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STABILITY OF SOME FUNCTIONAL EQUATIONS ON BOUNDED DOMAINS

B. NOORI, M. B. MOGHIMI, B. KHOSRAVI AND CHOONKIL PARK*

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Abstract. In this paper, we investigate the Hyers-Ulam stability of the functional equations

$$f(x+y)+f(x-y)=2f(x),$$
$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y),$$
$$f\big(px+(1-p)y\big)+f\big((1-p)x+py\big)=f(x)+f(y)$$

for $p=\frac{1}{3}$ and $p=\frac{1}{4}$, where f is a mapping from a bounded subset of $\mathbb{R}^{N\geqslant 1}$ into a Banach space E.

1. Introduction

It is well-known that the Hyers-Ulam stability problems of functional equations originated from a question of Ulam [12] in 1940, concerning the stability of group homomorphisms. In other words, the concept of stability for functional equations arises when we replace the functional equation by an inequality which outs as a perturbation of the equation. Hyers [1] gave a first affirmative partial answer to the question of Ulam for Banach spaces. It is interesting to consider a functional equation satisfying on a bounded domain or satisfying under a restricted condition. Skof [9] was the first author to solve Ulam problem for additive mapping on a bounded domain. Indeed, Skof proved that if a function f from $[0,c)$ into a Banach space E satisfies the functional inequality $\|f(x+y)-f(x)-f(y)\|\leqslant\delta$ for all $x,y\in[0,c)$ with $x+y\in[0,c)$, then there exists an additive function $A:\mathbb{R}\to E$ such that $\|f(x)-A(x)\|\leqslant 3\delta$ for all $x\in[0,c)$. Z. Kominek [5] extended this result on a bounded domain $[0,c)^N$ of $\mathbb{R}^N$ for any positive integer N. He also proved a more generalized theorem concerning the stability of the additive Cauchy equation and Jensen equation on a bounded domain of $\mathbb{R}^N$. Skof [331] also proved the Hyers–Ulam stability of the additive Cauchy equation on an unbounded and restricted domain. She applied this result and obtained an interesting asymptotic behavior of additive functions: *The function* $f:\mathbb{R}\to\mathbb{R}$ *is additive if and only if* $f(x+y)-f(x)-f(y)\to 0$ *as* $|x|+|y|\to+\infty$. F. Skof and S. Terracini [11] investigated the problem of stability of the quadratic functional equations for functions defined on bounded real domains with values in a Banach space. For more general information on this subject, we refer the reader to [3, 6, 8].

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* Corresponding author.

2. Stability of $f(x+y)+f(x-y)=2f(x)$ on bounded subsets of $\mathbb{R}$

In this section $r>0$ and $\delta \geqslant 0$ are real numbers and we assume that E is a Banach space.

THEOREM 1. *Let $f:[0,r)\to E$ be a function with $f(0)=0$ and satisfy*

$$\|f(x+y)+f(x-y)-2f(x)\| \leqslant \delta, \tag{1}$$

for some $\delta>0$ and all $(x,y)\in T(r)$, where

$$T(r)=\{(x,y)\in[0,r)\times[0,r):\ 0\leqslant x\pm y<r\}.$$

Then there exists an additive function $A:\mathbb{R}\to E$ such that

$$\|f(x)-A(x)\| \leqslant 11\delta, \quad x\in[0,r). \tag{2}$$

Proof. Let $u,v\in[0,r)$. We can choose $x,y\in[0,r)$ such that $x\pm y\in[0,r)$, $x+y=u$ and $x-y=v$. Then it follows from (1) that

$$\left\|f(u)+f(v)-2f\left(\frac{u+v}{2}\right)\right\| \leqslant \delta. \tag{3}$$

Letting $v=0$ in (3), we get

$$\left\|f(u)-2f\left(\frac{u}{2}\right)\right\| \leqslant \delta, \quad u\in[0,r). \tag{4}$$

We extend the function f to $[0,+\infty)$. For this we represent an arbitrary $x\geqslant 0$ by $x=n(r/2)+\alpha$, where n is an integer and $0\leqslant\alpha<r/2$. Then we define a function $\varphi:[0,+\infty)\to E$ by $\varphi(x)=nf(r/2)+f(\alpha)$. It is clear that $\varphi(x)=f(x)$ for all $x\in[0,r/2)$. If $x\in[r/2,r)$, then $\varphi(x)=f(r/2)+f(x-r/2)$, and we get from (3) and (4) that

$$\begin{aligned}\|\varphi(x)-f(x)\| &= \left\|f\left(\frac{r}{2}\right)+f\left(x-\frac{r}{2}\right)-f(x)\right\| \\ &\leqslant \left\|f\left(\frac{r}{2}\right)+f\left(x-\frac{r}{2}\right)-2f\left(\frac{x}{2}\right)\right\|+\left\|2f\left(\frac{x}{2}\right)-f(x)\right\| \\ &\leqslant 2\delta.\end{aligned}$$

So

$$\|\varphi(x)-f(x)\| \leqslant 2\delta, \quad x\in[0,r). \tag{5}$$

We now show that φ satisfies

$$\left\|\varphi(x)+\varphi(y)-2\varphi\left(\frac{x+y}{2}\right)\right\| \leqslant 3\delta, \quad x,y\in[0,+\infty). \tag{6}$$

For given $x,y \geqslant 0$, let $x = n(r/2)+\alpha$ and $y = m(r/2)+\beta$, where m and n are integers and $0 \leqslant \alpha,\beta < r/2$. Then

$$\frac{x+y}{2} = \frac{m+n}{2}\Big(\frac{r}{2}\Big) + \frac{\alpha+\beta}{2}, \quad m+n \text{ is even;}$$
$$\frac{x+y}{2} = \frac{m+n+1}{2}\Big(\frac{r}{2}\Big) + \frac{\alpha+\beta}{2} - \frac{r}{4}, \quad m+n \text{ is odd and } \alpha+\beta \geqslant \frac{r}{2};$$
$$\frac{x+y}{2} = \frac{m+n-1}{2}\Big(\frac{r}{2}\Big) + \frac{\alpha+\beta}{2} + \frac{r}{4}, \quad m+n \text{ is odd and } \alpha+\beta < \frac{r}{2}.$$

Therefore we have

$$\varphi\Big(\frac{x+y}{2}\Big) = \frac{m+n}{2} f\Big(\frac{r}{2}\Big) + f\Big(\frac{\alpha+\beta}{2}\Big), \quad m+n \text{ is even;}$$
$$\varphi\Big(\frac{x+y}{2}\Big) = \frac{m+n+1}{2} f\Big(\frac{r}{2}\Big) + f\Big(\frac{\alpha+\beta}{2} - \frac{r}{4}\Big), \quad m+n \text{ is odd and } \alpha+\beta \geqslant \frac{r}{2};$$
$$\varphi\Big(\frac{x+y}{2}\Big) = \frac{m+n-1}{2} f\Big(\frac{r}{2}\Big) + f\Big(\frac{\alpha+\beta}{2} + \frac{r}{4}\Big), \quad m+n \text{ is odd and } \alpha+\beta < \frac{r}{2}.$$

To prove (6) we have the following cases.

(i) If $m+n$ is even, then

$$\Big\|\varphi(x)+\varphi(y)-2\varphi\Big(\frac{x+y}{2}\Big)\Big\| = \Big\|f(\alpha)+f(\beta)-2f\Big(\frac{\alpha+\beta}{2}\Big)\Big\| \leqslant \delta.$$

(ii) If $m+n$ is odd and $\alpha+\beta \geqslant \frac{r}{2}$, then

$$\begin{aligned}\Big\|\varphi(x)+\varphi(y)-2\varphi\Big(\frac{x+y}{2}\Big)\Big\| &= \Big\|f(\alpha)+f(\beta)-f\Big(\frac{r}{2}\Big)-2f\Big(\frac{\alpha+\beta}{2}-\frac{r}{4}\Big)\Big\| \\ &\leqslant \Big\|f(\alpha)+f(\beta)-2f\Big(\frac{\alpha+\beta}{2}\Big)\Big\| \\ &\quad + \Big\|f\Big(\alpha+\beta-\frac{r}{2}\Big)-2f\Big(\frac{\alpha+\beta}{2}-\frac{r}{4}\Big)\Big\| \\ &\quad + \Big\|2f\Big(\frac{\alpha+\beta}{2}\Big)-f\Big(\frac{r}{2}\Big)-f\Big(\alpha+\beta-\frac{r}{2}\Big)\Big\| \\ &\leqslant 3\delta.\end{aligned}$$

(iii) If $m+n$ is odd and $\alpha+\beta < \frac{r}{2}$, then

$$\begin{aligned}\Big\|\varphi(x)+\varphi(y)-2\varphi\Big(\frac{x+y}{2}\Big)\Big\| &= \Big\|f(\alpha)+f(\beta)+f\Big(\frac{r}{2}\Big)-2f\Big(\frac{\alpha+\beta}{2}+\frac{r}{4}\Big)\Big\| \\ &\leqslant \Big\|f(\alpha)+f(\beta)-2f\Big(\frac{\alpha+\beta}{2}\Big)\Big\| \\ &\quad + \Big\|2f\Big(\frac{\alpha+\beta}{2}\Big)-f(\alpha+\beta)\Big\| \\ &\quad + \Big\|f(\alpha+\beta)+f\Big(\frac{r}{2}\Big)-2f\Big(\frac{\alpha+\beta}{2}+\frac{r}{4}\Big)\Big\| \\ &\leqslant 3\delta.\end{aligned}$$

Hence φ satisfies (6). Now, we define a function $g:\mathbb{R}\to E$ by

$$g(x)=\begin{cases}\varphi(x), & x\geqslant 0;\\ -\varphi(-x), & x<0.\end{cases}$$

We show that g satisfies

$$\left\|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right\|\leqslant 9\delta,\quad x,y\in\mathbb{R}. \tag{7}$$

For given $x,y\in\mathbb{R}$, since the left-hand side of (7) is symmetric in x and y, we may assume the following cases.

(i) If $x,y\geqslant 0$ or $x,y<0$, we get (7) from (6).

(ii) If $x\geqslant 0, y<0$ and $x+y\geqslant 0$, then (6) yields

$$\begin{aligned}\left\|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right\| &= \left\|\varphi(x)-\varphi(-y)-2\varphi\left(\frac{x+y}{2}\right)\right\|\\ &\leqslant \left\|\varphi(x)-2\varphi\left(\frac{x}{2}\right)\right\|+\left\|\varphi(x+y)-2\varphi\left(\frac{x+y}{2}\right)\right\|\\ &\quad+\left\|2\varphi\left(\frac{x}{2}\right)-\varphi(-y)-\varphi(x+y)\right\|\\ &\leqslant 9\delta.\end{aligned}$$

(iii) If $x\geqslant 0, y<0$ and $x+y<0$, then (6) yields

$$\begin{aligned}\left\|g(x)+g(y)-2g\left(\frac{x+y}{2}\right)\right\| &= \left\|\varphi(x)-\varphi(-y)+2\varphi\left(-\frac{x+y}{2}\right)\right\|\\ &\leqslant \left\|2\varphi\left(-\frac{y}{2}\right)-\varphi(-y)\right\|\\ &\quad+\left\|2\varphi\left(-\frac{x+y}{2}\right)-\varphi(-x-y)\right\|\\ &\quad+\left\|\varphi(-x-y)+\varphi(x)-2\varphi\left(-\frac{y}{2}\right)\right\|\\ &\leqslant 9\delta.\end{aligned}$$

Therefore g satisfies (7) and then according to [2], there exist an additive function $A:\mathbb{R}\to E$ such that $\|g(x)-A(x)\|\leqslant 9\delta$ for all $x\in\mathbb{R}$. Since $\varphi(x)=g(x)$ for all $x\geqslant 0$, it follows from (5) that

$$\|f(x)-A(x)\|\leqslant\|f(x)-g(x)\|+\|g(x)-A(x)\|\leqslant 11\delta,\quad x\in[0,r).$$

COROLLARY 1. *Let $f:[0,r)\to E$ be a function with $f(0)=0$ and satisfy*

$$\left\|f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right\|\leqslant\delta,$$

for some $\delta>0$ and all $(x,y)\in T(r)$. Then there exists an additive function $A:\mathbb{R}\to E$ such that

$$\|f(x)-A(x)\|\leqslant 11\delta,\quad x\in[0,r).$$

COROLLARY 2. *Let $f:(-r,r)\to E$ be a function with $f(0)=0$ and satisfy*

$$\|f(x+y)+f(x-y)-2f(x)\| \leqslant \delta, \tag{8}$$

for some $\delta>0$ and all $(x,y)\in T(r)$. Then there exists an additive function $A:\mathbb{R}\to E$ such that

$$\|f(x)-A(x)\| \leqslant 12\delta, \quad x\in(-r,r).$$

Proof. Letting $x=0$ in (8), we get $\|f(y)+f(-y)\|\leqslant\delta$ for all $y\in(-r,r)$. By Theorem 1, there exists an additive function $A:\mathbb{R}\to E$ such that $\|f(x)-A(x)\|\leqslant 11\delta$ for all $x\in[0,r)$. If $x\in(-r,0)$, then

$$\|f(x)-A(x)\| \leqslant \|f(x)+f(-x)\|+\|A(-x)-f(-x)\| \leqslant 12\delta.$$

This completes the proof.

THEOREM 2. *Let $f:(-r\sqrt{2},r\sqrt{2})\to E$ be a function with $f(0)=0$ and satisfy*

$$\|f(x+y)+f(x-y)-2f(x)\| \leqslant \delta, \tag{9}$$

for some $\delta>0$ and all $(x,y)\in\mathbb{R}^2$, where $x^2+y^2\leqslant r^2$. Then there exists an additive function $A:\mathbb{R}\to E$ such that

$$\|f(x)-A(x)\| \leqslant 19\delta, \quad x\in(-r\sqrt{2},r\sqrt{2}). \tag{10}$$

Proof. It is clear that if $|x\pm y|\leqslant r$, then $x^2+y^2\leqslant r^2$. Therefore f satisfies (1) for all $(x,y)\in T(r)$. By Theorem 1, there exist an additive function $A:\mathbb{R}\to E$ satisfying (2) for all $x\in[0,r)$. Let φ and g be given as in the proof of Theorem 1. Then

$$\varphi(x)=g(x), \quad \|\varphi(x)-f(x)\|\leqslant 2\delta, \quad x\in[0,r). \tag{11}$$

If $r\leqslant x<r\sqrt{2}$, then $(x/2)^2+(x/2)^2<r^2$, and we infer from (9) that

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| \leqslant \delta, \quad x\in[r,r\sqrt{2}).$$

Since $\varphi(x)=g(x)$ for all $x\geqslant 0$, we get from (6) that

$$\left\|g(x)-2g\left(\frac{x}{2}\right)\right\| \leqslant 3\delta, \quad x\in[0,+\infty).$$

Therefore from the above inequalities, we have

$$\begin{aligned}\|f(x)-g(x)\| &\leqslant \left\|f(x)-2f\left(\frac{x}{2}\right)\right\|+\left\|2g\left(\frac{x}{2}\right)-g(x)\right\|+2\left\|f\left(\frac{x}{2}\right)-g\left(\frac{x}{2}\right)\right\| \\ &\leqslant 8\delta, \quad x\in[r,r\sqrt{2}).\end{aligned}$$

For the case $-r\sqrt{2} < x < 0$, from the definition of g, (9) and (11), we have

$$\begin{aligned}\|f(x)-g(x)\| &= \|f(x)+\varphi(-x)\| \\ &\leqslant \Big\|f(x)-2f\Big(\frac{x}{2}\Big)\Big\| + 2\Big\|f\Big(\frac{x}{2}\Big)+f\Big(-\frac{x}{2}\Big)\Big\| \\ &\quad + 2\Big\|\varphi\Big(-\frac{x}{2}\Big)-f\Big(-\frac{x}{2}\Big)\Big\| + \Big\|\varphi(-x)-2\varphi\Big(-\frac{x}{2}\Big)\Big\| \\ &\leqslant 10\delta.\end{aligned}$$

Hence we get

$$\|f(x)-g(x)\| \leqslant 10\delta, \quad x \in (-r\sqrt{2}, r\sqrt{2}).$$

Since $\|g(x)-A(x)\| \leqslant 9\delta$ for all $x \in \mathbb{R}$ (see the proof of Theorem 1), it follows from the last inequality that

$$\|f(x)-A(x)\| \leqslant \|f(x)-g(x)\| + \|g(x)-A(x)\| \leqslant 19\delta, \quad x \in (-r\sqrt{2}, r\sqrt{2}),$$

which ends the proof.

THEOREM 3. *Let* $f:(-r,r)\to E$ *be a function with* $f(0)=0$ *and satisfy*

$$\|f(x+y)+f(x-y)-2f(x)\| \leqslant \delta, \tag{12}$$

for some $\delta > 0$ *and all* $(x,y) \in D(r)$, *where*

$$D(r) = \{(x,y) \in (-r,r)\times(-r,r) : \ |x\pm y| < r\}.$$

Then there exists an additive function $A:\mathbb{R}\to E$ *such that*

$$\|f(x)-A(x)\| \leqslant 5\delta, \quad x \in (-r,r). \tag{13}$$

Proof. Letting $y=x$ and $x=0$ in (12), respectively, we get

$$\|f(2x)-2f(x)\| \leqslant \delta, \quad \|f(y)+f(-y)\| \leqslant \delta, \quad |2x|,|y| < r. \tag{14}$$

For an arbitrary $x \in \mathbb{R}$, we set $x = n(r/2)+\mu$, where n is an integer and $0 \leqslant \mu < r/2$. Hence we can define a function $g:\mathbb{R}\to E$ by $g(x) = nf(r/2)+f(\mu)$. We show that $\|g(x)-f(x)\| \leqslant 2\delta$ for all $x \in (-r,r)$. For this we have the following cases:

1. For $0 \leqslant x < r/2$, we have $g(x) = f(x)$.

2. For $r/2 \leqslant x < r$, we have $x = r/2+\mu$. Then it follows from (12) and (14) that

$$\begin{aligned}\|g(x)-f(x)\| &= \Big\|f\Big(\frac{r}{2}\Big)+f(\mu)-f(x)\Big\| \\ &\leqslant \Big\|f\Big(\frac{r}{2}\Big)+f(\mu)-2f\Big(\frac{x}{2}\Big)\Big\| + \Big\|2f\Big(\frac{x}{2}\Big)-f(x)\Big\| \\ &\leqslant \delta+\delta = 2\delta.\end{aligned}$$

3. For $-(r/2) \leqslant x < 0$, we have $x = -(r/2) + \mu$. Then

$$\begin{aligned}\|g(x) - f(x)\| &= \left\| -f\left(\frac{r}{2}\right) + f(\mu) - f(x)\right\| \\ &\leqslant \left\|f(x) + f\left(\frac{r}{2}\right) - 2f\left(\frac{\mu}{2}\right)\right\| + \left\|2f\left(\frac{\mu}{2}\right) - f(\mu)\right\| \\ &\leqslant \delta + \delta = 2\delta.\end{aligned}$$

4. For $-r < x < -(r/2)$, we have $x = -2(r/2) + \mu$. Then

$$\begin{aligned}\|g(x) - f(x)\| &= \left\| -2f\left(\frac{r}{2}\right) + f(\mu) - f(x)\right\| \\ &\leqslant \left\|f(\mu) + f(-x) - 2f\left(\frac{r}{2}\right)\right\| + \|f(-x) + f(x)\| \\ &\leqslant \delta + \delta = 2\delta.\end{aligned}$$

We now show that g satisfies

$$\|g(x+y) + g(x-y) - 2g(x)\| \leqslant 3\delta, \quad x, y \in \mathbb{R}. \tag{15}$$

For given $x, y \in \mathbb{R}$, let $x = n(r/2) + \alpha$ and $y = m(r/2) + \beta$, where n and m are integers and $\alpha, \beta \in [0, r/2)$. Therefore

$$\begin{aligned} x + y &= (n+m)\frac{r}{2} + (\alpha + \beta), \quad 0 \leqslant \alpha + \beta < r, \\ x - y &= (n-m)\frac{r}{2} + (\alpha - \beta), \quad \frac{-r}{2} \leqslant \alpha - \beta < \frac{r}{2}. \end{aligned}$$

We consider following cases:

1. If $0 \leqslant \alpha \pm \beta < r/2$, then

$$\|g(x+y) + g(x-y) - 2g(x)\| = \|f(\alpha+\beta) + f(\alpha-\beta) - 2f(\alpha)\| \leqslant \delta.$$

2. If $0 \leqslant \alpha + \beta < r/2$ and $-r/2 \leqslant \alpha - \beta < 0$, then

$$\begin{aligned}\|g(x+y) + g(x-y) - 2g(x)\| &= \left\|f(\alpha+\beta) + f\left(\alpha - \beta + \frac{r}{2}\right) - f\left(\frac{r}{2}\right) - 2f(\alpha)\right\| \\ &\leqslant \|f(\alpha+\beta) + f(\alpha-\beta) - 2f(\alpha)\| \\ &\quad + \left\|f(\alpha-\beta) + f\left(\frac{r}{2}\right) - f\left(\alpha - \beta + \frac{r}{2}\right)\right\| \\ &= \|f(\alpha+\beta) + f(\alpha-\beta) - 2f(\alpha)\| \\ &\quad + \|f(\alpha-\beta) - g(\alpha-\beta)\| \\ &\leqslant \delta + 2\delta = 3\delta.\end{aligned}$$

3. If $r/2 \leqslant \alpha+\beta < r$ and $0 \leqslant \alpha-\beta < r/2$, then

$$\begin{aligned}\|g(x+y)+g(x-y)-2g(x)\| &= \left\|f\left(\frac{r}{2}\right)+f\left(\alpha+\beta-\frac{r}{2}\right)+f(\alpha-\beta)-2f(\alpha)\right\| \\ &\leqslant \|f(\alpha+\beta)+f(\alpha-\beta)-2f(\alpha)\| \\ &\quad + \left\|f\left(\frac{r}{2}\right)+f\left(\alpha+\beta-\frac{r}{2}\right)-f(\alpha+\beta)\right\| \\ &= \|f(\alpha+\beta)+f(\alpha-\beta)-2f(\alpha)\| \\ &\quad + \|g(\alpha+\beta)-f(\alpha+\beta)\| \\ &\leqslant \delta+2\delta = 3\delta.\end{aligned}$$

4. If $r/2 \leqslant \alpha+\beta < r$ and $-r/2 \leqslant \alpha-\beta < 0$, then

$$\|g(x+y)+g(x-y)-2g(x)\| = \left\|f\left(\alpha+\beta-\frac{r}{2}\right)+f\left(\alpha-\beta+\frac{r}{2}\right)-2f(\alpha)\right\| \leqslant \delta.$$

Therefore g satisfies (15). It is easy to show that

$$\left\|\frac{g(2^n x)}{2^n}-\frac{g(2^m x)}{2^m}\right\| \leqslant \sum_{i=m+1}^{n}\frac{3\delta}{2^i}, \quad n>m,\ x\in\mathbb{R}. \tag{16}$$

Hence $\{2^{-n}g(2^n x)\}$ is a Cauchy sequence for every $x\in\mathbb{R}$. Since E is a Banach space, we can define a function $A:\mathbb{R}\to E$ by

$$A(x)=\lim_{n\to\infty}\frac{g(2^n x)}{2^n}.$$

Letting $m=0$ and taking the limit as $n\to\infty$ in (16), we obtain

$$\|A(x)-g(x)\| \leqslant 3\delta, \quad x\in\mathbb{R}.$$

Since $\|g(x)-f(x)\| \leqslant 2\delta$ on $(-r,r)$, we get

$$\|f(x)-A(x)\| = \|f(x)-g(x)\|+\|g(x)-A(x)\| \leqslant 5\delta, \quad x\in(-r,r).$$

It follows from (15) that

$$\|g(2^n x+2^n y)+g(2^n x-2^n y)-2g(2^n x)\| \leqslant 3\delta, \quad x,y\in\mathbb{R},\ n\geqslant 1.$$

Dividing by 2^n and letting $n\to\infty$ in this inequality, we infer that A is an additive function.

3. Stability of Drygas functional equation on bounded subsets of $\mathbb{R}$

We now prove the stability of Drygas functional equation on a restricted domain. First, we introduce a theorem of Skof and Terracini [11].

THEOREM 4. [11] *Let E be a Banach space and let a function $f:(-r,r)\to E$ satisfy the inequality*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\|\leqslant\delta, \tag{17}$$

for some $\delta>0$ and all $x,y\in\mathbb{R}$ with $|x\pm y|<r$. Then there exists a quadratic function $Q:\mathbb{R}\to E$ such that

$$\|f(x)-Q(x)\|\leqslant\frac{81}{2}\delta,\quad x\in(-r,r).$$

Using ideas from [5], we can state the following proposition which is a generalization of Theorem 4.

PROPOSITION 1. *Let E be a Banach space and let D be a bounded subset of $\mathbb{R}$. Assume, moreover, that there exist a non-negative integer n and a positive number $c>0$ such that*

(i) $D\subseteq 2D$,

(ii) $(-c,c)\subseteq D$,

(iii) $D\subseteq(-2^nc,2^nc)$.

If a function $f:D\to E$ satisfies the functional inequality (17) for some $\delta\geqslant 0$ and for all $x,y\in D$ with $x\pm y\in D$, then there exists a quadratic function $Q:\mathbb{R}\to E$ such that

$$\|f(x)-Q(x)\|\leqslant\frac{82.4^n-1}{2}\delta,\quad x\in D.$$

Proof. By Theorem 4, there exists a quadratic function $Q:\mathbb{R}\to E$ such that

$$\|f(x)-Q(x)\|\leqslant\frac{81}{2}\delta,\quad x\in(-c,c).$$

For $x\in D$, the conditions (i) and (iii) imply that $2^{-k}x\in D$ for $k=1,2,\dots,n$ and $2^{-n}x\in(-c,c)$. It follows from (17) that for each $x\in D$

$$\left\|4^{k-1}f\left(\frac{x}{2^{k-1}}\right)-4^kf\left(\frac{x}{2^k}\right)+4^{k-1}f(0)\right\|\leqslant 4^{k-1}\delta,\quad k=1,2,\dots,n.$$

Therefore

$$\left\|f(x)-4^nf\left(\frac{x}{2^n}\right)+\frac{4^n-1}{3}f(0)\right\|\leqslant\frac{4^n-1}{3}\delta.$$

Using the above inequalities and $2\|f(0)\|\leqslant\delta$, we get

$$\begin{aligned}\|f(x)-Q(x)\|&\leqslant\left\|f(x)-4^nf\left(\frac{x}{2^n}\right)+\frac{4^n-1}{3}f(0)\right\|+\left\|4^nf\left(\frac{x}{2^n}\right)-Q(x)\right\|+\frac{4^n-1}{3}\|f(0)\|\\&\leqslant\frac{82.4^n-1}{2}\delta,\quad x\in D.\end{aligned}$$

This completes the proof.

THEOREM 5. *Let $f:(-r,r)\to E$ be a function with $f(0)=0$ and satisfy*

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)\|\leqslant\delta, \tag{18}$$

for some $\delta>0$ and all $(x,y)\in D(r)$, where

$$D(r)=\{(x,y)\in(-r,r)\times(-r,r):\ |x\pm y|<r\}.$$

Then there exist a quadratic function $Q:\mathbb{R}\to E$ and an additive function $A:\mathbb{R}\to E$ such that

$$\|f(x)-A(x)-Q(x)\|\leqslant\frac{91}{2}\delta,\quad x\in(-r,r). \tag{19}$$

Proof. We denote by g and h the even and odd part of f, respectively. i.e.,

$$g,h:(-r,r)\to E,\quad g(x)=\frac{f(x)+f(-x)}{2},\quad h(x)=\frac{f(x)-f(-x)}{2}.$$

It is clear that g and h satisfy in (18) for all $(x,y)\in D(r)$. Since g is even and h is odd, we have

$$\|g(x+y)+g(x-y)-2g(x)-2g(y)\|\leqslant\delta,\quad x,y\in D(r), \tag{20}$$

$$\|h(x+y)+h(x-y)-2h(x)\|\leqslant\delta,\quad x,y\in D(r). \tag{21}$$

By Theorems 3 and 4, there exist an additive function $A:\mathbb{R}\to E$ and a quadratic function $Q:\mathbb{R}\to E$ such that

$$\|g(x)-Q(x)\|\leqslant\frac{81}{2}\delta,\quad \|h(x)-A(x)\|\leqslant 5\delta,\quad x\in(-r,r).$$

Since $f=g+h$, we get (19).

PROPOSITION 2. *Let E be a Banach space and let D be a symmetric bounded subset of $\mathbb{R}$. Assume, moreover, that there exist a non-negative integer n and a positive number $c>0$ such that*

(*i*) $D\subseteq 2D$,

(*ii*) $(-c,c)\subseteq D$,

(*iii*) $D\subseteq(-2^nc,2^nc)$.

If a function $f:D\to E$ satisfies the functional inequality (18) for some $\delta\geqslant 0$ and for all $x,y\in D$ with $x\pm y\in D$, then there exist a quadratic function $Q:\mathbb{R}\to E$ and an additive function $A:\mathbb{R}\to E$ such that

$$\|f(x)-A(x)-Q(x)\|\leqslant\left[6.2^n+41.4^n-\frac{3}{2}\right]\delta,\quad x\in D.$$

Proof. Let g and h be the even and odd part of f, respectively. Since D is symmetric, g satisfies (20) and h satisties (21) for all $x,y \in D$ with $x \pm y \in D$. By Proposition 1, there exists a quadratic function $Q : \mathbb{R} \to E$ such that

$$\|g(x) - Q(x)\| \leqslant \frac{82.4^n - 1}{2}\delta, \quad x \in D. \tag{22}$$

Similarly, as in the proof of Proposition 1, it follows from (21) that for each $x \in D$

$$\left\|2^{k-1}h\left(\frac{x}{2^{k-1}}\right) - 2^k h\left(\frac{x}{2^k}\right)\right\| \leqslant 2^{k-1}\delta, \quad k = 1,2,\ldots,n.$$

Therefore

$$\left\|h(x) - 2^n h\left(\frac{x}{2^n}\right)\right\| \leqslant (2^n - 1)\delta, \quad x \in D.$$

On the other hand, by Theorem 3, there exists an additive function $A : \mathbb{R} \to E$ such that $\|h(x) - A(x)\| \leqslant 5\delta$ for all $x \in (-c,c)$. Using the above inequalities, we get

$$\begin{aligned} \|h(x) - A(x)\| &\leqslant \left\|h(x) - 2^n h\left(\frac{x}{2^n}\right)\right\| + \left\|2^n h\left(\frac{x}{2^n}\right) - A(x)\right\| \\ &\leqslant (6.2^n - 1)\delta, \quad x \in D. \end{aligned} \tag{23}$$

Since $f = g + h$, the result follows from (22) and (23).
Theorem 4 was generalized by Jung and Kim [4]. They proved the following result:

THEOREM 6. *Let E be a Banach space and let $r,\delta > 0$ be given constants. If a function $f : [-r,r]^n \to E$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leqslant \delta$$

for all $x,y \in [-r,r]^n$ with $x \pm y \in [-r,r]^n$, then there exists a quadratic function $Q : \mathbb{R}^n \to E$ such that

$$\|f(x) - Q(x)\| \leqslant (2912n^2 + 1872n + 334)\delta,$$

for any $x \in [-r,r]^n$.

4. Stability of $f\big(px + (1-p)y\big) + f\big((1-p)x + py\big) = f(x) + f(y)$ on bounded subsets of $\mathbb{R}^{N \geqslant 1}$ for $p = \frac{1}{3}$ and $p = \frac{1}{4}$

In this section $r > 0$ and $\delta \geqslant 0$ are real numbers and we assume that E is a normed space. We will now start this section with the following lemma presented by Kominek [5] (see also [3]).

LEMMA 1. *Let E be a Banach space and let N be a positive integer. Suppose D is a bounded subset of $\mathbb{R}^N$ containing zero in its interior. Assume, moreover, that there exist a nonnegative integer n and a positive number $c > 0$ such that*

(i) $D \subseteq 2D$,

(ii) $(-c,c)^N \subseteq D$,

(iii) $D \subseteq (-2^n c, 2^n c)^N$.

If a function $f : D \to E$ *satisfies the functional inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \delta$$

for some $\delta \geqslant 0$ *and for all* $x,y \in D$ *with* $x+y \in D$*, then there exists an additive function* $A : \mathbb{R}^N \to E$ *such that*

$$\|f(x) - A(x)\| \leqslant (2^n.5N-1)\delta, \quad x \in D.$$

THEOREM 7. *Let* $f : (-r,r) \to E$ *be a function with* $f(0) = 0$ *and satisfy*

$$\left\|f\left(\frac{1}{3}x+\frac{2}{3}y\right)+f\left(\frac{2}{3}x+\frac{1}{3}y\right)-f(x)-f(y)\right\| \leqslant \delta, \quad x,y \in (-r,r). \tag{24}$$

Then

$$\|f(x+y) - f(x) - f(y)\| \leqslant 9\delta, \quad x,y \in (-\frac{2r}{9}, \frac{2r}{9}).$$

Proof. Replacing x by $3x$ and y by $3y$ in (24), we have

$$\|f(x+2y)+f(2x+y)-f(3x)-f(3y)\| \leqslant \delta, \quad x,y \in (-\frac{r}{3},\frac{r}{3}). \tag{25}$$

By replacing x by $\dfrac{2y-x}{3}$ and y by $\dfrac{2x-y}{3}$ in (25), we get

$$\|f(x)+f(y)-f(2x-y)-f(2y-x)\| \leqslant \delta, \quad x,y \in (-\frac{r}{3},\frac{r}{3}). \tag{26}$$

Replacing y by $-y$ in (26), we have

$$\|f(2x+y)+f(-2y-x)-f(x)-f(-y)\| \leqslant \delta, \quad x,y \in (-\frac{r}{3},\frac{r}{3}). \tag{27}$$

Replacing $y = 0$ in (25), we infer

$$\|f(x)+f(2x)-f(3x)\| \leqslant \delta, \quad x \in (-\frac{r}{3},\frac{r}{3}), \tag{28}$$

and replacing x by $-x$ in (28), we have

$$\|f(-x)+f(-2x)-f(-3x)\| \leqslant \delta, \quad x \in (-\frac{r}{3},\frac{r}{3}). \tag{29}$$

Letting $y = -x$ in (25), we have

$$\|f(-x)+f(x)-f(3x)-f(-3x)\| \leqslant \delta, \quad x \in (-\frac{r}{3},\frac{r}{3}). \tag{30}$$

Using (28), (29) and (30), we have $\|f(2x)+f(-2x)\| \leqslant 3\delta$, for all $x \in (-\frac{r}{3}, \frac{r}{3})$. Therefore

$$\|f(x)+f(-x)\| \leqslant 3\delta, \quad x \in (-\frac{2r}{3}, \frac{2r}{3}). \tag{31}$$

Putting $y=-2x$ in (25), we get

$$\|f(-3x)-f(3x)-f(-6x)\| \leqslant \delta, \quad x \in (-\frac{r}{6}, \frac{r}{6}). \tag{32}$$

Using the triangle inequality, it follows from (31) and (32) that

$$\|2f(-3x)-f(-6x)\| \leqslant 4\delta, \quad x \in (-\frac{r}{6}, \frac{r}{6}).$$

Then

$$\|2f(x)-f(2x)\| \leqslant 4\delta, \quad x \in (-\frac{r}{2}, \frac{r}{2}). \tag{33}$$

It follows from (31) that $\|f(-2y-x)+f(2y+x)\| \leqslant 3\delta$ for all $x,y \in (-\frac{2r}{9}, \frac{2r}{9})$. Hence (25), (27) and (28) imply

$$\|2f(2x+y)-f(2x)-2f(x)-f(2y)-f(y)-f(-y)\| \leqslant 7\delta, \quad x,y \in (-\frac{2r}{9}, \frac{2r}{9}).$$

Using this inequality and applying (31) and (33), we obtain

$$\|f(2x+y)-f(2x)-f(y)\| \leqslant 9\delta, \quad x,y \in (-\frac{2r}{9}, \frac{2r}{9}). \tag{34}$$

Then we have

$$\|f(x+y)-f(x)-f(y)\| \leqslant 9\delta, \quad x,y \in (-\frac{2r}{9}, \frac{2r}{9}).$$

A similar argument as in the proof of Theorem 7 yields the following results in the case of functions defined on certain subsets of $\mathbb{R}^N$ (N is a positive integer) with values in a normed space.

THEOREM 8. *Suppose that D is a symmetric and bounded subset of $\mathbb{R}^N$ containing zero. Let $f: D \to E$ be a function with $f(0)=0$ and satisfy*

$$\left\|f\left(\frac{1}{3}x+\frac{2}{3}y\right)+f\left(\frac{2}{3}x+\frac{1}{3}y\right)-f(x)-f(y)\right\| \leqslant \delta, \tag{35}$$

for some $\delta \geqslant 0$ and for all $x,y \in D$ with $2x+y \in 3D$. Then

$$\|f(x+y)-f(x)-f(y)\| \leqslant 9\delta, \quad x,y \in (2/9)D.$$

COROLLARY 3. *Let $f: (-r,r)^N \to E$ be a function with $f(0)=0$ and satisfy*

$$\left\|f\left(\frac{1}{3}x+\frac{2}{3}y\right)+f\left(\frac{2}{3}x+\frac{1}{3}y\right)-f(x)-f(y)\right\| \leqslant \delta, \quad x,y \in (-r,r)^N.$$

Then

$$\|f(x+y)-f(x)-f(y)\| \leqslant 9\delta, \quad x,y \in (-\frac{2r}{9}, \frac{2r}{9})^N.$$

Using Lemma 1 and Theorem 8 we prove the stability of the functional equation $f\Big(\frac{1}{3}x+\frac{2}{3}y\Big)+f\Big(\frac{2}{3}x+\frac{1}{3}y\Big)=f(x)+f(y)$ on a restricted domain.

THEOREM 9. *Let E be a Banach space and let $f:(-r,r)^N\to E$ be a function with $f(0)=0$ and satisfy (35) for all $x,y\in(-r,r)^N$. Then there exists an additive function $A:\mathbb{R}^N\to E$ such that*

$$\|f(x)-A(x)\|\leqslant 9(5N-1)\delta,\quad x\in(-\frac{2r}{9},\frac{2r}{9})^N.$$

THEOREM 10. *Let E be a Banach space and let N be a positive integer. Suppose D is a symmetric and bounded subset of $\mathbb{R}^N$ containing zero in its interior. Assume, moreover, that there exist a nonnegative integer n and a positive number $c>0$ such that*

(i) $D\subseteq 2D$,

(ii) $(-c,c)^N\subseteq D$,

(iii) $D\subseteq(-2^nc,2^nc)^N$.

If a function $f:D\to E$ satisfies $f(0)=0$ and the functional inequality

$$\Big\|f\Big(\frac{1}{3}x+\frac{2}{3}y\Big)+f\Big(\frac{2}{3}x+\frac{1}{3}y\Big)-f(x)-f(y)\Big\|\leqslant\delta,$$

for some $\delta\geqslant 0$ and for all $x,y\in D$ with $2x+y\in 3D$, then there exists an additive function $A:\mathbb{R}^N\to E$ such that

$$\|f(x)-A(x)\|\leqslant 9(2^n.5N-1)\delta,\quad x\in(2/9)D.$$

Proof. Let $G=(2/9)D$ and $r=(2/9)c$. Then $G\subseteq 2G$, $(-r,r)^N\subseteq G$ and $D\subseteq(-2^nr,2^nr)^N$. By Theorem 8, f satisfies

$$\|f(x+y)-f(x)-f(y)\|\leqslant 9\delta,\quad x,y\in G.$$

Therefore on account of Lemma 1, we get the result.

THEOREM 11. *Let $f:(-r,r)\to E$ be a function with $f(0)=0$ and satisfy*

$$\Big\|f\Big(\frac{1}{4}x+\frac{3}{4}y\Big)+f\Big(\frac{3}{4}x+\frac{1}{4}y\Big)-f(x)-f(y)\Big\|\leqslant\delta,\quad x,y\in(-r,r).\tag{36}$$

Then

$$\|f(x+y)-f(x)-f(y)\|\leqslant 9\delta,\quad x,y\in(-\frac{3r}{16},\frac{3r}{16}).$$

Proof. Replacing x by $4x$ and y by $4y$ in (36), we have

$$\|f(x+3y)+f(3x+y)-f(4x)-f(4y)\| \leqslant \delta, \quad x,y \in (-\frac{r}{4},\frac{r}{4}). \tag{37}$$

By replacing x by $\frac{3y-x}{4}$ and y by $\frac{3x-y}{4}$ in (37), we have

$$\|f(2x)+f(2y)-f(3x-y)-f(3y-x)\| \leqslant \delta, \quad x,y \in (-\frac{r}{4},\frac{r}{4}).$$

If we replace y by $-y$ in the last inequality, we obtain

$$\|f(3x+y)+f(-3y-x)-f(2x)-f(-2y)\| \leqslant \delta, \quad x,y \in (-\frac{r}{4},\frac{r}{4}). \tag{38}$$

Putting $x=0$ in (38), we get

$$\|f(y)+f(-3y)-f(-2y)\| \leqslant \delta, \quad y \in (-\frac{r}{4},\frac{r}{4}). \tag{39}$$

Putting $y=0$ in (37), we have

$$\|f(x)+f(3x)-f(4x)\| \leqslant \delta, \quad x \in (-\frac{r}{4},\frac{r}{4}). \tag{40}$$

If we put $y=-x$ in (37), we obtain

$$\|f(-2x)+f(2x)-f(-4x)-f(4x)\| \leqslant \delta, \quad x \in (-\frac{r}{4},\frac{r}{4}), \tag{41}$$

and then

$$\|f(-x)+f(x)-f(-2x)-f(2x)\| \leqslant \delta, \quad x \in (-\frac{r}{2},\frac{r}{2}). \tag{42}$$

It follows from (40) that

$$\|f(-x)+f(x)+f(-3x)+f(3x)-f(-4x)-f(4x)\| \leqslant 2\delta, \quad x \in (-\frac{r}{4},\frac{r}{4}). \tag{43}$$

Hence we get from (42) and (43) that

$$\|f(-2x)+f(2x)+f(-3x)+f(3x)-f(-4x)-f(4x)\| \leqslant 3\delta, \quad x \in (-\frac{r}{4},\frac{r}{4}). \tag{44}$$

Using the triangle inequality for (41) and (44), we obtain

$$\|f(-3x)+f(3x)\| \leqslant 4\delta, \quad x \in (-\frac{r}{4},\frac{r}{4}). \tag{45}$$

Therefore

$$\begin{aligned} &\|f(-x)+f(x)\| \leqslant 4\delta, \quad x \in (-\frac{3r}{4},\frac{3r}{4}), \\ &\|f(-3y-x)+f(3y+x)\| \leqslant 4\delta, \quad x,y \in (-\frac{3r}{16},\frac{3r}{16}). \end{aligned} \tag{46}$$

Using the last inequality (46) and inequalities (37) and (38), we get

$$\|2f(3x+y)-f(4x)-f(4y)-f(2x)-f(-2y)\| \leqslant 6\delta, \quad x,y \in (-\frac{3r}{16},\frac{3r}{16}). \tag{47}$$

If we consider (40) with x and y, then it follows by (47) that

$$\|2f(3x+y)-f(3x)-f(3y)-f(x)-f(y)-f(2x)-f(-2y)\| \leqslant 8\delta,$$

for all $x,y \in (-\frac{3r}{16},\frac{3r}{16})$. Consider the inequality (39) for y and $-x$, and using the above inequality, we obtain

$$\|2f(3x+y)-2f(3x)-f(3y)-f(-3y)-f(x)-f(-x)-2f(y)\| \leqslant 10\delta,$$

for all $x,y \in (-\frac{3r}{16},\frac{3r}{16})$. Hence this inequality with the inequalities (45) and (46) imply

$$\|2f(3x+y)-2f(3x)-2f(y)\| \leqslant 18\delta, \quad x,y \in (-\frac{3r}{16},\frac{3r}{16}).$$

Therefore

$$\|f(x+y)-f(x)-f(y)\| \leqslant 9\delta, \quad x,y \in (-\frac{3r}{16},\frac{3r}{16}).$$

By a similar way as in the proof of Theorem 11 we obtain the following results on restricted domains of $\mathbb{R}^N$.

THEOREM 12. *Suppose that D is a symmetric and bounded subset of $\mathbb{R}^N$ containing zero. Let $f: D \to E$ be a function with $f(0)=0$ and satisfy*

$$\left\|f\left(\frac{1}{4}x+\frac{3}{4}y\right)+f\left(\frac{3}{4}x+\frac{1}{4}y\right)-f(x)-f(y)\right\| \leqslant \delta,$$

for some $\delta \geqslant 0$ and for all $x,y \in D$ with $3x+y \in 4D$. Then

$$\|f(x+y)-f(x)-f(y)\| \leqslant 9\delta, \quad x,y \in (3/16)D.$$

THEOREM 13. *Let $f:(-r,r)^N \to E$ be a function with $f(0)=0$ and satisfy*

$$\left\|f\left(\frac{1}{4}x+\frac{3}{4}y\right)+f\left(\frac{3}{4}x+\frac{1}{4}y\right)-f(x)-f(y)\right\| \leqslant \delta, \quad x,y \in (-r,r)^N. \tag{48}$$

Then

$$\|f(x+y)-f(x)-f(y)\| \leqslant 9\delta, \quad x,y \in (-\frac{3r}{16},\frac{3r}{16})^N.$$

Using Lemma 1 and Theorem 13 we prove the stability of the functional equation $f\left(\frac{1}{3}x+\frac{2}{3}y\right)+f\left(\frac{2}{3}x+\frac{1}{3}y\right)=f(x)+f(y)$ on a restricted domain.

THEOREM 14. *Let E be a Banach space and let $f:(-r,r)^N \to E$ be a function with $f(0)=0$ and satisfy (48) for all $x,y \in (-r,r)^N$. Then there exists an additive function $A:\mathbb{R}^N \to E$ such that*

$$\|f(x)-A(x)\| \leqslant 9(5N-1)\delta, \quad x,y \in (-\frac{3r}{16},\frac{3r}{16})^N.$$

REFERENCES

[1] D. H. HYERS, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. **27** (1941), 222–224.

[2] S. JUNG, *Hyers-Ulam-Rassias stability of Jensen's equation and its application*, Proc. Am. Math. Soc. **126** (1998), 3137–3143.

[3] S. JUNG, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, Dordrecht, Heidelberg, London, 2011.

[4] S. JUNG AND B. KIM, *On the stability of the quadratic functional equation on bounded domains*, Abh. Math. Sem. Univ. Hamburg **69** (1999), 293–308.

[5] Z. KOMINEK, *On a local stability of the Jensen functional equation*, Demonstratio Math. **22** (1989), 499–507.

[6] D. MOLAEI AND A. NAJATI, *Hyperstability of the general linear equation on restricted domains*, Acta Math. Hungar. **149** (2016), 238–253.

[7] A. NAJATI AND TH. M. RASSIAS, *Stability of the Pexiderized Cauchy and Jensen's equations on restricted domains*, Commun. Math. Anal. **8** (2010), 125–135.

[8] A. NAJATI AND S. JUNG, *Approximately quadratic mappings on restricted domains*, J. Inequal. Appl. **2010**, Art. ID 503458, 10 pp. (2010).

[9] F. SKOF, *Sullapprossimazione delle applicazioni localmente δ-additive*, Atti Accad. Sci. Torino **117** (1983), 377–389.

[10] F. SKOF, *Proprietá locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.

[11] F. SKOF AND S. TERRACINI, *Sulla stabilita dell'equazione funzionale quadratica su un dominio ristretto*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **121** (1987), 153–167.

[12] S. M. ULAM, *A Collection of the Mathematical Problems*, Interscience Publication, NewYork, 1960.

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B. Noori
Department of Mathematics, Faculty of Sciences
University of Mohaghegh Ardabili
Ardabil, Iran
e-mail: noori.batool@yahoo.com

M. B. Moghimi
Department of Mathematics, Faculty of Sciences
University of Mohaghegh Ardabili
Ardabil, Iran
e-mail: mbfmoghimi@yahoo.com

B. Khosravi
Department of Mathematics, Faculty of Sciences
University of Mohaghegh Ardabili
Ardabil, Iran
e-mail: b.khosravi88@yahoo.com

Choonkil Park
Research Institute for Natural Sciences
Hanyang University
Seoul 04763, Korea
e-mail: baak@hanyang.ac.kr