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ON SOME NEW NONLINEAR VOLTERRA–FREDHOLM TYPE DISCRETE INEQUALITIES AND ITS APPLICATIONS

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Abstract. In this paper, we develop some extensions and generalizations of some new nonlinear Volterra-Fredholm type discrete inequalities. These inequalities can be used as handy tools in the study of class of nonlinear Volterra-Fredholm sum-difference equations and its variants to obtain bound on the unknown function and analysis of various properties of solutions.

1. Introduction

The prominence of discrete and integral inequalities is diversified due to their application in the study of qualitative and quantitative properties of solutions of various linear and nonlinear difference, differential and integral equations. The Gronwall-Bellman inequality [8, 9] and its numerous extensions and genralizations play a central role in the analysis of properties like boundedness, uniqueness, stability etc. of solutions of such equations, *see* [1, 2, 4, 5, 6, 7, 11, 12, 13, 14, 15, 16, 17] and references cited therein.

The necessity of such integral inequalities of more general kind has always been felt in the study of boundedness of solutions of second order linear differential equations of the type $y'' + A(t)y' = 0$. For the first time, Ou-Iang [10] investigated an integral inequality to study this class of differential equations. This inequality is currently known as Ou-Iang's inequality in the branch of inequalities. Recently Ma[3] have developed the discrete version of Ou-Iang's inequality in two variables of Volterra-Fredholm type. This version has soon became a powerfool tool in the study of large variety of Volterra-Fredholm difference and sum-difference equations.

In this manuscript, we extend and improve some of the results reported in [3] to obtain new generalizations of Volterra-fredholm type discrete inequalities. These inequalities can be used in the study of more general kind of nonlinear Volterra-Fredholm sum-difference equations. Some examples are also given to exhibit the importance of our results.

Throughout this article, we let $N_0 = 0, 1, \cdots$, $\mathscr{N}_S = [0, S) \cap \mathbb{Z}$ and $\mathscr{N}_T = [0, T) \cap \mathbb{Z}$, where $S, T \in N_0$. We denote $\mathbb{R}_0 = [0, \infty)$, $\mathbb{R}_+ = (0, \infty)$ and as usual $\mathbb{R}$ denote

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the set of all real numbers. Further we denote $\mathscr{N} = \mathscr{N}_S \times \mathscr{N}_T \subset \mathbb{Z}^2$ and sublattice $\mathscr{N}_{(s,t)}$ of $\mathscr{N}$ by $\mathscr{N}_{(s,t)} = ([0,s] \times [0,t]) \cap \mathscr{N}$, for any $(s,t) \in \mathscr{N}$. We denote the set of all nonnegative real valued fuctions on N_0 by $\mathscr{F}_+(N_0)$ and the collection of all continuously differentiable functions from X to Y by $C(X,Y)$. For $\mathscr{U} \subset N_0^2$, the set of all real valued fuctions on $\mathscr{U}$ is denoted by $\mathscr{F}(\mathscr{U})$ and set of all nonnegative real valued fuctions on $\mathscr{U}$ is denoted by $\mathscr{F}_+(\mathscr{U})$.

The partial difference operators Δ_1, Δ_2 and $\Delta_2\Delta_1$ on $x \in \mathscr{F}(N_0^2)$ or $x \in \mathscr{F}_+(N_0^2)$ are defined as $\Delta_1 x(s,t) = \Delta_s x(s,t) = x(s+1,t) - x(s,t), \Delta_2 x(s,t) = \Delta_t x(s,t) = x(s,t+1) - x(s,t)$, and $\Delta_2\Delta_1 x(s,t) = \Delta_{st}^2 x(s,t)$ for any $(s,t) \in \mathscr{N}$.

Before proceeding to the statement of our main result, we state here some important finite difference inequalities and definitions that will be used in further discussion.

THEOREM 1.1. *(Ma[3]) Suppose that $u,a \in \mathscr{F}_+(\mathscr{N})$ and $w \in C(R_+,R_+)$ be nondecreasing function such that $w(m) > 0$ for $m > 0$.*

If $u(m,n)$ satisfies

$$u(m,n) \leqslant k + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)w(u(s,t)) + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} a(s,t)w(u(s,t))$$

for $(m,n) \in \mathscr{N}$, then

$$u(m,n) \leqslant G_1^{-1}\left\{G_1\left[H_1^{-1}\left(\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} a(s,t)\right)\right] + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\right\}$$

for $(m,n) \in \mathscr{N}$, where

$$G_1(v) = \int_{v_0}^{v} \frac{\mathrm{d}s}{w(s)}, \quad v \geqslant v_0 > 0,\ G_1(\infty) = \infty,\ H_1(t) = G_1(2t-k) - G_1(t)$$

with $H_1(t)$ strictly increasing for $t \geqslant k$ and G_1^{-1} and H_1^{-1} are inverse functions of G_1 and H_1 respectively.

THEOREM 1.2. *(Pachpatte [1], p.371) Let the functions $u(x,y) \geqslant 0, a(x,y) \geqslant 0$, $b(x,y) \geqslant 0$ be defined for $x,y \in N_0$ and $u(x,0) = u(0,y) = k$, where $k \geqslant 0$ is a constant. If*

$$\Delta_2\Delta_1 u(x,y) \leqslant a(x,y)u(x,y) + b(x,y) \tag{1.1}$$

for $x,y \in N_0$, then

$$u(x,y) \leqslant p(x,y)\prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1} a(s,t)\right] \tag{1.2}$$

for $x,y \in N_0$, where

$$p(x,y) = k + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s,t) \tag{1.3}$$

for $x,y \in N_0$.

THEOREM 1.3. *(Salem and Raslan[2]) Let $u(m,n), a(m,n), b(m,n)$ be nonnegative functions and $a(m,n)$ nondecreasing for $m,n \in \mathbb{N}$. If*

$$u(m,n) \leqslant a(m,n) + \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} b(s,t)x(s,t)$$

for $m,n \in \mathbb{N}$, then

$$u(m,n) \leqslant a(m,n) \prod_{t=0}^{n-1} \left[1 + \sum_{s=0}^{m-1} b(s,t) \right].$$

THEOREM 1.4. *(Lees[17]) Let $u(t), a(t)$ and $b(t)$ be nonnegative functions defined on N_0 and $a(t)$ is nondecreasing for $t \in N_0$. If*

$$u(t) \leqslant a(t) + \sum_{s=0}^{t-1} b(s)u(s)$$

for $t \in N_0$, then

$$u(t) \leqslant a(t) \prod_{s=0}^{t-1} [1 + b(s)] \leqslant a(t) \exp \left(\sum_{s=0}^{t-1} b(s) \right)$$

for $t \in N_0$.

DEFINITION 1. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be superadditive if $\alpha(x+y) \geqslant \alpha(x) + \alpha(y)$ for all $x, y \in \mathbb{R}_+$.

DEFINITION 2. A function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is called as submultiplicative if $\beta(xy) \leqslant \beta(x)\beta(y)$ for all $x, y \in \mathbb{R}_+$.

2. Main results

In present section, we state and prove some new nonlinear Volterra-Fredholm type discrete inequalities. These inequalities can be used as powerfool tool in the analysis of behavior of solution and determining the explicit bound on various nonlinear difference and sum-difference equations of Volterra-Fredholm type.

THEOREM 2.1. *Let $x, a, \Delta_s x(s,t), \Delta_t x(s,t), \Delta_{st}^2 x(s,t) \in \mathscr{F}_+(\mathscr{U})$ and $h, \omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ be continuous nondecreasing functions such that $h(m) > 0, \omega(m) > 0$ for $m \in \mathbb{R}_+$. Let $k \in \mathbb{R}_0$ be a constant and $x(0,t) = x(s,0) = k'$. If*

$$\Delta_{st}^2 x(s,t) \leqslant k + h \left(\sum_{m=0}^{s-1} \sum_{n=0}^{t-1} a(m,n)\omega(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1} \sum_{n=0}^{T-1} a(m,n)\omega(\Delta_{mn}^2 x(m,n)) \right) \tag{2.1}$$

for $(s,t) \in \mathscr{N}$ *, then*

$$x(s,t) \leqslant k' + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1} k + h\left\{Q^{-1}\left[Q\left(P^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\right]\right) + \sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1} a(m,n)\right]\right\} \tag{2.2}$$

for $(s,t) \in \mathscr{N}$*, where* $Q(r) = \int_{r_0}^{r} \frac{ds}{\omega(k+h(s))}$, $r \geqslant r_0 > 0$, $P(r) = Q(2r) - Q(r)$ *is strictly increasing function and* Q^{-1}, P^{-1} *are inverse functions of* Q *and* P *respectively.*

Proof. Define a function $z(s,t)$ by

$$z(s,t) = \sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\omega(\Delta^2_{mn}x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\omega(\Delta^2_{mn}x(m,n)). \tag{2.3}$$

Then we have

$$z(0,t) = \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\omega(\Delta^2_{mn}x(m,n)) \tag{2.4}$$

and

$$\Delta^2_{st}x(s,t) \leqslant k + h(z(s,t)), \quad (s,t) \in \mathscr{N}. \tag{2.5}$$

From equation (2.3), we obtain

$$\begin{aligned}\Delta_s z(s,t) = \sum_{n=0}^{t-1} a(s,n)\omega(\Delta^2_{sn}x(s,n)) &\leqslant \sum_{n=0}^{t-1} a(s,n)\omega(k+h(z(s,n))) \\ &\leqslant \omega(k+h(z(s,t)))\sum_{n=0}^{t-1} a(s,n).\end{aligned} \tag{2.6}$$

We also observe that

$$Q(z(s+1,t)) - Q(z(s,t)) = \int_{z(s,t)}^{z(s+1,t)} \frac{ds}{\omega(k+h(s))} \leqslant \frac{\Delta_s z(s,t)}{\omega(k+h(z(s,t)))} \leqslant \sum_{n=0}^{t-1} a(s,n). \tag{2.7}$$

Substitute $s = m$ in (2.7) and sum it over m from 0 to $s-1$ to get

$$Q(z(s,t)) \leqslant Q(z(0,t)) + \sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n). \tag{2.8}$$

As Q is increasing, (2.8) gives an estimate

$$z(s,t) \leqslant Q^{-1}\left[Q(z(0,t)) + \sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\right]. \tag{2.9}$$

From (2.3) and (2.9), it is clear that

$$\begin{aligned} 2z(0,t) = z(S,T) &\leqslant Q^{-1}\left[Q(z(0,T)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\right] \\ &= Q^{-1}\left[Q(z(0,t)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\right]. \end{aligned} \tag{2.10}$$

The inequality (2.10) results into

$$Q(2z(0,t)) - Q(z(0,t)) \leqslant \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n). \tag{2.11}$$

As $P(t) = Q(2t) - Q(t)$ is strictly increasing function, we have

$$z(0,t) \leqslant P^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\right]. \tag{2.12}$$

Using (2.12) in (2.9), we get

$$z(s,t) \leqslant Q^{-1}\left[Q\left(P^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\right]\right) + \sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\right]. \tag{2.13}$$

Making use of (2.13) alongwith (2.5), we have

$$\Delta_{st}^2 x(s,t) \leqslant k + h\left(Q^{-1}\left[Q\left(P^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\right]\right) + \sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\right]\right), \quad (s,t) \in \mathscr{N}. \tag{2.14}$$

Application of Theorem 1.2 implies the bound

$$x(s,t) \leqslant k' + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1} k + h\left(Q^{-1}\left[Q\left(P^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\right]\right) + \sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1} a(m,n)\right]\right). \tag{2.15}$$

This proves our theorem.

COROLLARY 2.1. *Let $x, \Delta_s x(s,t), \Delta_t x(s,t), \Delta_{st}^2 x(s,t), h, \omega, k, k'$ be as defined in Theorem 2.1 and $b, c \in \mathscr{F}_+(\mathscr{U})$. If*

$$\Delta_{st}^2 x(s,t) \leqslant k + h\left(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} b(m,n)\omega_1(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} c(m,n)\omega_2(\Delta_{mn}^2 x(m,n))\right), \tag{2.16}$$

for $(s,t) \in \mathscr{N}$ *and there is a nondecreasing function* $\mathscr{W}_1(r) \in C(\mathbb{R}_+,\mathbb{R}_+)$ *with the property that both* ω_1 *and* ω_2 *are less than or equal to* $\mathscr{W}_1(r)$*, then*

$$x(s,t) \leqslant k' + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1} k + h\left(Q_2^{-1}\left[Q_2\left(P_2^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1}\tilde{d}(m,n)\right]\right) + \sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1}\tilde{d}(m,n)\right]\right), \tag{2.17}$$

for $(s,t) \in \mathscr{N}$*, where*

$$Q_2(r) = \int_{r_0}^{r} \frac{ds}{\mathscr{W}_1(k+h(s))},\ r \geqslant r_0 > 0,\ P_2(r) = Q_2(2r) - Q_2(r)$$

and Q_2^{-1}, P_2^{-1} *are inverse functions of* Q_2 *and* P_2 *respectively and* $\tilde{d}(m,n) \in \mathscr{F}_+(\mathscr{U})$ *is such that* $b(s,t)$ *and* $c(s,t)$ *both are less than or equal to* $\tilde{d}(s,t)$.

Proof. Proof can be easily carried out by following the proof of previous theorem with little changes.

COROLLARY 2.2. *Let* $x, \Delta_s x(s,t), \Delta_t x(s,t), \Delta_{st}^2 x(s,t), h, \omega, k'$ *be as defined in Theorem* 2.1*,* $k_1, k_2 \in \mathbb{R}_0$ *be constants and* $c, d \in \mathscr{F}_+(\mathscr{U})$*. If*

$$\Delta_{st}^2 x(s,t) \leqslant k_1 + k_2\, h\left(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} c(m,n)\omega_1(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} d(m,n)\omega_2(\Delta_{mn}^2 x(m,n))\right), \tag{2.18}$$

for $(s,t) \in \mathscr{N}$ *and there is a nondecreasing function* $\mathscr{W}_2(r) \in C(\mathbb{R}_+,\mathbb{R}_+)$ *with the property that both* ω_1 *and* ω_2 *are less than or equal to* $\mathscr{W}_2(r)$*, then*

$$x(s,t) \leqslant k' + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1} k_1 + k_2\left[h\left(Q_3^{-1}\left[Q_3\left(P_3^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1}\tilde{e}(m,n)\right]\right) + \sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1}\tilde{e}(m,n)\right]\right)\right], \tag{2.19}$$

for $(s,t) \in \mathscr{N}$*, where*

$$Q_3(r) = \int_{r_0}^{r} \frac{ds}{\mathscr{W}_2(k_1 + k_2 h(s))},\ r \geqslant r_0 > 0,\ P_3(r) = Q_3(2r) - Q_3(r)$$

and Q_3^{-1}, P_3^{-1} *are inverse functions of* Q_3 *and* P_3 *respectively and* $\tilde{e}(m,n) \in \mathscr{F}_+(\mathscr{U})$ *is such that* $c(s,t)$ *and* $d(s,t)$ *both are less than or equal to* $\tilde{e}(s,t) \in \mathscr{F}_+(\mathscr{U})$.

Proof. Proof can be finished by minutely observing and following the proof of Theorem 2.1.

THEOREM 2.2. *Let $x, a, \Delta_s x(s,t), \Delta_t x(s,t), \Delta_{st}^2 x(s,t), h, \omega, k, k'$ be as defined in Theorem 2.1 and $b \in \mathscr{F}_+(\mathscr{U})$. If $\omega(r)$ is a submultiplicative function and*

$$\Delta_{st}^2 x(s,t) \leqslant k + \sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\Delta_{mn}^2 x(m,n) + h\Bigg(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} b(m,n)\omega(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} b(m,n)\omega(\Delta_{mn}^2 x(m,n))\Bigg) \tag{2.20}$$

for $(s,t) \in \mathscr{N}$, then

$$x(s,t) \leqslant k' + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1} \mathrm{J}(m',n')\Bigg\{k + h\Bigg(Q^{-1}\Bigg[Q\Bigg(P^{-1}\Bigg[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} b(m,n)\mathrm{J}(m,n)\Bigg]\Bigg) + \sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1} b(m,n)\mathrm{J}(m,n)\Bigg]\Bigg)\Bigg\} \tag{2.21}$$

where P, Q, P^{-1}, Q^{-1} are as defined in Theorem 2.1 and

$$\mathrm{J}(s,t) = \prod_{n=0}^{t-1}\left[1 + \sum_{m=0}^{s-1} a(m,n)\right] \tag{2.22}$$

for $(s,t) \in \mathscr{N}$.

Proof. Define a function $z(s,t)$ as

$$z(s,t) = k + h\Bigg(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} b(m,n)\omega(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} b(m,n)\omega(\Delta_{mn}^2 x(m,n))\Bigg). \tag{2.23}$$

Then inequality (2.20) takes the form

$$\Delta_{st}^2 x(s,t) \leqslant z(s,t) + \sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\Delta_{mn}^2 x(m,n). \tag{2.24}$$

As $z(s,t)$ is nonnegative nondecreasing in each variable $s,t \in N_0$, a straightforward application of Theorem 1.3 to the inequality (2.24) implies the estimate

$$\Delta_{st}^2 x(s,t) \leqslant z(s,t)J(s,t), \tag{2.25}$$

where $J(s,t)$ is as defined in (2.22). Using (2.25) in (2.23) and submultiplicativity of ω, we get

$$z(s,t) \leqslant k + h\Bigg(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} b(m,n)\omega(J(m,n))\omega(z(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} b(m,n)\omega(J(m,n))\omega(z(m,n))\Bigg). \tag{2.26}$$

Making use of Theorem 2.1 in inequality (2.26) with appropriate modifications, we obtain

$$z(s,t) \leqslant k + h\left(Q^{-1}\left[Q\left(P^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} b(m,n)\omega(J(m,n))\right]\right) + \sum_{m=0}^{s-1}\sum_{n=0}^{t-1} b(m,n)\omega(J(m,n))\right]\right). \tag{2.27}$$

From (2.25) and (2.27), we get

$$\Delta_{st}^2 x(s,t) \leqslant J(s,t)\left\{k + h\left(Q^{-1}\left[Q\left(P^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} b(m,n)J(m,n)\right]\right) + \sum_{m=0}^{s-1}\sum_{n=0}^{t-1} b(m,n)J(m,n)\right]\right)\right\}. \tag{2.28}$$

Applying Theorem 1.2 to the inequality (2.28), we get the desired inequality in (2.21).

REMARK 2.1. The inequality mentioned in Theorem 2.1 can be obtained as a particular case of the above inequality if we substitute $a(s,t) = 0, (s,t) \in \mathscr{N}$.

THEOREM 2.3. *Let* $x, a, b, \Delta_s x(s,t), \Delta_t x(s,t), \Delta_{st}^2 x(s,t), h, \omega, k, k', P, Q, P^{-1}, Q^{-1}$ *be as defined in Theorem* 2.2.

I. *If*

$$\Delta_{st}^2 x(s,t) \leqslant k + \sum_{m=0}^{s-1} a(m,t)\Delta_{mt}^2 x(m,t) + h\left(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} b(m,n)\omega(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} b(m,n)\omega(\Delta_{mn}^2 x(m,n))\right), \tag{2.29}$$

for $(s,t) \in \mathscr{N}$, *then*

$$x(s,t) \leqslant k' + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1} \mathrm{J}_1(m',n')\left\{k + h\left(Q^{-1}\left[Q\left(P^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} b(m,n)\omega(\mathrm{J}_1(m,n))\right]\right) + \sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1} b(m,n)\omega(\mathrm{J}_1(m,n))\right]\right)\right\}, \tag{2.30}$$

where $\mathrm{J}_1(s,t) = \prod_{s'=0}^{s-1}[1 + a(s',t)]$ *for* $(s,t) \in \mathscr{N}$.

II. *If*

$$\Delta_{st}^2 x(s,t) \leqslant k + \sum_{n=0}^{t-1} a(s,n)\Delta_{sn}^2 x(s,n) + h\Bigg(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} b(m,n)\omega(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} b(m,n)\omega(\Delta_{mn}^2 x(m,n))\Bigg), \tag{2.31}$$

for $(s,t) \in \mathscr{N}$, *then*

$$x(s,t) \leqslant k' + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1} \mathrm{J}_2(m',n')\Bigg\{k + h\Bigg(Q^{-1}\Bigg[Q\Bigg(P^{-1}\Bigg[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} b(m,n)\omega(\mathrm{J}_2(m,n))\Bigg]\Bigg) + \sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1} b(m,n)\omega(\mathrm{J}_2(m,n))\Bigg]\Bigg)\Bigg\}, \tag{2.32}$$

where $\mathrm{J}_2(s,t) = \prod_{t'=0}^{t-1}[1 + a(s,t')]$ *for* $(s,t) \in \mathscr{N}$.

Proof. To prove I, define a function $z(s,t)$ as

$$z(s,t) = k + h\Bigg(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} b(m,n)\omega(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} b(m,n)\omega(\Delta_{mn}^2 x(m,n))\Bigg). \tag{2.33}$$

Then inequality (2.20) takes the form

$$\Delta_{st}^2 x(s,t) \leqslant z(s,t) + \sum_{m=0}^{s-1} a(m,t)\Delta_{mt}^2 x(m,t). \tag{2.34}$$

Here $z(s,t)$ is nonnegative and nondecreasing in $s \in N_0$ for $t \in N_0$. Fixing t in (2.34) and making use of Theorem 1.4, we get

$$\Delta_{st}^2 x(s,t) \leqslant z(s,t)J_1(s,t), \tag{2.35}$$

where $J_1(s,t)$ is as defined in theorem. Further using submultiplicativity of $\omega(r)$ and (2.35) in (2.33), we get

$$z(s,t) \leqslant k + h\Bigg(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} b(m,n)\omega(J_1(m,n))\omega(z(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} b(m,n)\omega(J_1(m,n))\omega(z(m,n))\Bigg). \tag{2.36}$$

A starightforward application of Theorem 2.1 to the inequality (2.36) with appropriate modifications concludes the proof. To prove II, follow the proof of I.

THEOREM 2.4. *Let* $x, a, b, \Delta_s x(s,t), \Delta_t x(s,t), \Delta^2_{st} x(s,t), k, k', P, Q, P^{-1}, Q^{-1}$ *be as defined in Theorem* 2.2. *Let* $h, g, \omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing functions such that* h *is superadditive,* $g \leqslant h$ *and* $h(m), g(m), \omega(m) \in \mathbb{R}_+$ *for* $m \in \mathbb{R}_+$. *If*

$$\Delta^2_{st} x(s,t) \leqslant k + h\left(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\omega(\Delta^2_{mn} x(m,n))\right) + g\left(\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} b(m,n)\omega(\Delta^2_{mn} x(m,n))\right) \tag{2.37}$$

for $(s,t) \in \mathscr{N}$ *and if there is a function* $\tilde{f}(s,t) \in \mathscr{F}_+(\mathscr{U})$ *such that* $\tilde{f}(s,t)$ *is greater than or equal to both* $a(s,t)$ *and* $b(s,t)$ *then*

$$x(s,t) \leqslant k' + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1} k + h\left(Q^{-1}\left[Q\left(P^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} \tilde{f}(m,n)\right]\right) + \sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1} \tilde{f}(m,n)\right]\right) \tag{2.38}$$

for $(s,t) \in \mathscr{N}$.

Proof. From inequality (2.37) and assumptions, we have

$$\Delta^2_{st} x(s,t) \leqslant k + h\left(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} \tilde{f}(m,n)\omega(\Delta^2_{mn} x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} \tilde{f}(m,n)\omega(\Delta^2_{mn} x(m,n))\right). \tag{2.39}$$

Then by reapeating the same arguments as in the proof of Theorem 2.1 we obtain the desired inequality.

THEOREM 2.5. *Let* $x, a, \Delta_s x(s,t), \Delta_t x(s,t), \Delta^2_{st} x(s,t), k, k'$ *be as defined in Theorem* 2.2. *Let* $h_i, \omega_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be nondecreasing functions for* $1 \leqslant i \leqslant l$ *such that* $h_i(m) > 0, \omega_i(m) > 0$ *for* $m \in \mathbb{R}_+$. *Further assume that for each* i, h_i *is superadditive,* $h = \max_{1 \leqslant i \leqslant l}\{h_i\}$ *and* $\omega = \max_{1 \leqslant i \leqslant l}\{\omega_i\}$. *If*

$$\begin{aligned}\Delta^2_{st} x(s,t) \leqslant k + \sum_{i=1}^{l} h_i\Bigg[\sum_{j=1}^{i}\Bigg\{&\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\omega_j(\Delta^2_{mn} x(m,n)) \\ &+ \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\omega_j(\Delta^2_{mn} x(m,n))\Bigg\}\Bigg]\end{aligned} \tag{2.40}$$

for $(s,t) \in \mathscr{N}$ *then*

$$x(s,t) \leqslant k' + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1} k + h\left\{Q_l^{-1}\left[Q_l\left(P_l^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\right]\right) + \sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1} a(m,n)\right]\right\} \tag{2.41}$$

for $(s,t) \in \mathscr{N}$, *where* $Q_l(r) = \int_{r_0}^{r} \frac{ds}{\omega\left(k + h\left(\frac{l(l+1)}{2}s\right)\right)}$, $r \geqslant r_0 > 0$, $P_l(r) = Q_l(2r) - Q_l(r)$ *is strictly increasing function and* Q_l^{-1}, P_l^{-1} *are inverse functions of* Q_l *and* P_l *respectively.*

Proof. We have

$$\begin{aligned}
&\Delta_{st}^2 x(s,t) \\
&\leqslant k + \sum_{i=1}^{l} h_i\left(\sum_{j=1}^{i}\left\{\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\omega_j(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\omega_j(\Delta_{mn}^2 x(m,n))\right\}\right) \\
&= k + h_1\left(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\omega_1(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\omega_1(\Delta_{mn}^2 x(m,n))\right) \\
&\quad + h_2\Bigg[\left(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\omega_1(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\omega_1(\Delta_{mn}^2 x(m,n))\right) \\
&\qquad + \left(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\omega_2(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\omega_2(\Delta_{mn}^2 x(m,n))\right)\Bigg] \\
&\qquad\qquad \vdots \\
&\quad + h_l\Bigg[\left(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\omega_1(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\omega_1(\Delta_{mn}^2 x(m,n))\right) \\
&\qquad + \ldots + \left(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\omega_l(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\omega_l(\Delta_{mn}^2 x(m,n))\right)\Bigg].
\end{aligned} \tag{2.42}$$

Under assumptions, this inequality turns to the form

$$\begin{aligned}
\Delta_{st}^2 x(s,t) \leqslant k + h\Bigg\{\frac{l(l+1)}{2}\Bigg[&\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a(m,n)\omega(\Delta_{mn}^2 x(m,n)) \\
&+ \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\omega(\Delta_{mn}^2 x(m,n))\Bigg]\Bigg\}.
\end{aligned} \tag{2.43}$$

By similar steps as carried out in Theorem 2.1, we obtain the bound as

$$x(s,t) \leqslant k' + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1} k + h\left\{Q_l^{-1}\left[Q_l\left(P_l^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\right]\right) + \sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1} a(m,n)\right]\right\} \tag{2.44}$$

for $(s,t) \in \mathscr{N}$. This completes the proof of our theorem.

3. Applications

EXAMPLE 3.1. Consider the following nonlinear Volterra-Fredholm sum-difference equation

$$\Delta_{st}^2 x(s,t) = \sqrt{\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} \frac{1}{3^m}\Delta_{mn}^2 x(m,n) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1}\left(3^{-m}-4^{-m}\right)\Delta_{mn}^2 x(m,n)} \tag{3.1}$$

for $(s,t)\in\mathscr{N}$, with initial condition $x(0,t)=x(s,0)=1$.

If we let $a^*(s,t)=3^{-s}$ and $\mathscr{W}(u)=u$, then equation (3.1) can be rewritten as

$$\begin{aligned}\Delta_{st}^2 x(s,t) &\leqslant \left(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} a^*(m,n)\mathscr{W}(\Delta_{mn}^2 x(m,n)) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a^*(m,n)\mathscr{W}(\Delta_{mn}^2 x(m,n))\right)^{\frac{1}{2}}\\ &= \left(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1} 3^{-m}\Delta_{mn}^2 x(m,n) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1} 3^{-m}\Delta_{mn}^2 x(m,n)\right)^{\frac{1}{2}}\end{aligned} \tag{3.2}$$

for $(s,t)\in\mathscr{N}$.

Consider $h(u)=\sqrt{u}$, then applying Theorem 2.1 to the inequality (3.2) we get

$$\begin{aligned}x(s,t) &\leqslant 1+\sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1}\sqrt{Q^{-1}\left[Q\left(P^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} a(m,n)\right]\right)+\sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1} a(m,n)\right]},\\ &= 1+\sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1}\sqrt{Q^{-1}\left[Q\left(P^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1} 3^{-m}\right]\right)+\sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1} 3^{-m}\right]},\end{aligned} \tag{3.3}$$

where

$$\left.\begin{aligned} Q(r) &= \int_1^r \frac{ds}{\mathscr{W}(h(s))} = \int_1^r \frac{ds}{\sqrt{s}} = 2(\sqrt{r}-1),\\ P(r) &= Q(2r)-Q(r) = \alpha\sqrt{r},\\ Q^{-1}(r) &= \left(\frac{r}{2}+1\right)^2,\\ P^{-1}(r) &= \frac{r^2}{\alpha^2}\end{aligned}\right\} r\geqslant r_0>0, \alpha = 0.82842.$$

Using these in (3.3) we obtain explicit bound on $x(s,t)$ as

$$x(s,t)\leqslant 1+\sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1}\left\{Q^{-1}\left[Q\left(P^{-1}\left[\frac{T\,3^{1-S}(3^S-1)}{2}\right]\right)+\frac{n'\,3^{1-m'}(3^{m'}-1)}{2}\right]\right\}^{\frac{1}{2}}$$

$$= 1 + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1}\left\{Q^{-1}\left[Q\left(\frac{T^2\, 3^{2-2S}(3^S-1)^2}{4\alpha^2}\right) + \frac{n'\, 3^{1-m'}(3^{m'}-1)}{2}\right]\right\}^{\frac{1}{2}}$$

$$= 1 + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1}\left\{Q^{-1}\left[\frac{T\, 3^{1-S}(3^S-1)-2\alpha}{\alpha} + \frac{n'\, 3^{1-m'}(3^{m'}-1)}{2}\right]\right\}^{\frac{1}{2}}$$

$$= 1 + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1}\left\{\left(\frac{T\, 3^{1-S}(3^S-1)-2\alpha}{2\alpha} + \frac{n'\, 3^{1-m'}(3^{m'}-1)}{4} + 1\right)^2\right\}^{\frac{1}{2}}$$

$$= 1 + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1}\left(\frac{T\, 3^{1-S}(3^S-1)}{2\alpha} + \frac{n'\, 3^{1-m'}(3^{m'}-1)}{4}\right)$$

$$= 1 + \frac{1}{16}t\left(2s(8c+3t-3) - 3^{2-s}(3^s-1)(t-1)\right), \tag{3.4}$$

where $c = \dfrac{T\, 3^{1-S}(3^S-1)}{2\alpha} < \infty$ for fixed $S, T \in N_0$.
This shows that solution of equation (3.1) is bounded for each $(s,t) \in \mathcal{N}$.

EXAMPLE 3.2. Consider the following Volterra-Fredholm sum-difference equation

$$\Delta_{st}^2 x(s,t) = \sum_{m=0}^{s-1}\sum_{n=0}^{t-1}\frac{1}{2}\,\Delta_{mn}^2 x(m,n) + \left[\sum_{m=0}^{s-1}\sum_{n=0}^{t-1}\left(1+\frac{m}{2}\right)^{-n}\frac{\Delta_{mn}^2 x(m,n)}{10000} + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1}\left(1+\frac{m}{2}\right)^{-n}(\Delta_{mn}^2 x(m,n))\right]^{\frac{1}{3}} \tag{3.5}$$

for $(s,t) \in \mathcal{N}$, with initial condition $x(0,t) = x(s,0) = \dfrac{1}{2}$.

If we let $a^*(s,t) = (1+\frac{s}{2})^{-t}$ and $\mathscr{W}(u) = u$, then equation (3.5) can be rewritten as

$$\Delta_{st}^2 x(s,t) \leqslant \sum_{m=0}^{s-1}\sum_{n=0}^{t-1}\frac{1}{2}\,\Delta_{mn}^2 x(m,n) + \left[\sum_{m=0}^{s-1}\sum_{n=0}^{t-1}\left(1+\frac{m}{2}\right)^{-n}\Delta_{mn}^2 x(m,n) + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1}\left(1+\frac{m}{2}\right)^{-n}(\Delta_{mn}^2 x(m,n))\right]^{\frac{1}{3}} \tag{3.6}$$

for $(s,t) \in \mathcal{N}$.

Consider $h(u) = \sqrt[3]{u}$, then applying Theorem 2.2 to the inequality (3.6), we get

$$x(s,t) \leqslant \frac{1}{3} + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1} J(m',n')\left\{\left(Q^{-1}\left[Q\left(P^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1}\left(1+\frac{m}{2}\right)^{-n} J(m,n)\right]\right)\right.\right.\right.$$

$$+\sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1}\left(1+\frac{m}{2}\right)^{-n}J(m,n)\Big]\Big)\Big\}^{\frac{1}{3}}, \quad (3.7)$$

where

$$J(s,t)=\left(1+\frac{s}{2}\right)^{t}$$

for $(s,t)\in\mathscr{N}$. Where

$$\left.\begin{aligned} Q(r)&=\int_1^r\frac{ds}{\mathscr{W}(h(s))}=\int_1^r\frac{ds}{\sqrt[3]{s}}=\frac{3(r^{\frac{2}{3}}-1)}{2}\\ P(r)&=Q(2r)-Q(r)=(0.8811)r^{\frac{2}{3}}\\ Q^{-1}(r)&=\left(\frac{2r}{3}+1\right)^{\frac{3}{2}}\\ P^{-1}(r)&=[(1.135)r]^{\frac{3}{2}}\end{aligned}\right\} r>1.$$

Using these in (3.7) we obtain explicit bound on $x(s,t)$ as

$$x(s,t)\leqslant\frac{1}{2}+\sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1}\left(1+\frac{m'}{2}\right)^{n'}\sqrt{1.135\,ST+0.7m'n'}. \quad (3.8)$$

For instance, if we take $S=4$ and $T=5$ then $\mathscr{N}=[0,4)\times[0,5)$ and bounds on $x(s,t)$ for $(s,t)\in\mathscr{N}$ can be tabulated as follows:

s	**t**	**x(s,t)**	**s**	**t**	**x(s,t)**	**s**	**t**	**x(s,t)**	**s**	**t**	**x(s,t)**
0	0	**0.5**	1	0	**0.5**	2	0	**0.5**	3	0	**0.5**
0	1	**0.5**	1	1	**5.26445**	2	1	**10.0289**	3	1	**14.7934**
0	2	**0.5**	1	2	**10.0289**	2	2	**22.0494**	3	2	**36.6322**
0	3	**0.5**	1	3	**14.7934**	2	3	**37.8595**	3	3	**72.6413**
0	4	**0.5**	1	4	**19.5578**	2	4	**59.4313**	3	4	**135.705**

Table 3.1

From Table 3.1 we can easily see that $x(s,t)$ has concrete boundary for each $(s,t)\in\mathscr{N}$. In general, the inequality (3.8) gives us the bound on the solution of (3.5).

EXAMPLE 3.3. Consider the following nonlinear Volterra-Fredholm sum-difference equation

$$\Delta_{st}^2x(s,t)=\sum_{m=0}^{s-1}\sum_{n=0}^{t-1}\frac{m+n}{2}\sqrt{\Delta_{mn}^2x(m,n)}+\sum_{m=0}^{S-1}\sum_{n=0}^{T-1}\frac{m+n}{3}\sqrt[3]{\Delta_{mn}^2x(m,n)} \quad (3.9)$$

for $(s,t) \in \mathcal{N}$, with initial condition $x(0,t) = x(s,0) = 1$.

If we let $a^*(s,t) = s+t$ and $\mathscr{W}(u) = \sqrt{u}$, then equation (3.9) can be rewritten as

$$\Delta^2_{st}x(s,t) \leqslant \sum_{m=0}^{s-1}\sum_{n=0}^{t-1}(m+n)\sqrt{\Delta^2_{mn}x(m,n)} + \sum_{m=0}^{S-1}\sum_{n=0}^{T-1}(m+n)\sqrt{\Delta^2_{mn}x(m,n)} \tag{3.10}$$

for $(s,t) \in \mathcal{N}$.

Consider $h(u) = u$, then applying Theorem 2.1 to the inequality (3.10) we get

$$x(s,t) \leqslant 1 + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1}\left\{Q^{-1}\left[Q\left(P^{-1}\left[\sum_{m=0}^{S-1}\sum_{n=0}^{T-1}(m+n)\right]\right) + \sum_{m=0}^{m'-1}\sum_{n=0}^{n'-1}(m+n)\right]\right\}, \tag{3.11}$$

where Q, Q^{-1}, P, P^{-1} are as defined in Example 3.1. Using these in (3.11), we obtain explicit bound on $x(s,t)$ as

$$x(s,t) \leqslant 1 + \sum_{m'=0}^{s-1}\sum_{n'=0}^{t-1}\left(\frac{m'n'(m'+n'-2)}{4} + \frac{ST(S+T-2)}{1.65684}\right)^2. \tag{3.12}$$

For instance, if we take $S=5$ and $T=6$ then $\mathcal{N} = [0,5) \times [0,6)$ and bounds on $x(s,t)$ for $(s,t) \in \mathcal{N}$ can be tabulated as follows:

s	t	x(s,t)	s	t	x(s,t)	s	t	x(s,t)	s	t	x(s,t)	s	t	x(s,t)
0	0	**1**	1	0	**1**	2	0	**1**	3	0	**1**	4	0	**1**
0	1	**1**	1	1	**26557**	2	1	**53112.9**	3	1	**79668.9**	4	1	**106225**
0	2	**1**	1	2	**53112.9**	2	2	**106225**	3	2	**159500**	4	2	**213103**
0	3	**1**	1	3	**79668.9**	2	3	**159500**	3	3	**239987**	4	3	**321633**
0	4	**1**	1	4	**106225**	2	4	**213103**	3	4	**321633**	4	4	**432849**
0	5	**1**	1	5	**132781**	2	5	**267202**	3	5	**404959**	4	5	**547845**

Table 3.2

Table 3.2 shows that on any such domain the solution of (3.9) has a concrete boundary.

EXAMPLE 3.4. Consider the following nonlinear Volterra-Fredholm sum-difference equation

$$\Delta^2_{st}x(s,t) = \left(\sum_{m=0}^{s-1}\sum_{n=0}^{t-1}N_1\sqrt[6]{\Delta^2_{mn}x(m,n)}\right)^2 + \left(\sum_{m=0}^{S-1}\sum_{n=0}^{T-1}N_2\sqrt[6]{\Delta^2_{mn}x(m,n)}\right) \tag{3.13}$$

for $(s,t) \in \mathcal{N}$, with initial condition $x(0,t) = x(s,0) = \frac{1}{4}$.

On comparing (3.13) with (2.37), we get $h(x) = x^2$, $g(x) = x$, $a(m,n) = N_1$, $b(m,n) =$

N_2 and $\omega(x) = \sqrt[6]{x}$. It is clear that $g(x) \leqslant h(x)$ and h is superadditive for $x \in \mathbb{R}_+$. If we choose N such that N_1 and N_2 both are less than or equal to N then (3.13) can be rewritten as

$$\Delta_{st}^2 x(s,t) \leqslant \left(\sum_{m=0}^{s-1} \sum_{n=0}^{t-1} N \sqrt[6]{\Delta_{mn}^2 x(m,n)} + \sum_{m=0}^{S-1} \sum_{n=0}^{T-1} N \sqrt[6]{\Delta_{mn}^2 x(m,n)} \right)^2 \tag{3.14}$$

for $(s,t) \in \mathscr{N}$.

Now applying Theorem 2.4 to the inequality (3.14), we get

$$x(s,t) \leqslant \frac{1}{4} + \sum_{m'=0}^{s-1} \sum_{n'=0}^{t-1} \left\{ Q^{-1} \left[Q \left(P^{-1} \left[\sum_{m=0}^{S-1} \sum_{n=0}^{T-1} N \right] \right) + \sum_{m=0}^{m'-1} \sum_{n=0}^{n'-1} N \right] \right\}, \tag{3.15}$$

where Q, Q^{-1}, P, P^{-1} are as defined in Example 3.2. Using these in (3.15), we obtain explicit bound on $x(s,t)$ as

$$x(s,t) \leqslant \frac{1}{4} + \sum_{m'=0}^{s-1} \sum_{n'=0}^{t-1} N^3 (1.135ST + 0.7m'n')^3. \tag{3.16}$$

From inequality (3.16), we can easily conclude that the solution of equation (3.13) is bounded for each $(s,t) \in \mathscr{N}$.

4. Conclusions

In this paper, some more general extensions of existing Volterra-Fredholm type discrete inequalities have been investigated. These inequalities are designed in order to solve some crucial finite difference equations where direct application of previous inequalities is not possible. However in literature we come across more general Volterra-Fredholm difference equations which involve critical nonlinear functions. Hence, the above inequalities can be extended and generalised to study various types of nonlinear difference equations.

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