SOME MULTIPLE INTEGRAL INEQUALITIES
VIA THE DIVERGENCE THEOREM

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Abstract. In this paper, by the use of the divergence theorem, we establish some inequalities for functions defined on closed and bounded subsets of the Euclidean space $\mathbb{R}^n$, $n \geq 2$.

1. Introduction

Let $\partial D$ be a simple, closed counterclockwise curve bounding a region $D$ and $f$ defined on an open set containing $D$ and having continuous partial derivatives on $D$. In the recent paper [4], by the use of Green’s identity, we have shown among others that

$$
\int \int_D f(x,y) \, dx \, dy - \frac{1}{2} \oint_{\partial D} \left[ (\beta - y) f(x,y) \, dx + (x - \alpha) f(x,y) \, dy \right] 
\leq \frac{1}{2} \int \int_D \left[ |\alpha - x| \left| \frac{\partial f(x,y)}{\partial x} \right| + |\beta - y| \left| \frac{\partial f(x,y)}{\partial y} \right| \right] \, dx \, dy =: M(\alpha, \beta ; f) \tag{1.1}
$$

for all $\alpha, \beta \in \mathbb{C}$ and

$$
M(\alpha, \beta ; f) \leq \left\{ \begin{array}{l}
\left\| \frac{\partial f}{\partial x} \right\|_{D,\infty} \int\int_D |\alpha - x| \, dx \, dy + \left\| \frac{\partial f}{\partial y} \right\|_{D,\infty} \int\int_D |\beta - y| \, dx \, dy;

\left\| \frac{\partial f}{\partial x} \right\|_{D,p} \left( \int\int_D |\alpha - x|^q \, dx \, dy \right)^{1/q} + \left\| \frac{\partial f}{\partial y} \right\|_{D,p} \left( \int\int_D |\beta - y|^q \, dx \, dy \right)^{1/q}
\end{array} \right.
$$

where $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$;

$$
sup_{(x,y) \in D} |\alpha - x| \left\| \frac{\partial f}{\partial x} \right\|_{D,1} + sup_{(x,y) \in D} |\beta - y| \left\| \frac{\partial f}{\partial y} \right\|_{B,1}, \tag{1.2}
$$

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where \( \| \cdot \|_{D,p} \) are the usual Lebesgue norms, we recall that
\[
\| g \|_{D,p} := \begin{cases} \left( \iint_D |g(x,y)|^p \, dxdy \right)^{1/p}, & p \geq 1; \\ \sup_{(x,y) \in D} |g(x,y)|, & p = \infty. \end{cases}
\]

Applications for rectangles and disks were also provided in [4]. For some recent double integral inequalities see [1], [2] and [3].

We also considered similar inequalities for 3-dimensional bodies as follows, see [5]. Let \( B \) be a solid in the three dimensional space \( \mathbb{R}^3 \) bounded by an orientable closed surface \( \partial B \). If \( f : B \to \mathbb{C} \) is a continuously differentiable function defined on a open set containing \( B \), then by making use of the Gauss-Ostrogradsky identity, we have obtained the following inequality
\[
\begin{aligned}
\iiint_B f(x,y,z) \, dxdydz - \frac{1}{3} \left[ \int \int_{\partial B} (x-\alpha) f(x,y,z) \, dy \wedge dz \\
+ \int \int_{\partial B} (y-\beta) f(x,y,z) \, dz \wedge dx + \int \int_{\partial B} (z-\gamma) f(x,y,z) \, dx \wedge dy \right]
\leq \frac{1}{3} \iiint_B \left[ |\alpha-x| \left| \frac{\partial f(x,y,z)}{\partial x} \right| + |\beta-y| \left| \frac{\partial f(x,y,z)}{\partial y} \right| + |\gamma-z| \left| \frac{\partial f(x,y,z)}{\partial z} \right| \right] dxdydz
=: M(\alpha,\beta,\gamma,f)
\end{aligned}
\]
for all \( \alpha, \beta, \gamma \) complex numbers. Moreover, we have the bounds
\[
M(\alpha,\beta,\gamma,f) \leq \begin{cases} \left\| \frac{\partial f}{\partial x} \right\|_{B,\infty} \iiint_B |\alpha-x| \, dxdydz + \left\| \frac{\partial f}{\partial y} \right\|_{B,\infty} \iiint_B |\beta-y| \, dxdydz \\
+ \left\| \frac{\partial f}{\partial z} \right\|_{B,\infty} \iiint_B |\gamma-z| \, dxdydz; \\
\leq \frac{1}{3} \left( \iiint_B |\alpha-x|^q \, dxdydz \right)^{1/q} + \left( \iiint_B |\beta-y|^q \, dxdydz \right)^{1/q} \\
+ \left( \iiint_B |\gamma-z|^q \, dxdydz \right)^{1/q}, \quad p, q > 1; \quad \frac{1}{p} + \frac{1}{q} = 1; \\
\sup_{(x,y,z) \in B} |\alpha-x| \left\| \frac{\partial f}{\partial x} \right\|_{B,1} + \sup_{(x,y,z) \in B} |\beta-y| \left\| \frac{\partial f}{\partial y} \right\|_{B,1} \\
+ \sup_{(x,y,z) \in B} |\gamma-z| \left\| \frac{\partial f}{\partial z} \right\|_{B,1}. \end{cases}
\]

Applications for 3-dimensional balls were also given in [5]. For some triple integral inequalities see [6] and [9].

Motivated by the above results, in this paper we establish several similar inequalities for multiple integrals for functions defined on bonded subsets of \( \mathbb{R}^n \) \((n \geq 2)\) with smooth (or piecewise smooth) boundary \( \partial B \). To achieve this goal we make use of the well known divergence theorem for multiple integrals as summarized below.
2. Some preliminary facts

Let $B$ be a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$ with smooth (or piecewise smooth) boundary $\partial B$. Let $F = (F_1, ..., F_n)$ be a smooth vector field defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$. Let $\mathbf{n}$ be the unit outward-pointing normal of $\partial B$. Then the Divergence Theorem states, see for instance [8]:

$$\int_B \text{div} F \, dV = \int_{\partial B} F \cdot n \, dA,$$

(2.1)

where

$$\text{div} F = \nabla \cdot F = \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k},$$

$dV$ is the element of volume in $\mathbb{R}^n$ and $dA$ is the element of surface area on $\partial B$.

If $\mathbf{n} = (\mathbf{n}_1, ..., \mathbf{n}_n), \ x = (x_1, ..., x_n) \in B$ and use the notation $dx$ for $dV$ we can write (2.1) more explicitly as

$$\sum_{k=1}^{n} \int_B \frac{\partial F_k(x)}{\partial x_k} \, dx = \sum_{k=1}^{n} \int_{\partial B} F_k(x) \mathbf{n}_k(x) \, dA.$$  

(2.2)

By taking the real and imaginary part, we can extend the above equality for complex valued functions $F_k, \ k \in \{1, ..., n\}$ defined on $B$.

If $n = 2$, the normal is obtained by rotating the tangent vector through 90° (in the correct direction so that it points out). The quantity $tds$ can be written $(dx_1, dx_2)$ along the surface, so that

$$ndA := nds = (dx_2, -dx_1)$$

Here $t$ is the tangent vector along the boundary curve and $ds$ is the element of arc-length.

From (2.2) we get for $B \subset \mathbb{R}^2$ that

\[
\int_B \frac{\partial F_1(x_1, x_2)}{\partial x_1} \, dx_1 \, dx_2 + \int_B \frac{\partial F_2(x_1, x_2)}{\partial x_2} \, dx_1 \, dx_2 = \int_{\partial B} F_1(x_1, x_2) \, dx_2 - \int_{\partial B} F_2(x_1, x_2) \, dx_1,
\]

(2.3)

which is Green’s theorem in plane.

If $n = 3$ and if $\partial B$ is described as a level-set of a function of 3 variables i.e. $\partial B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid G(x_1, x_2, x_3) = 0\}$, then a vector pointing in the direction of $\mathbf{n}$ is $\text{grad} G$. We shall use the case where $G(x_1, x_2, x_3) = x_3 - g(x_1, x_2), \ (x_1, x_2) \in D$, a domain in $\mathbb{R}^2$ for some differentiable function $g$ on $D$ and $B$ corresponds to the inequality $x_3 < g(x_1, x_2)$, namely

$$B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 < g(x_1, x_2)\}.$$  

Then

$$\mathbf{n} = \frac{(-g_{x_1}, -g_{x_2}, 1)}{\sqrt{1 + g_{x_1}^2 + g_{x_2}^2}}, \ dA = (1 + g_{x_1}^2 + g_{x_2}^2)^{1/2} \, dx_1 \, dx_2$$
and 

$$n dA = (-g_{x_1}, -g_{x_2}, 1) dx_1 dx_2.$$ 

From (2.2) we get

$$\int_B \left( \frac{\partial F_1(x_1, x_2, x_3)}{\partial x_1} + \frac{\partial F_2(x_1, x_2, x_3)}{\partial x_2} + \frac{\partial F_3(x_1, x_2, x_3)}{\partial x_3} \right) dx_1 dx_2 dx_3$$

$$= - \int_D F_1(x_1, x_2, g(x_1, x_2)) g_{x_1}(x_1, x_2) dx_1 dx_2 - \int_D F_1(x_1, x_2, g(x_1, x_2)) g_{x_2}(x_1, x_2) dx_1 dx_2$$

$$+ \int_D F_3(x_1, x_2, g(x_1, x_2)) dx_1 dx_2 \quad (2.4)$$

which is the Gauss-Ostrogradsky theorem in space.

3. Identities of interest

We have the following identity of interest:

**Theorem 1.** Let $B$ be a bounded open subset of $\mathbb{R}^n$ ($n \geq 2$) with smooth (or piecewise smooth) boundary $\partial B$. Let $f$ be a continuously differentiable function defined in $\mathbb{R}^n$, or at least in $B \cup \partial B$ and with complex values. If $\alpha_k, \beta_k \in \mathbb{C}$ for $k \in \{1, \ldots, n\}$ with $\sum_{k=1}^n \alpha_k = 1$, then

$$\int_B f(x) dx = \sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} dx + \sum_{k=1}^n \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) dA. \quad (3.1)$$

We also have

$$\int_B f(x) dx = \frac{1}{n} \sum_{k=1}^n \int_B (\gamma_k - x_k) \frac{\partial f(x)}{\partial x_k} dx + \frac{1}{n} \sum_{k=1}^n \int_{\partial B} (x_k - \gamma_k) f(x) n_k(x) dA \quad (3.2)$$

for all $\gamma_k \in \mathbb{C}$, where $k \in \{1, \ldots, n\}$.

**Proof.** Let $x = (x_1, \ldots, x_n) \in B$. We consider

$$F_k(x) = (\alpha_k x_k - \beta_k) f(x), \quad k \in \{1, \ldots, n\}$$

and take the partial derivatives $\frac{\partial F_k(x)}{\partial x_k}$ to get

$$\frac{\partial F_k(x)}{\partial x_k} = \alpha_k f(x) + (\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k}, \quad k \in \{1, \ldots, n\}.$$ 

If we sum this equality over $k$ from 1 to $n$ we get

$$\sum_{k=1}^n \frac{\partial F_k(x)}{\partial x_k} = \sum_{k=1}^n \alpha_k f(x) + \sum_{k=1}^n (\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} = f(x) + \sum_{k=1}^n (\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} \quad (3.3)$$
for all \( x = (x_1, \ldots, x_n) \in B \).

Now, if we take the integral in the equality (3.3) over \((x_1, \ldots, x_n) \in B\) we get

\[
\int_B \left( \sum_{k=1}^n \frac{\partial F_k(x)}{\partial x_k} \right) \, dx = \int_B f(x) \, dx + \sum_{k=1}^n \int_B \left[ (\alpha_k x_k - \beta_k) \frac{\partial f(x)}{\partial x_k} \right] \, dx. \tag{3.4}
\]

By the Divergence Theorem (2.2) we also have

\[
\int_B \left( \sum_{k=1}^n \frac{\partial F_k(x)}{\partial x_k} \right) \, dx = \sum_{k=1}^n \int_{\partial B} \left( \alpha_k x_k - \beta_k \right) f(x) n_k(x) \, dA \tag{3.5}
\]

and by making use of (3.4) and (3.5) we get

\[
\int_B f(x) \, dx + \sum_{k=1}^n \int_B \left( \alpha_k x_k - \beta_k \right) \frac{\partial f(x)}{\partial x_k} \, dx = \sum_{k=1}^n \int_{\partial B} \left( \alpha_k x_k - \beta_k \right) f(x) n_k(x) \, dA
\]

which gives the desired representation (3.1).

The identity (3.2) follows by (3.1) for \( \alpha_k = \frac{1}{n} \) and \( \beta_k = \frac{1}{n} \gamma_k \), \( k \in \{1, \ldots, n\} \).

For the body \( B \) we consider the coordinates for the centre of gravity \( G(x_B,1, \ldots, x_B,n) \) defined by

\[
\overline{x}_{B,k} := \frac{1}{V(B)} \int_B x_k \, dx, \ k \in \{1, \ldots, n\},
\]

where

\[
V(B) := \int_B x \, dx
\]

is the volume of \( B \).

**Corollary 1.** With the assumptions of Theorem 1 we have

\[
\int_B f(x) \, dx = \sum_{k=1}^n \int_B \alpha_k \left( \overline{x}_{B,k} - x_k \right) \frac{\partial f(x)}{\partial x_k} \, dx + \sum_{k=1}^n \int_{\partial B} \alpha_k \left( x_k - \overline{x}_{B,k} \right) f(x) n_k(x) \, dA
\]

and, in particular,

\[
\int_B f(x) \, dx = \frac{1}{n} \sum_{k=1}^n \int_B \left( \overline{x}_{B,k} - x_k \right) \frac{\partial f(x)}{\partial x_k} \, dx + \frac{1}{n} \sum_{k=1}^n \int_{\partial B} \left( x_k - \overline{x}_{B,k} \right) f(x) n_k(x) \, dA. \tag{3.7}
\]

The proof follows by (3.1) on taking \( \beta_k = \alpha_k \overline{x}_{B,k}, \ k \in \{1, \ldots, n\} \).

For a function \( f \) as in Theorem 1 above, we define the points

\[
x_{B,\partial f,k} := \frac{\int_B x_k \frac{\partial f(x)}{\partial x_k} \, dx}{\int_B \frac{\partial f(x)}{\partial x_k} \, dx}, \ k \in \{1, \ldots, n\},
\]

provided that all denominators are not zero.
COROLLARY 2. With the assumptions of Theorem 1 we have

\[ \int_B f(x) \, dx = \sum_{k=1}^n \int_{\partial B} \alpha_k \left(x_k - x_{B, \partial f, k}\right) f(x) n_k(x) \, dA \] (3.8)

and, in particular,

\[ \int_B f(x) \, dx = \frac{1}{n} \sum_{k=1}^n \int_{\partial B} \left(x_k - x_{B, \partial f, k}\right) f(x) n_k(x) \, dA. \] (3.9)

The proof follows by (3.1) on taking \( \beta_k = \alpha_k x_{B, \partial f, k}, \quad k \in \{1, \ldots, n\} \) and observing that

\[ \sum_{k=1}^n \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} \, dx = \sum_{k=1}^n \alpha_k \int_B \left(x_{B, \partial f, k} - x_k\right) \frac{\partial f(x)}{\partial x_k} \, dx = 0. \]

For a function \( f \) as in Theorem 1 above, we define the points

\[ x_{\partial B, f, k} := \frac{\int_{\partial B} x_k f(x) n_k(x) \, dA}{\int_{\partial B} f(x) n_k(x) \, dA}, \quad k \in \{1, \ldots, n\} \]

provided that all denominators are not zero.

COROLLARY 3. With the assumptions of Theorem 1 we have

\[ \int_B f(x) \, dx = \sum_{k=1}^n \int_B \alpha_k \left(x_{\partial B, f, k} - x_k\right) \frac{\partial f(x)}{\partial x_k} \, dx \] (3.10)

and, in particular,

\[ \int_B f(x) \, dx = \frac{1}{n} \sum_{k=1}^n \int_B \left(x_{\partial B, f, k} - x_k\right) \frac{\partial f(x)}{\partial x_k} \, dx. \] (3.11)

The proof follows by (3.1) on taking \( \beta_k = \alpha_k x_{\partial B, f, k}, \quad k \in \{1, \ldots, n\} \) and observing that

\[ \sum_{k=1}^n \int_{\partial B} \left(\alpha_k x_k - \beta_k\right) f(x) n_k(x) \, dA = 0. \]

4. Some integral inequalities

We have the following result generalizing the inequalities from the introduction:

THEOREM 2. Let \( B \) be a bounded open subset of \( \mathbb{R}^n \) \((n \geq 2)\) with smooth (or piecewise smooth) boundary \( \partial B \). Let \( f \) be a continuously differentiable function defined in \( \mathbb{R}^n \), or at least in \( B \cup \partial B \) and with complex values. If \( \alpha_k, \beta_k \in \mathbb{C} \) for
\[ k \in \{1, \ldots, n\} \text{ with } \sum_{k=1}^{n} \alpha_k = 1, \text{ then} \]

\[
\left| \int_B f(x) \, dx - \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) \, dA \right| \\
\leq \sum_{k=1}^{n} \int_B |\beta_k - \alpha_k x_k| \left| \frac{\partial f(x)}{\partial x_k} \right| \, dx \leq \begin{cases} \\
\sum_{k=1}^{n} \int_B |\beta_k - \alpha_k x_k| \, dx \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,\infty} \\
\sum_{k=1}^{n} (\int_B |\beta_k - \alpha_k x_k|^q \, dx)^{1/q} \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,p} \\
\sum_{k=1}^{n} \sup_{x \in B} |\beta_k - \alpha_k x_k| \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,1} \\
\end{cases} \tag{4.1}
\]

We also have

\[
\left| \int_B f(x) \, dx - \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} (x_k - \gamma_k) f(x) n_k(x) \, dA \right| \\
\leq \frac{1}{n} \sum_{k=1}^{n} \int_B |\gamma_k - x_k| \left| \frac{\partial f(x)}{\partial x_k} \right| \, dx \leq \frac{1}{n} \begin{cases} \\
\sum_{k=1}^{n} \int_B |\gamma_k - x_k| \, dx \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,\infty} \\
\sum_{k=1}^{n} (\int_B |\gamma_k - x_k|^q \, dx)^{1/q} \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,p} \\
\sum_{k=1}^{n} \sup_{x \in B} |\gamma_k - x_k| \left\| \frac{\partial f(x)}{\partial x_k} \right\|_{B,1} \\
\end{cases} \tag{4.2}
\]

for all \( \gamma_k \in \mathbb{C} \), where \( k \in \{1, \ldots, n\} \).

**Proof.** By the identity (3.1) we have

\[
\left| \int_B f(x) \, dx - \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) \, dA \right| \\
= \sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} \, dx \leq \sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} \, dx \\
\leq \sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} \, dx,
\]

which proves the first inequality in (4.1).
By Hölder’s integral inequality for multiple integrals we have

\[
\int_B \left| \beta_k - \alpha_k x_k \right| \frac{\partial f(x)}{\partial x_k} \, dx
\]

\[
\leq \left\{ \begin{array}{ll}
\sup_{x \in B} \left| \beta_k - \alpha_k x_k \right| \int_B \left| f(x) \right| \, dx & \\
\left( \int_B \left| \frac{\partial f(x)}{\partial x_k} \right|^p \right)^{1/p} \left( \int_B \left| \beta_k - \alpha_k x_k \right|^q \, dx \right)^{1/q} & \\
where \, p, \, q > 1, \, \frac{1}{p} + \frac{1}{q} = 1;
\end{array} \right.
\]

which proves the last part of (4.1).

**Corollary 4.** **With the assumptions of Theorem 2** we have

\[
\left| \int_B f(x) \, dx - \frac{1}{n} \sum_{k=1}^n \int_{\partial B} \left( x_k - x_{B,k} \right) f(x) \, n_k(x) \, dA \right|
\]

\[
\leq \frac{1}{n} \sum_{k=1}^n \int_B \left| \frac{\partial f(x)}{\partial x_k} \right| \, dx \leq \frac{1}{n} \left\{ \begin{array}{ll}
\sum_{k=1}^n \int_B \left| x_{B,k} - x_k \right| \left| \frac{\partial f(x)}{\partial x_k} \right| \, dx & \\
\left( \int_B \left| x_{B,k} - x_k \right|^q \, dx \right)^{1/q} & \\
where \, p, \, q > 1, \, \frac{1}{p} + \frac{1}{q} = 1;
\end{array} \right.
\]

and

\[
\left| \int_B f(x) \, dx \right| \leq \frac{1}{n} \sum_{k=1}^n \int_B \left| x_{\partial B,f,k} - x_k \right| \left| \frac{\partial f(x)}{\partial x_k} \right| \, dx
\]

\[
\leq \frac{1}{n} \left\{ \begin{array}{ll}
\sum_{k=1}^n \int_B \left| x_{\partial B,f,k} - x_k \right| \left| \frac{\partial f(x)}{\partial x_k} \right| \, dx & \\
\left( \int_B \left| x_{\partial B,f,k} - x_k \right|^q \, dx \right)^{1/q} & \\
where \, p, \, q > 1, \, \frac{1}{p} + \frac{1}{q} = 1;
\end{array} \right.
\]

We also have the dual result:
\textbf{Theorem 3.} With the assumption of Theorem 2 we have

\[
\left| \int_B f(x) \, dx - \sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} \, dx \right| \leq \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} |\gamma_k - x_k| |n_k(x)| \, dA
\]

where

\[
\|f\|_{\partial B, p} := \begin{cases} (\int_{\partial B} |f(x)|^p \, dA)^{1/p}, & p \geq 1; \\ \sup_{x \in \partial B} |f(x)|, & p = \infty. \end{cases}
\]

In particular,

\[
\left| \int_B f(x) \, dx - \frac{1}{n} \sum_{k=1}^{n} \int_B (\gamma_k - x_k) \frac{\partial f(x)}{\partial x_k} \, dx \right| \leq \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} |\gamma_k - x_k| |n_k(x)| \, dA
\]

where \( p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \)

\[
\|f\|_{\partial B, 1} \sup_{x \in \partial B} |\gamma_k - x_k| |n_k(x)|.
\]

(4.5)

\[
\left| \int_B f(x) \, dx - \sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} \, dx \right| = \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) \, dA \leq \sum_{k=1}^{n} \int_{\partial B} (\alpha_k x_k - \beta_k) f(x) n_k(x) \, dA
\]

which proves the first inequality in (4.5).

\textbf{Proof.} From the identity (3.1) we have

\[
\left| \int_B f(x) \, dx - \sum_{k=1}^{n} \int_B (\beta_k - \alpha_k x_k) \frac{\partial f(x)}{\partial x_k} \, dx \right|
\]

(4.6)
By Hölder’s inequality for functions defined on $\partial B$ we have

$$\int_{\partial B} |\alpha_k x_k - \beta_k| |n_k(x)| |f(x)| dA \leq \begin{cases}
\|f\|_{\partial B, \infty} \int_{\partial B} |\alpha_k x_k - \beta_k| |n_k(x)| dA |
\|f\|_{\partial B, p} \left(\int_{\partial B} |\alpha_k x_k - \beta_k|^q |n_k(x)|^q dA\right)^{1/q}
\sup_{x \in \partial B} |\alpha_k x_k - \beta_k| |n_k(x)| \|f\|_{\partial B, 1},
\end{cases}$$

which proves the second part of the inequality (4.5).

We also have:

**Corollary 5.** With the assumptions of Theorem 2 we have

$$\left| \int_{B} f(x) dx - \frac{1}{n} \sum_{k=1}^{n} \int_{B} (x_{\partial B, k}^{-} - x_k) \frac{\partial f(x)}{\partial x_k} dx \right| \leq \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} |x_{\partial B, k}^{-} - x_k| |n_k(x)| |f(x)| dA \leq \frac{1}{n} \int_{\partial B} |f| \sum_{k=1}^{n} \int_{\partial B} |x_{\partial B, k}^{-} - x_k| |n_k(x)| dA,$$

and

$$\left| \int_{B} f(x) dx \right| \leq \frac{1}{n} \sum_{k=1}^{n} \int_{\partial B} |x_{\partial B, \partial f, k}^{-} - x_k| |n_k(x)| |f(x)| dA \leq \frac{1}{n} \int_{\partial B} |f| \sum_{k=1}^{n} \int_{\partial B} |x_{\partial B, \partial f, k}^{-} - x_k| |n_k(x)| dA,$$

where $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$.

If we take $n = 2$ in Theorem 3, then we get other results from [4], while for $n = 3$ we recapture some results from [5].

**References**


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