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A NOTE ON GAUSSIAN INTEGRAL MEANS OF ENTIRE FUNCTIONS

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Abstract. In this paper, we investigate the convexity of function $r \mapsto \ln M_{2,1}(f(z), r)$ in $\ln r$ on $(0, \infty)$, where $f(z) = z^3 + C$.

1. Introduction

Let dA be the Euclidean area measure on the finite complex plane $\mathbb{C}$. Suppose α is real and $0 < p < \infty$. For any entire function $f : \mathbb{C} \mapsto \mathbb{C}$, we recall that the Gaussian integral means of f is defined by

$$M_{p,\alpha}(f,r) = \frac{\int_{|z|<r} |f(z)|^p e^{-\alpha|z|^2} dA(z)}{\int_{|z|<r} e^{-\alpha|z|^2} dA(z)}, \quad \forall r \in (0,\infty).$$

The Gaussian integral means is derived from the related theory of Fock spaces, see [1, 2, 3, 4].

In [1], suppose k is a nonnegative integer and $0 < p < \infty$. If $0 < \alpha < \infty$, then the function $r \mapsto \ln M_{p,\alpha}(z^k, r)$ is concave in $\ln r$. If $-\infty < \alpha \leqslant 0$, then there exists some c (depending on k and α) on $(0,\infty)$ such that the function $r \mapsto \ln M_{p,\alpha}(z^k, r)$ is convex in $\ln r$ on $(0, c]$ and concave in $\ln r$ on $[c, \infty)$.

Recall the Remark 9 in [1] we have that the integral means of all monomials are logarithmically concave when $\alpha > 0$. However, this is not true for all entire functions, even for linear mappings. And we have an instance in [1], and just choose $p = 2$, $\alpha = 1$ and $f(z) = a + z$ which proved $M_{2,1}(a + z, r)$ is logarithmically concave on $(\sqrt{\lambda}, \infty)$, for any $a \in \mathbb{C}$.

In this paper, we consider the convexity of function $r \mapsto \ln M_{2,1}(f, r)$ in $\ln r$, when $f(z) = z^3 + C$. The main result of this paper is the following theorem.

THEOREM. *Suppose* $p = 2$, $\alpha = 1$, $f(z) = z^3 + C$ *and* $|C|^2 = c$. *There exists a point* $P_1 \in (0, \infty)$ *(depending on* c*) on* $(0, \infty)$ *such that the function* $r \mapsto \ln M_{2,1}(f(z), r)$ *is convex in* $\ln r$ *on* $(0, P_1)$. *And the function* $r \mapsto \ln M_{2,1}(f(z), r)$ *is concave in* $\ln r$ *on* $(2\sqrt{2}, \infty)$.

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2. Preliminaries

The proof of the Theorem is elementary but very laborious, it requires several preliminary results which we present in this section. Throughout the paper we use the symbol $\equiv$ when a new notation is being introduced. We will also use the notation $A \sim B$ to mean that A and B have the same sign as a convention in [5].

In order to give the proof of our main theorem,we need the following three lemmas. The first one comes from [6]. For convenience, the proof is given.

LEMMA 1. *If we write every analytic function* $f : \mathbb{C} \longmapsto \mathbb{C}$ *in the form of a power series*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

then we can immediately obtain that

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta = \sum_{k=0}^{\infty} |a_k|^2 r^{2k}.$$

Proof.

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta &= \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\overline{f(re^{i\theta})} d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=0}^{\infty} a_k (re^{i\theta})^k \sum_{j=0}^{\infty} \overline{a_j} (\overline{re^{i\theta}})^j d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=0}^{\infty} a_k r^k e^{ik\theta} \sum_{j=0}^{\infty} \overline{a_j} r^j e^{-ij\theta} d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_k \overline{a_j} r^{k+j} e^{(k-j)i\theta} d\theta \\
&= \frac{1}{2\pi}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_k \overline{a_j} r^{k+j} \int_0^{2\pi} e^{(k-j)i\theta} d\theta = \sum_{k=0}^{\infty} |a_k|^2 r^{2k}.
\end{aligned}$$

This completes the proof of the lemma. □

The second lemma comes directly from [5] with $(0,1)$ being replaced by $(0,\infty)$. For completeness, we give the proof of this lemma.

LEMMA 2. (*i*) *Suppose* f *is twice differentiable on* $(0,\infty)$. *Then* $f(x)$ *is convex in* $\ln x$ *if and only if* $f'(x) + xf''(x) \geqslant 0$ *on* $(0,\infty)$.

(*ii*) *Suppose* f *is twice differentiable on* $(0,\infty)$. *Then* $f(x)$ *is convex in* $\ln x$ *if and only if* $f(x^2)$ *is convex in* $\ln x$ *and* $f(x)$ *is concave in* $\ln x$ *if and only if* $f(x^2)$ *is concave in* $\ln x$.

(*iii*) *Suppose f is positive twice differentiable on $(0,\infty)$. Let*

$$D(f(x)) \equiv \frac{f'(x)}{f(x)} + x\frac{f''(x)}{f(x)} - x\left(\frac{f'(x)}{f(x)}\right)^2.$$

Then the function $\ln f(x)$ is convex in $\ln x$ if and only if $D(f(x)) \geqslant 0$ on $(0,\infty)$ and the function $\ln f(x)$ is concave in $\ln x$ if and only if $D(f(x)) \leqslant 0$ on $(0,\infty)$.

Proof. (i) Let $t = \ln x$, It follows easily from the Chain Rule that

$$\frac{d^2 f}{dt^2} = x[f'(x) + xf''(x)].$$

Thus f is convex in $\ln x$ if and only if $f'(x) + xf''(x) \geqslant 0$ on $(0,\infty)$.

(ii) For the function $g(x) = f(x^2)$, we easily compute that

$$g'(x) + xg''(x) = 4x[f'(x^2) + x^2 f''(x^2)].$$

The desired result then follows from (i).

(iii) Apply (i) to the function $g(x) = \ln f(x)$. The desired result follows immediately.

The third lemma comes from [7] and [8].

LEMMA 3. *If $f = \frac{f_1}{f_2}$ is a quotient of two positive and twice differentiable functions f_1 and f_2 on $(0,\infty)$, then*

$$D(f(x)) = D(f_1(x)) - D(f_2(x)), \quad \forall x \in (0,\infty).$$

3. Proof of Theorem

This section is devoted to the proof of the theorem.

Proof. Let $p = 2$, $\alpha = 1$ and $f(z) = z^3 + C$. According to

$$M_{p,\alpha}(f,r) = \frac{\int_{|z|<r} |f(z)|^p e^{-\alpha|z|^2} dA(z)}{\int_{|z|<r} e^{-\alpha|z|^2} dA(z)}, \forall r \in (0,\infty),$$

by polar coordinates, an obvious change of variables and Lemma 1, we have

$$\begin{aligned} M_{2,1}(f,r) &= \frac{\int_{|z|<r} |z^3 + C|^2 e^{-|z|^2} dA(z)}{\int_{|z|<r} e^{-|z|^2} dA(z)} \\ &= \frac{\int_0^r (\rho^6 + |C|^2) e^{-\rho^2} \rho\, d\rho}{\int_0^r e^{-\rho^2} \rho\, d\rho} \\ &= \frac{\int_0^r (\rho^6 + |C|^2) e^{-\rho^2} d\rho^2}{\int_0^r e^{-\rho^2} d\rho^2} \end{aligned}$$

$$= \frac{\int_0^x (t^3 + |C|^2) e^{-t} dt}{\int_0^x e^{-t} dt}.$$

Let $|C|^2 = c > 0,$ we rewrite

$$\frac{\int_0^x (t^3 + c) e^{-t} dt}{\int_0^x e^{-t} dt} \equiv F(x).$$

Since

$$\int_0^x e^{-t} dt = 1 - e^{-x},$$

$$\int_0^x t^3 e^{-t} dt = -e^{-x}(x^3 + 3x^2 + 6x + 6) + 6,$$

so

$$\begin{aligned} \int_0^x (t^3 + c) e^{-t} dt &= \int_0^x t^3 e^{-t} dt + \int_0^x c e^{-t} dt \\ &= -e^{-x}(x^3 + 3x^2 + 6x + 6) + 6 + c(1 - e^{-x}) \\ &= -e^{-x}(x^3 + 3x^2 + 6x + 6 + c) + (6 + c). \end{aligned}$$

Therefore

$$F(x) = \frac{-e^{-x}(x^3 + 3x^2 + 6x + 6 + c) + (6 + c)}{1 - e^{-x}} \equiv \frac{g(x)}{h(x)}.$$

We immediately obtain

$$g'(x) = e^{-x}(x^3 + c),$$

$$g''(x) = e^{-x}(-x^3 + 3x^2 - c),$$

$$\begin{aligned} g'(x)g(x) &= -e^{-2x}(x^6 + 3x^5 + 6x^4 + (6 + 2c)x^3 + 3cx^2 + 6cx + 6c + c^2) \\ &\quad + e^{-x}((6 + c)x^3 + 6c + c^2), \end{aligned}$$

$$\begin{aligned} xg''(x)g(x) &= -xe^{-2x}(-x^6 + 3x^4 + (12 - 2c)x^3 + 18x^2 - 6cx - (6c + c^2)) \\ &\quad + xe^{-x}(-(6 + c)x^3 + (18 + 3c)x^2 - (6c + c^2)), \end{aligned}$$

$$x(g'(x))^2 = xe^{-2x}(x^6 + 2cx^3 + c^2),$$

$$\begin{aligned} g^2(x) &= e^{-2x}(x^6 + 21x^4 + 6x^5 + (48 + 2c)x^3 + (72 + 6c)x^2 + (72 + 12c)x \\ &\quad + (36 + 12c + c^2)) - e^{-x}((12 + 2c)x^3 + (36 + 6c)x^2 + (72 + 12c)x \\ &\quad + (72 + 24c + 2c^2)) + (36 + 12c + c^2). \end{aligned}$$

Employing the D-notation in Lemma 2 (iii), we have

$$\begin{aligned} D(g(x)) &= \frac{g'(x)}{g(x)} + x\frac{g''(x)}{g(x)} - x\left(\frac{g'(x)}{g(x)}\right)^2 \\ &= \frac{g'(x)g(x) + xg''(x)g(x) - x(g'(x))^2}{g^2(x)} \\ &= \frac{m(x)}{g^2(x)}, \end{aligned}$$

where

$$\begin{aligned} m(x) = & \, e^{-2x}(-x^6 - 6x^5 - 18x^4 - (24+2c)x^3 + 3cx^2 - (6c+c^2)) \\ & + e^{-x}(-(6+c)x^4 + (24+4c)x^3 - (6c+c^2)x + (6c+c^2)). \end{aligned}$$

From the following calculations

$$\begin{aligned} & h(x) = 1 - e^{-x}, h'(x) = e^{-x}, h''(x) = -e^{-x}, h(x)h'(x) = e^{-x} - e^{-2x}, \\ & h''(x)h(x) = -e^{-x} + e^{-2x}, (h'(x))^2 = e^{-2x}, h^2(x) = 1 - 2e^{-x} + e^{-2x}, \end{aligned}$$

we can get

$$\begin{aligned} D(h(x)) &= \frac{h'(x)}{h(x)} + x\frac{h''(x)}{h(x)} - x\left(\frac{h'(x)}{h(x)}\right)^2 \\ &= \frac{h'(x)h(x) + xh''(x)h(x) - x(h'(x))^2}{h^2(x)} \\ &= \frac{e^{-x}(1 - x - e^{-x})}{h^2(x)}. \end{aligned}$$

Hence, by Lemma 3,

$$\begin{aligned} D(F(x)) &= D(g(x)) - D(h(x)) \\ &= \frac{m(x)(1 - 2e^{-x} + e^{-2x}) - e^{-x}(1 - x - e^{-x})g^2(x)}{g^2(x)h^2(x)} \\ &\sim m(x)(1 - 2e^{-x} + e^{-2x}) - e^{-x}(1 - x - e^{-x})g^2(x) \\ &\equiv n(x). \end{aligned}$$

By calculation, we have

$$\begin{aligned} n(x) = & \, e^{-4x}(3x^4 + 24x^3 + (72+9c)x^2 + (72+12c)x + (36+6c)) \\ & + e^{-3x}(x^7 + 7x^6 + 27x^5 + (57+c)x^4 + (84+10c)x^3 \\ & - (36+6c)x^2 - (108+18c)x - (108+18c)) + e^{-2x}(-x^6 \\ & - 6x^5 - 18x^4 - (96+14c)x^3 - (36+3c)x^2 + (108+18c)) \\ & + e^{-x}(-(6+c)x^4 + (24+4c)x^3 + (36+6c)x - (36+6c)). \end{aligned}$$

We can get

$$n(x) \sim n(x) \cdot e^{4x} \equiv G(x).$$

Therefore,

$$\begin{aligned} G'(x) = {} & 12x^3 + 72x^2 + (144 + 18c)x + (72 + 12c) + e^x(x^7 + 14x^6 \\ & + 69x^5 + (192 + c)x^4 + (312 + 14c)x^3 + (216 + 24c)x^2 \\ & - (180 + 30c)x - (216 + 36c)) + e^{2x}(-2x^6 - 18x^5 \\ & - 66x^4 - (264 + 28c)x^3 - (360 + 48c)x^2 - (72 + 6c)x \\ & + (216 + 36c)) + e^{3x}(-(18 + 3c)x^4 + (48 + 8c)x^3 \\ & + (72 + 12c)x^2 + (108 + 18c)x - (72 + 12c)), \end{aligned}$$

$$\begin{aligned} G''(x) = {} & 36x^2 + 144x + (144 + 18c) + e^x(x^7 + 21x^6 + 153x^5 + (537 + c)x^4 \\ & + (1080 + 18c)x^3 + (1152 + 66c)x^2 + (252 + 18c)x - (396 + 66c)) \\ & + e^{2x}(-4x^6 - 48x^5 - 222x^4 - (792 + 56c)x^3 - (1512 + 180c)x^2 \\ & - (864 + 108c)x + (360 + 66c)) + e^{3x}(-(54 + 9c)x^4 \\ & + (72 + 12c)x^3 + (360 + 60c)x^2 + (468 + 78c)x \\ & - (108 + 18c)), \end{aligned}$$

$$\begin{aligned} G'''(x) = {} & 72x + 144 + e^x(x^7 + 28x^6 + 279x^5 + (1302 + c)x^4 \\ & + (3228 + 22c)x^3 + (4392 + 120c)x^2 + (2556 + 150c)x - (144 + 48c)) \\ & + e^{2x}(-8x^6 - 120x^5 - 684x^4 - (2472 + 112c)x^3 - (5400 + 528c)x^2 \\ & - (4752 + 576c)x - (144 - 24c)) + e^{3x}(-(162 + 27c)x^4 + (1296 + 216c)x^2 \\ & + (2124 + 354c)x + (144 + 24c)), \end{aligned}$$

$$\begin{aligned} G^{(4)}(x) = {} & 72 + e^x(x^7 + 35x^6 + 447x^5 + (2697 + c)x^4 + (8436 + 26c)x^3 \\ & + (14076 + 186c)x^2 + (11340 + 390c)x + (2412 + 102c)) \\ & + e^{2x}(-16x^6 - 288x^5 - 1968x^4 - (7680 + 224c)x^3 \\ & - (18216 + 1392c)x^2 - (20304 + 2208c)x - (5040 + 528c)) \\ & + e^{3x}(-(486 + 81c)x^4 - (648 + 108c)x^3 + (3888 + 648c)x^2 \\ & + (8964 + 1494c)x + (2556 + 426c)). \end{aligned}$$

$$\begin{aligned} G^{(5)}(x) = {} & e^x(x^7 + 42x^6 + 657x^5 + (4932 + c)x^4 + (19224 + 30c)x^3 \\ & + (39384 + 264c)x^2 + (39492 + 762c)x + (13752 + 492c)) \\ & + e^{2x}(-32x^6 - 672x^5 - 5376x^4 - (23232 + 448c)x^3 \\ & - (59472 + 3456c)x^2 - (77040 + 7200c)x - (30384 + 3264c)) \\ & + e^{3x}(-(1458 + 243c)x^4 - (3888 + 648c)x^3 + (9720 + 1620c)x^2 \end{aligned}$$

$$+(34668+5778c)x+(16632+2772c)).$$

Since

$$G^{(5)}(x)\sim G^{(5)}(x)\cdot e^{-x}\equiv H(x),$$

then

$$\begin{aligned}H'(x) &= 7x^6+252x^5+3285x^4+(19728+4c)x^3+(57672+90c)x^2\\ &+(78768+528c)x+(39492+762c)+e^x(-32x^6-864x^5-8736x^4\\ &-(44736+448c)x^3-(129168+4800c)x^2-(195984+14112c)x\\ &-(107424+10464c))+e^{2x}(-(2916+486c)x^4-(13608+2268c)x^3\\ &+(7776-1296c)x^2+(88776+14796c)x+(67932+11322c)).\end{aligned}$$

After calculation,

$$G(0)=G'(0)=G''(0)=G'''(0)=G^{(4)}(0)=G^{(5)}(0)=H(0)=0,$$

$$H'(0)=39492+726c-(107424+10464c)+67932+11322c=1620c.$$

Since $H'(0)>0$, there exists a point $x_1\in(0,\infty)$ such that $H'(x)>0$ for $x\in(0,x_1)$, therefore $H(x)$ is increasing on $x\in(0,x_1)$. Since $H(0)=0$, there exists a point $x_2\in(x_1,\infty)$ such that $H(x)>0$ for $x\in(0,x_2)$, that is, $G^{(5)}(x)>0$ for $x\in(0,x_2)$, therefore, $G^{(4)}(x)$ is increasing on $x\in(0,x_2)$. Since $G^{(4)}(0)=G'''(0)=G''(0)=G'(0)=G(0)=0$, it have same properties. So, there exists a point $x^*\in(x_2,\infty)$ such that $G(x)>0$ for $x\in(0,x^*)$, that is, $n(x)>0$ for $x\in(0,x^*)$. Thus we can obtain $D(F(x))>0$ for $r\in(0,x^*)$, which means that there exists a point $P_1\in(0,\infty)$ (depending on c), function $r\mapsto \ln M_{2,1}((z^3+C),r)$ is convex in $\ln r$ for $r\in(0,P_1)$.

Next, we consider the case of bigger r. Since $x=r^2$, that is the case of bigger x. We change $n(x)$ into $n_1(x)$

$$n_1(x)=e^{-4x}a_1+e^{-3x}a_2+e^{-2x}a_3+e^{-x}a_4,$$

where
$a_1=3x^4+24x^3+(72+9c)x^2+(72+12c)x+(36+6c)$,

$$\begin{aligned}a_2 &= x^7+7x^6+27x^5+(57+c)x^4+(84+10c)x^3-(36+6c)x^2-(108+18c)x\\ &-(108+18c),\end{aligned}$$

$a_3=-x^6-6x^5-18x^4-(96+14c)x^3-(36+3c)x^2+(108+18c)$,
$a_4=-(6+c)x^4+(24+4c)x^3+(36+6c)x-(36+6c)$.

Suppose $x\geqslant 1$, since e^x is monotone increasing on $x\in[1,\infty)$, then $e^{-4x}\leqslant e^{-3x}\leqslant e^{-2x}\leqslant e^{-x}$. Since $c\geqslant 0$, $x\leqslant e^x=1+x+\cdots$, we have $xe^{-3x}\leqslant e^{-2x}$, $xe^{-2x}\leqslant e^{-x}$.

$$\begin{aligned}n(x) &= e^{-4x}a_1+e^{-3x}a_2+e^{-2x}a_3+e^{-x}a_4\\ &\leqslant e^{-3x}(x^7+7x^6+27x^5+(60+c)x^4+(108+10c)x^3+(36+3c)x^2\end{aligned}$$

$$
\begin{aligned}
&-(36+6c)x-(72+12c))+e^{-2x}a_3+e^{-x}a_4\\
\leqslant\ & xe^{-3x}(x^6+7x^5+27x^4+(60+c)x^3+(108+10c)x^2+(36+3c)x)\\
&+e^{-2x}a_3+e^{-x}a_4\\
\leqslant\ & e^{-2x}(x^5+9x^4-(36+13c)x^3+(72+7c)x^2+(36+3c)x\\
&+(108+18c))+e^{-x}a_4\\
\leqslant\ & e^{-2x}(x^5+9x^4+(72+7c)x^2+(36+3c)x+(108+18c))+e^{-x}a_4\\
\leqslant\ & xe^{-2x}(x^4+9x^3+(72+7c)x+(144+21c))+e^{-x}a_4\\
\leqslant\ & e^{-x}(-(5+c)x^4+(33+4c)x^3+(108+13c)x+(108+15c))\\
\sim\ & -(5+c)x^4+(33+4c)x^3+(108+13c)x+(108+15c)\equiv N(x).
\end{aligned}
$$

Also we have

$$
\begin{aligned}
&N'(x)=-(20+4c)x^3+(99+12c)x^2+(108+13c),\\
&N''(x)=-(60+12c)x^2+(198+24c)x,\\
&N'''(x)=-(120+24c)x+(198+24c),\\
&N^{(4)}(x)=-(120+24c).
\end{aligned}
$$

Since $N^{(4)}(x)<0$, $N'''(x)$ is monotone decreasing on $x\in(1,\infty)$. We notice that $N'''(2)=-240-48c+198+24c=-42-24c<0$, so we can get $N'''(x)<0$ on $x\in(2,\infty)$, so $N''(x)$ is monotone decreasing on $x\in(2,\infty)$. When $x=4$, $N''(4)=-168-96c<0$, so $N''(x)<0$ on $x\in(4,\infty)$, that is, $N'(x)$ is monotone decreasing on $x\in(4,\infty)$. When $x=6$, $N'(6)=-648-419c<0$, so $N'(x)<0$ on $x\in(6,\infty)$, that is, $N(x)$ is monotone decreasing on $x\in(6,\infty)$. When $x=8$, $N(8)=-2612-1929c<0$, so $N(x)<0$ on $x\in(8,\infty)$, that is, the function $r\mapsto\ln M_{2,1}((z^3+C),r)$ is concave in $\ln r$ for $r\in(2\sqrt{2},\infty)$.

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