2^q}T\left(2^qx,z\right)\right\|\\ &\leqslant\left\|\frac{1}{2^q}P\left(2^qx,z\right)-\frac{1}{2^q}f\left(2^qx,z\right)\right\|+\left\|\frac{1}{2^q}T\left(2^qx,z\right)-\frac{1}{2^q}f\left(2^qx,z\right)\right\|\\ &\leqslant\frac{1}{2^q}\Psi\left(2^qx,2^qx\right)\varphi(z,0),\end{aligned}$$

which tends to zero as $q\to\infty$ for all $x,z\in X$. So we can conclude that $P(x,z)=T(x,z)$ for all $x,z\in X$. This proves the uniqueness of P.

It follows from (2.15) that

$$\|f(x,2z)-4f(x,z)\|\leqslant\varphi(x,0)\varphi(z,z) \tag{2.15}$$

and so

$$\left\|f(x,z)-\frac{1}{4}f\left(x,2z\right)\right\|\leqslant\frac{1}{4}\varphi(x,0)\varphi\left(2z,2z\right)$$

for all $x,z\in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f(x,2^lz)-\frac{1}{4^m}f(x,2^mz)\right\|&\leqslant\sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(x,2^jz\right)-\frac{1}{4^{j+1}}f\left(x,2^{j+1}z\right)\right\| \\ &\leqslant\frac{1}{4}\sum_{j=l}^{m-1}\frac{1}{4^j}\varphi(x,0)\varphi\left(2^jz,2^jz\right)\end{aligned} \tag{2.16}$$

for all nonnegative integers m and l with $m > l$ and all $x, z \in X$. It follows from (2.16) that the sequence $\{\frac{1}{4^k} f(x, 2^k z)\}$ is Cauchy for all $x, z \in X$. Since Y is a Banach space, the sequence $\{\frac{1}{4^k} f(x, 2^k z\}$ converges. So one can define the mapping $Q : X^2 \to Y$ by

$$Q(x,z) := \lim_{k\to\infty} \frac{1}{4^k} f\left(x, 2^k z\right)$$

for all $x, z \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.16), we get

$$\|f(x,z) - Q(x,z)\| \leqslant \frac{1}{4}\varphi(x,0)\Phi(z,z) \tag{2.17}$$

for all $x, z \in X$.

It follows from (2.3) and (2.11) that

$$\begin{aligned}
&\|Q(x+y,z+w) + Q(x-y,z-w) - 2Q(x,z) - 2Q(x,w)\| \\
&= \lim_{n\to\infty} \left\| \frac{1}{4^n} \left(f(x+y, 2^n(z+w)) + f(x-y, 2^n(z-w)) - 2f(x, 2^n z) - 2f(x, 2^n w)\right) \right\| \\
&\leqslant \lim_{n\to\infty} \frac{1}{4^n} \varphi(x,y)\varphi(2^n z, 2^n w) = 0
\end{aligned}$$

for all $x, y, z, w \in X$. So

$$Q(x+y,z+w) + Q(x-y,z-w) - 2Q(x,z) - 2Q(x,w) = 0$$

for all $x, y, z, w \in X$. By Lemma 2.1, the mapping $Q : X^2 \to Y$ is additive in the first variable and quadratic in second variable.

Now, let $T : X^2 \to Y$ be another additive-quadratic mapping satisfying (2.17). Then we have

$$\begin{aligned}
\|Q(x,z) - T(x,z)\| &= \left\| \frac{1}{4^q} Q(x, 2^q z) - \frac{1}{4^q} T(x, 2^q z) \right\| \\
&\leqslant \left\| \frac{1}{4^q} Q(x, 2^q z) - \frac{1}{4^q} f(x, 2^q z) \right\| + \left\| \frac{1}{4^q} T(x, 2^q z) - \frac{1}{4^q} f(x, 2^q z) \right\| \\
&\leqslant \frac{1}{2\cdot 4^q} \varphi(x,0)\Phi(2^q z, 2^q z),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x, z \in X$. So we can conclude that $Q(x,z) = T(x,z)$ for all $x, z \in X$. This proves the uniqueness of Q.

It follows from (2.17) that

$$\frac{1}{2^n} \|f(2^n x, z) - Q(2^n z, z)\| \leqslant \frac{1}{4\cdot 2^n} \varphi(2^n x, 0)\Phi(z,z),$$

which tends to zero as $n \to \infty$ for all $x, z \in X$. Since $Q : X^2 \to Y$ is additive in the first variable, we get $\|P(x,z) - Q(x,z)\| = 0$, i.e., $F(x,z) := P(x,z) = Q(x,z)$ for all $x, z \in X$. Thus there is an additive-quadratic mapping $F : X^2 \to Y$ such that

$$\|f(x,z) - F(x,z)\| \leqslant \min\left\{ \frac{1}{2}\Psi(x,x)\varphi(z,0), \frac{1}{4}\varphi(x,0)\Phi(z,z) \right\}$$

for all $x, z \in X$.

COROLLARY 2.5. *Let* $r < 1$ *and* θ *be nonnegative real numbers and* $f : X^2 \to Y$ *be a mapping satisfying* (2.10) *and* $f(x,0) = f(0,z) = 0$ *for all* $x,z \in X$. *Then there exists a unique additive-quadratic mapping* $F : X^2 \to Y$ *such that*

$$\|f(x,z) - F(x,z)\| \leqslant \frac{2\theta}{4-2^r}\|x\|^r\|z\|^r$$

for all $x,z \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x,y \in X$, since $\min\{\frac{2\theta}{2-2^r}\|x\|^r\|z\|^r, \frac{2\theta}{4-2^r}\|x\|^r\|z\|^r\} = \frac{2\theta}{4-2^r}\|x\|^r\|z\|^r$ for all $x,z \in X$.

3. Hyers-Ulam stability of the additive-quadratic functional equation (0.1): fixed point method

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (0.1) in complex Banach spaces.

THEOREM 3.1. *Let* $\varphi : X^2 \to [0,\infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right) \leqslant \frac{L}{4}\varphi(x,y) \leqslant \frac{L}{2}\varphi(x,y) \tag{3.1}$$

for all $x,y \in X$. *Let* $f : X^2 \to Y$ *be a mapping satisfying satisfying* (2.3) *and* $f(x,0) = f(0,z) = 0$ *for all* $x,z \in X$. *Then there exists a unique additive-quadratic mapping* $F : X^2 \to Y$ *such that*

$$\|f(x,z) - F(x,z)\| \leqslant \min\left\{\frac{L}{2(1-L)}\varphi(x,x)\varphi(z,0), \frac{L}{4(1-L)}\varphi(x,0)\varphi(z,z)\right\} \tag{3.2}$$

for all $x,z \in X$.

Proof. Letting $w = 0$ and $y = x$ in (2.3), we get

$$\|f(2x,z) - 2f(x,z)\| \leqslant \varphi(x,x)\varphi(z,0) \tag{3.3}$$

for all $x,z \in X$.

Consider the set

$$S := \{h : X^2 \to Y,\ \ h(x,0) = h(0,z) = 0\ \ \forall x,z \in X\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x,z) - h(x,z)\| \leqslant \mu\varphi(x,x)\varphi(z,0),\ \ \forall x,z \in X\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S,d) is complete (see [17]).

Now we consider the linear mapping $J: S \to S$ such that

$$Jg(x,z) := 2g\left(\frac{x}{2},z\right)$$

for all $x,z \in X$.

Let $g,h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x,z) - h(x,z)\| \leqslant \varepsilon\varphi(x,x)\,\varphi(z,0)$$

for all $x,z \in X$. Hence

$$\begin{aligned}\|Jg(x,z) - Jh(x,z)\| &= \left\|2g\left(\frac{x}{2},z\right) - 2h\left(\frac{x}{2},z\right)\right\| \leqslant 2\varepsilon\varphi\left(\frac{x}{2},\frac{x}{2}\right)\varphi(z,0) \\ &\leqslant 2\varepsilon\frac{L}{2}\varphi(x,x)\,\varphi(z,0) = L\varepsilon\varphi(x,x)\,\varphi(z,0)\end{aligned}$$

for all $x,z \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leqslant L\varepsilon$. This means that

$$d(Jg,Jh) \leqslant Ld(g,h)$$

for all $g,h \in S$.

It follows from (3.3) that

$$\left\|f(x,z) - 2f\left(\frac{x}{2},z\right)\right\| \leqslant \varphi\left(\frac{x}{2},\frac{x}{2}\right)\varphi(z,0) \leqslant \frac{L}{2}\varphi(x,x)\varphi(z,0)$$

for all $x,z \in X$. So $d(f,Jf) \leqslant \frac{L}{2}$.

By Theorem 1.3, there exists a mapping $P: X^2 \to Y$ satisfying the following:

(1) P is a fixed point of J, i.e.,

$$P(x,z) = 2P\left(\frac{x}{2},z\right) \tag{3.4}$$

for all $x,z \in X$. The mapping P is a unique fixed point of J. This implies that P is a unique mapping satisfying (3.4) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x,z) - P(x,z)\| \leqslant \mu\varphi(x,x)\,\varphi(z,0)$$

for all $x,z \in X$;

(2) $d(J^l f,P) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^l f\left(\frac{x}{2^l},z\right) = P(x,z)$$

for all $x,z \in X$;

(3) $d(f,P) \leqslant \frac{1}{1-L} d(f,Jf)$, which implies

$$\|f(x,z)-P(x,z)\| \leqslant \frac{L}{2(1-L)} \varphi(x,x)\varphi(z,0)$$

for all $x,z \in X$.

By the same reasoning as in the proof of Theorem 2.2, one can show that the mapping $P: X^2 \to Y$ is additive in the first variable and quadratic in the second variable.

Letting $z=w$ and $y=0$ in (2.3), we get

$$\|f(x,2z)-4f(x,z)\| \leqslant \varphi(x,0)\varphi(z,z) \tag{3.5}$$

for all $x,z \in X$.

Consider the set

$$S := \{h: X^2 \to Y,\ \ h(x,0)=h(0,z)=0\ \ \forall x,z \in X\}$$

and introduce the generalized metric on S:

$$d'(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x,z)-h(x,z)\| \leqslant \mu\varphi(x,0)\varphi(z,z),\ \ \forall x,z \in X\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S,d') is complete (see [17]).

Now we consider the linear mapping $J': S \to S$ such that

$$J'g(x,z) := 4g\left(\frac{x}{2},z\right)$$

for all $x,z \in X$.

Let $g,h \in S$ be given such that $d'(g,h)=\varepsilon$. Then

$$\|g(x,z)-h(x,z)\| \leqslant \varepsilon\varphi(x,0)\varphi(z,z)$$

for all $x,z \in X$. Hence

$$\begin{aligned}\|J'g(x,z)-J'h(x,z)\| &= \left\|4g\left(x,\frac{z}{2}\right)-4h\left(x,\frac{z}{2}\right)\right\| \leqslant 4\varepsilon\varphi(x,0)\varphi\left(\frac{z}{2},\frac{z}{2}\right)\\ &\leqslant 4\varepsilon\frac{L}{4}\varphi(x,0)\varphi(z,z) = L\varepsilon\varphi(x,0)\varphi(z,z)\end{aligned}$$

for all $x,z \in X$. So $d'(g,h)=\varepsilon$ implies that $d'(J'g,J'h) \leqslant L\varepsilon$. This means that

$$d'(J'g,J'h) \leqslant Ld'(g,h)$$

for all $g,h \in S$.

It follows from (3.5) that

$$\left\|f(x,z)-4f\left(x,\frac{z}{2}\right)\right\| \leqslant \varphi(x,0)\varphi\left(\frac{z}{2},\frac{z}{2}\right) \leqslant \frac{L}{4}\varphi(x,0)\varphi(z,z)$$

for all $x,z \in X$. So $d'(f,J'f) \leqslant \frac{L}{4}$.

By Theorem 1.3, there exists a mapping $Q: X^2 \to Y$ satisfying the following:

(1) Q is a fixed point of J', i.e.,

$$Q(x,z) = 4Q\left(x, \frac{z}{2}\right) \tag{3.6}$$

for all $x,z \in X$. The mapping Q is a unique fixed point of J'. This implies that Q is a unique mapping satisfying (3.6) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x,z) - Q(x,z)\| \leqslant \mu\varphi(x,0)\varphi(z,z)$$

for all $x,z \in X$;

(2) $d(J'^l f, Q) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 4^l f\left(x, \frac{z}{2^l}\right) = Q(x,z)$$

for all $x,z \in X$;

(3) $d(f,Q) \leqslant \frac{1}{1-L} d(f,J'f)$, which implies

$$\|f(x,z) - Q(x,z)\| \leqslant \frac{L}{4(1-L)}\varphi(x,0)\varphi(z,z)$$

for all $x,z \in X$.

By the same reasoning as in the proof of Theorem 2.2, one can show that the mapping $Q : X^2 \to Y$ is additive in the first variable and quadratic in the second variable.

By the same reasoning as in the proof of Theorem 2.2, we get $\|P(x,z) - Q(x,z)\| = 0$, i.e., $F(x,z) := P(x,z) = Q(x,z)$ for all $x,z \in X$. Thus there is an additive-quadratic mapping $F : X^2 \to Y$ such that

$$\|f(x,z) - F(x,z)\| \leqslant \min\left\{\frac{L}{2(1-L)}\varphi(x,x)\varphi(z,0), \frac{L}{4(1-L)}\varphi(x,0)\varphi(z,z)\right\}$$

for all $x,z \in X$.

COROLLARY 3.2. *Let $r > 2$ and θ be nonnegative real numbers and $f : X^2 \to Y$ be a mapping satisfying (2.10) and $f(x,0) = f(0,z) = 0$ for all $x,z \in X$. Then there exists a unique additive-quadratic mapping $F : X^2 \to Y$ such that*

$$\|f(x,z) - F(x,z)\| \leqslant \frac{2\theta}{2^r - 4}\|x\|^r\|z\|^r$$

for all $x,z \in X$.

Proof. The proof follows from Theorem 3.1 by taking $L = 2^{2-r}$ and $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x,y \in X$, since $\min\{\frac{2\theta}{2^r-4}\|x\|^r\|z\|^r, \frac{4\theta}{2^r-4}\|x\|^r\|z\|^r\} = \frac{2\theta}{2^r-4}\|x\|^r\|z\|^r$ for all $x,z \in X$.

THEOREM 3.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y) \leqslant 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right) \leqslant 4L\varphi\left(\frac{x}{2},\frac{y}{2}\right) \tag{3.7}$$

for all $x,y \in X$. Let $f : X^2 \to Y$ be a mapping satisfying satisfying (2.3) and $f(x,0) = f(0,z) = 0$ for all $x,z \in X$. Then there exists a unique additive-quadratic mapping $F : X^2 \to Y$ such that

$$\|f(x,z) - F(x,z)\| \leqslant \min\left\{\frac{1}{2(1-L)}\varphi(x,x)\varphi(z,0), \frac{1}{4(1-L)}\varphi(x,0)\varphi(z,z)\right\}$$

for all $x,z \in X$.

Proof. Consider the complete metric spaces (S,d) and (S,d') given in the proof of Theorem 3.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x,z) := \frac{1}{2}g(2x,z)$$

for all $x,z \in X$.

It follows from (3.3) that

$$\left\|f(x,z) - \frac{1}{2}f(2x,z)\right\| \leqslant \frac{1}{2}\varphi(x,x)\varphi(z,0)$$

for all $x,z \in X$. So $d(f,Jf) \leqslant \frac{1}{2}$.

By the same reasoning as in the proof of Theorem 2.2, one can show that there exists a unique additive-quadratic mapping $P : X^2 \to Y$ such that

$$\|f(x,z) - P(x,z)\| \leqslant \frac{1}{2(1-L)}\varphi(x,x)\varphi(z,0)$$

for all $x,z \in X$.

By the same reasoning as in the proof of Theorem 2.2, one can show that the mapping $P : X^2 \to Y$ is additive in the first variable and quadratic in the second variable.

Now we consider the linear mapping $J' : S \to S$ such that

$$J'g(x,z) := 4g\left(\frac{x}{2},z\right)$$

for all $x,z \in X$.

It follows from (3.5) that

$$\left\|f(x,z) - \frac{1}{4}f(x,2z)\right\| \frac{1}{4}\varphi(x,0)\varphi(z,z)$$

for all $x,z \in X$. So $d'(f,J'f) \leqslant \frac{1}{4}$.

By the same reasoning as in the proof of Theorem 2.2, one can show that there exists a unique additive-quadratic mapping $Q : X^2 \to Y$ such that

$$\|f(x,z) - Q(x,z)\| \leqslant \frac{1}{4(1-L)} \varphi(x,0)\, \varphi(z,z)$$

for all $x,z \in X$.

By the same reasoning as in the proof of Theorem 2.2, one can show that the mapping $Q : X^2 \to Y$ is additive in the first variable and quadratic in the second variable.

By the same reasoning as in the proof of Theorem 2.2, we get $\|P(x,z) - Q(x,z)\| = 0$, i.e., $F(x,z) := P(x,z) = Q(x,z)$ for all $x,z \in X$. Thus there is an additive-quadratic mapping $F : X^2 \to Y$ such that

$$\|f(x,z) - F(x,z)\| \leqslant \min\left\{ \frac{1}{2(1-L)} \varphi(x,x)\varphi(z,0), \frac{1}{4(1-L)} \varphi(x,0)\varphi(z,z) \right\}$$

for all $x,z \in X$.

COROLLARY 3.4. *Let $r < 1$ and θ be nonnegative real numbers and $f : X^2 \to Y$ be a mapping satisfying (2.10) and $f(x,0) = f(0,z) = 0$ for all $x,z \in X$. Then there exists a unique additive-quadratic mapping $F : X^2 \to Y$ such that*

$$\|f(x,z) - F(x,z)\| \leqslant \frac{\theta}{2-2^r} \|x\|^r \|z\|^r$$

for all $x,z \in X$.

Proof. The proof follows from Theorem 3.3 by taking $L = 2^{r-1}$ and $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x,y \in X$, since $\min\{\frac{2\theta}{2-2^r}\|x\|^r\|z\|^r, \frac{\theta}{2-2^r}\|x\|^r\|z\|^r\} = \frac{\theta}{2-2^r}\|x\|^r\|z\|^r$ for all $x,z \in X$.

Competing interests

The authors declare that they have no competing interests.

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