Journal of Mathematical Inequalities
Volume 14, Number 4 (2020), 951–966
doi:10.7153/jmi-2020-14-62

KOROVKIN TYPE THEOREMS AND ITS APPLICATIONS VIA $\alpha\beta-$ STATISTICALLY CONVERGENCE

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(*Communicated by M. Mursaleen*)

Abstract. In this paper we will introduce the generalized concept of the weighted $\alpha\beta-$ statistical convergence, introduced by Aktuglu. We will show a new $\alpha\beta-$weighted statistical convergence and based on this definition we will prove a kind of the Korovkin type theorems. Also we will show the rate of the convergence for this kind of weighted $\alpha\beta-$ statistical convergence and Voronovskaya type theorem.

1. Introduction

We shall denote by $\mathbb{N}$ the set of all natural numbers. Let $K \in \mathbb{N}$ and $K_n = \{k \leqslant n : k \in K\}$. Then the natural density of K is defined by $\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$ if the limit exists, where the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$, the set $K_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}$ has natural density zero (cf. [12, 17]), i.e. for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : |x_k - L| \geqslant \varepsilon\}| = 0.$$

In this case, we write $L = st - \lim x$. Note that every convergent sequence is statistically convergent but not conversely. Statistical convergence is extended to the weighted $\alpha\beta-$ statistically convergence by Aktuglu (see [1]) as follows.
Let $\alpha(n)$ and $\beta(n)$ be two sequences of positive numbers which satisfy the following conditions

1. α, β are both non-decreasing numerical sequences
2. $\beta(n) \geqslant \alpha(n)$,
3. $\beta(n) - \alpha(n) \to \infty$ as $n \to \infty$

Mathematics subject classification (2010): 40G15, 41A36.
Keywords and phrases: Weighted $\alpha\beta-$ statistical convergence, generalized Norlund summability, statistical convergence, Korovkin theorem, Vornovskaya theorem.

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and let us denote with Λ the set of all pairs (α,β) which satisfying conditions (1)-(3). For each pair $(\alpha,\beta)\in\Lambda$, $0<\gamma\leqslant 1$ and $K\subset\mathbb{N}$, we define $\delta^{\alpha,\beta}(K,\gamma)$ in the following way

$$\delta^{\alpha,\beta}(K,\gamma)=\lim_n \frac{|K\cap I_n^{\alpha,\beta}|}{(\beta(n)-\alpha(n)+1)^{\gamma}},$$

where $I_n^{(\alpha,\beta)}$ is the closed interval $[\alpha(n),\beta(n)]$. A sequence (x_n) is said to be weighted $\alpha\beta-$ statistically convergent of order γ to L, if

$$\delta^{\alpha,\beta}(\{k:|x_k-L|\geqslant\varepsilon\},\gamma)=\lim_n \frac{\left|\left\{k\in I_n^{\alpha,\beta}:|x_k-L|\geqslant\varepsilon\right\}\right|}{(\beta(n)-\alpha(n)+1)^{\gamma}}=0.$$

The statistical convergence of the numerical sequences is extensively studied by many authors (see [3, 7, 4, 17, 12]). In this paper we will define the weighted $\alpha\beta-$ statistical convergence of order γ and for this kind of statistical convergence we will prove the second Korovkin type theorems, rate of convergence and Voronovskaya type theorem. In what follows we will give the concept of the weighted $\alpha\beta-$ statistical convergence. Let (p_n), (q_n) be any two positive real sequences, such that

$$P_n=p_1+p_2+\cdots+p_n,\quad p_{-1}=0,$$
$$Q_n=q_1+q_2+\cdots+q_n,\quad q_{-1}=0.$$

Convolution of the above sequences we will denote by:

$$R_n=\sum_{k\in I_n^{(\alpha,\beta)}} p_k\cdot q_{\beta(n)-k}\to\infty\quad\text{as}\quad n\to\infty$$

and

$$N_{n,p,q}^{\gamma}(x)=\frac{1}{R_n^{\gamma}}\sum_{k\in I_n^{(\alpha,\beta)}} p_k q_{\beta(n)-k}x_k.$$

DEFINITION 1.1. A sequence (x_n) is said to be weighted $\alpha\beta-$ statistically convergent of order γ to a number L if for every $\varepsilon>0$

$$\lim_n \frac{1}{R_n^{\gamma}}\left|\left\{k\leqslant R_n: p_k q_{\beta(n)-k}|x_k-L|\geqslant\varepsilon\right\}\right|=0,$$

and we will denote it by $st_{(N_{n,p,q}^{\gamma})}-x_k\to L$.

In this example we will prove that above definition is generalization of the statistical convergence,hence, generalization of the ordinary convergence.

EXAMPLE 1.2. Let us consider that $p_k=1$, $\beta(n)=n$, $\alpha(n)=1$, $\gamma=1$ and

$$q_k=\begin{cases}1 \text{ if } & k=m^2, m=2,3,\cdots\\ 0 \text{ otherwise}\end{cases}$$

and

$$x_k = \begin{cases} 1 & \text{if} \quad k = m^2 - m, m^2 - m + 1, \cdots, m^2 - 1 \\ -m & \text{if} \quad k = m^2, m = 2, 3, \cdots \\ 0 & \text{otherwise} \end{cases}.$$

It is known that sequence $(m^2; m = 2, 3 \cdots,)$ is statistically convergent to 0. On the other hand $st - \liminf_n x_n = 0$ and $st - \limsup_n x_n = 1$. Thus $x = (x_n)$ is not statistically convergent. But

$$\lim_{n\to\infty} \frac{1}{R_n} |\{k \leqslant R_n : p_{n-k} q_k |x_k - L| \geqslant \varepsilon\}| = \lim_{n\to\infty} \frac{1}{R_n} |\{k \leqslant R_n : 1 \cdot q_k |x_k - L| \geqslant \varepsilon\}| = 0.$$

Hence, $st_{(N^{\gamma}_{n,p,q})} - x_k \to 0$.

DEFINITION 1.3. A sequence (x_n) is said to be strongly weighted $\alpha\beta-$ summable of order γ to a number L if

$$\lim_n \frac{1}{R_n^{\gamma}} \sum_{k \in I_n^{(\alpha,\beta)}} p_k q_{\beta(n)-k} |x_k - L|^r = 0,$$

for every $0 < r < \infty$ and we will denote it by $x_k \to L(N^{\gamma}_{n,p,q,r})$.

REMARK 1.4. In case where $r = 1$, the sequence (x_n) is said to be weighted $\alpha\beta-$ summable of order γ to a number L if

$$\lim_n \frac{1}{R_n^{\gamma}} \sum_{k \in I_n^{(\alpha,\beta)}} p_k q_{\beta(n)-k} |x_k - L| = 0,$$

and we will denote it by $x_k \to L(N^{\gamma}_{n,p,q})$.

REMARK 1.5. If $p_n = 1,\ q_n = 1,\ \gamma = 1,\ \alpha(n) = 1,\ \beta(n) = n+1$, then from above summability method we get Cesáro summability method.

2. Results

THEOREM 2.1. *Let $p_k q_{\beta(n)-k} |x_k - L| \leqslant M$, for all $k \in \mathbb{N}$. If a sequence $x = (x_k)$ is $st_{(N^{\gamma}_{n,p,q})}-$ statistically convergent to L then it is statistically summable $(N^{\gamma}_{n,p,q})-$ to L, but conversely is not true.*

Proof. Since $x = (x_k)$ is $st_{(N^{\gamma}_{n,p,q})}-$ statistically convergent to L, it means that

$$\lim_n \frac{1}{R_n^{\gamma}} |\{k \leqslant R_n : p_k q_{\beta(n)-k} |x_k - L| \geqslant \varepsilon\}| = 0.$$

Let us denote by $K=\{k\leqslant R_n : p_k q_{\beta(n)-k}|x_k-L|\geqslant\varepsilon\}$ and $K^c=\{k\leqslant R_n : p_k q_{\beta(n)-k}|x_k-L|<\varepsilon\}$. Then we have:

$$\begin{aligned}
&\frac{1}{R_n^\gamma}\sum_{k\in I_n^{(\alpha,\beta)}} p_k q_{\beta(n)-k}|x_k-L| \\
&\leqslant\frac{1}{R_n^\gamma}\sum_{\substack{k\in I_n^{(\alpha,\beta)}\\ k\in K}} p_k q_{\beta(n)-k}|x_k-L|+\frac{1}{R_n^\gamma}\sum_{\substack{k\in I_n^{(\alpha,\beta)}\\ k\in K^c}} p_k q_{\beta(n)-k}|x_k-L| \\
&\leqslant\frac{1}{R_n^\gamma}\sum_{\substack{k\in I_n^{(\alpha,\beta)}\\ k\in K}} p_k q_{\beta(n)-k}|x_k-L|+\frac{1}{R_n^\gamma}\sum_{\substack{k\in I_n^{(\alpha,\beta)}\\ k\in K^c}} p_k q_{\beta(n)-k}|x_k-L| \\
&\leqslant\frac{1}{R_n^\gamma}\cdot M\cdot|K|+\frac{1}{R_n^\gamma}\sum_{\substack{k\in I_n^{(\alpha,\beta)}\\ k\in K^c}}^{n}\varepsilon\to 0+\varepsilon\cdot 1=\varepsilon
\end{aligned}$$

as $n\to\infty$. Which implies that $N_{n,p,q}^\gamma(x)\to L$. That is $x=(x_n)$ is $N_{n,p,q}^\gamma-$ summable to L, in ordinary sense, which implies the $N_{n,p,q}^\gamma-$ statistically summability to L. To prove that converse is not true we will construct this

EXAMPLE 2.2. Let us consider that $p_n=q_n=1$ and $\beta(n)-\alpha(n)=n+1$, then the summability method $(N_{n,p,q}^\gamma)$ reduce to the $(C,1)-$ summability method. In this case the $st_{(N_{n,p,q}^\gamma)}-$ statistically convergence is reduced to the statistically convergence. We define the sequence $x=(x_n)$ as follows:

$$x_k=\begin{cases}1 & \text{if} \quad k=m^2-m, m^2-m+1,\cdots,m^2-1\\ -m & \text{if} \quad k=m^2, m=2,3,\cdots \\ 0 & \text{otherwise}\end{cases}.$$

Under this conditions we get:

$$\begin{aligned}
N_{n,p,q}^\gamma&=\frac{1}{R_n^\gamma}\sum_{k\in I_n^{(\alpha,\beta)}} p_k q_{\beta(n)-k}=\frac{1}{n+1}\sum_{k=0}^{n} x_k \\
&=\begin{cases}\frac{l+1}{n+1} \text{ if} \quad n=m^2-m+l; l=0,1,\cdots,m-1; m=2,3,\cdots\\ 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise}\end{cases}
\end{aligned}$$

whence it follows that $\lim_{n\to\infty}N_{n,p,q}^\gamma=0$, and hence $st_{N_{n,p,q}^\gamma}-\lim_{n\to\infty}N_{n,p,q}^\gamma=0$, i.e., $x=(x_n)$ is $(N_{n,p,q}^\gamma)-$summable to 0. On the other hand, the sequence $(m^2;m=2,3\cdots,)$ is statistically convergent to 0, it is clear that $st-\liminf_n x_n=0$ and $st-\limsup_n x_n=1$. Thus $x=(x_n)$ is not statistically convergent, nor $st_{(N_{n,p,q}^\gamma)}-$ statistically convergent.

THEOREM 2.3. *Let $0<\gamma\leqslant\tau\leqslant 1$. Then, we have $(N_{n,p,q,r}^\gamma)\subset(N_{n,p,q,r}^\tau)$ and the inclusion is strict for some $\gamma<\tau$.*

Proof. Let $x=(x_n)\in(N^{\gamma}_{n,p,q,1})$, and γ,τ be given such that $0<\gamma\leqslant\tau\leqslant 1$. Then, we obtain

$$\frac{1}{R^{\tau}_n}\sum_{k\in I^{(\alpha,\beta)}_n} p_k q_{\beta(n)-k}|x_k-L|^r \leqslant \frac{1}{R^{\gamma}_n}\sum_{k\in I^{(\alpha,\beta)}_n} p_k q_{\beta(n)-k}|x_k-L|^r.$$

Hence, $(N^{\gamma}_{n,p,q,1})\subset(N^{\tau}_{n,p,q,1})$. To prove that inclusion is strict, we will consider that $r=1$ and it is shown by the following

EXAMPLE 2.4. Let us consider that $p_k=q_k=1$, then $R^{\gamma}_n=(\beta(n)-\alpha(n)+1)^{\gamma}$.

$$x=(x_k)=\begin{cases}1, & \beta(n)-\sqrt{\beta(n)-\alpha(n)+1}\leqslant k\leqslant\beta(n)\\ 0, & \text{otherwise}\end{cases}.$$

For $0<\gamma<\frac{1}{2}$ we get

$$\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}}\sum_{k\in I^{(\alpha,\beta)}_n}|x_k-0|\geqslant\frac{\sqrt{\beta(n)-\alpha(n)+1}-1}{(\beta(n)-\alpha(n)+1)^{\gamma}}\to\infty,\quad\text{as}\quad n\to\infty.$$

On the other hand, for $\frac{1}{2}<\tau\leqslant 1$, we have

$$\frac{1}{(\beta(n)-\alpha(n)+1)^{\tau}}\sum_{k\in I^{(\alpha,\beta)}_n}|x_k-0|\leqslant\frac{\sqrt{\beta(n)-\alpha(n)+1}}{(\beta(n)-\alpha(n)+1)^{\tau}}\to 0,\quad\text{as}\quad n\to\infty.$$

So we have find a sequence $x=(x_n)\in(N^{\tau}_{n,p,q,1})\setminus(N^{\gamma}_{n,p,q,1})$. Which proves theorem.

In what follows we will show under which conditions from $st_{(N^{\gamma}_{n,p,q})}-$ statistical convergence, follows $(N^{\gamma}_{n,p,q,r})-$ summability and the conversely.

PROPOSITION 2.5. *Let us suppose that $x=(x_n)$ is $(N^{\gamma}_{n,p,q,r})$ summable convergent to L. If*

1. $0<r<1$ *and* $0\leqslant|x_k-L|<1$,
2. $1\leqslant r<\infty$ *and* $1\leqslant|x_k-L|<\infty$

then x is $st_{(N^{\gamma}_{n,p,q})}-$ statistically convergent to L.

Proof. Let us suppose that $x=(x_n)$ is $(N^{\gamma}_{n,p,q,r})-$ summable convergent to L. Under above conditions we get(in both cases):

$$p_k q_{\beta(n)-k}|x_k-L|^r\geqslant p_k q_{\beta(n)-k}|x_k-L|.$$

Let us denote by $A=\{k\leqslant R_n : p_kq_{\beta(n)-k}|x_k-L|\geqslant\varepsilon\}$, then

$$\lim_n \frac{\varepsilon|A|}{R_n^\gamma}=\lim_n \frac{1}{R_n^\gamma}\sum_{\substack{k\in I_n^{(\alpha,\beta)}\\ k\in A}}\varepsilon\leqslant\frac{1}{R_n^\gamma}\sum_{\substack{k\in I_n^{(\alpha,\beta)}\\ k\in A}}p_kq_{\beta(n)-k}|x_k-L|\leqslant\frac{1}{R_n^\gamma}\sum_{k\in I_n^{(\alpha,\beta)}}p_kq_{\beta(n)-k}|x_k-L|$$
$$\leqslant\frac{1}{R_n^\gamma}\sum_{k\in I_n^{(\alpha,\beta)}}p_kq_{\beta(n)-k}|x_k-L|^r=0.$$

Hence $x=(x_n)$ is $st_{(N^\gamma_{n,p,q})}-$ statistically convergent to L.

PROPOSITION 2.6. *Let us suppose that $x=(x_n)$ is $st_{(N^\gamma_{n,p,q})}-$ statistically convergent to L and $p_kq_{\beta(n)-k}|x_k-L|\leqslant M(k\in\mathbb{N})$. If*

1. $0<r<1$ *and* $0\leqslant M<1$,
2. $1\leqslant r<\infty$ *and* $1\leqslant M<\infty$

then x is $(N^\gamma_{n,p,q,r})-$ summable convergent to L.

Proof of the theorem is similar to Theorem 2.1, and we omit it.

PROPOSITION 2.7. **(i)** *If $x=(x_k)\to L$ statistically convergent, it is $st_{(N^\gamma_{n,p,q})}-$ $x_k\to L$.*

(ii) *If $\left(\frac{R_n^\gamma}{n}\right)$ is a bounded sequence, then statistical convergence is equivalent to $st_{(N^\gamma_{n,p,q})}$ convergence.*

3. Korovkin type theorem

Let $C[a,b]$ be the space of all functions f continuous on a $[a,b]$ of the real numbers. Also, $C[a,b]$ is a Banach space with norm

$$||f||=\sup_{x\in[a,b]}\{|f(x)|\}, f\in C[a,b].$$

The classical Korovkin first theorem is given as follows (see [10, 11, 2]):

THEOREM 3.1. *Let (B_n) be a sequence of positive linear operators from $C[0,1]$ into $C[0,1]$. Then*

$$\lim_{n\to\infty}||B_n(f,x)-f(x)||_\infty=0,$$

for all $f\in C[0,1]$ if and only if

$$\lim_{n\to\infty}||B_n(f_i,x)-f_i(x)||_\infty=0,$$

for $i\in\{0,1,2\}$ where $f_0(x)=1,\ f_1(x)=x$ and $f_2(x)=x^2$, where B_n is a sequence of positive linear operators from $[a,b] \to [a,b]$, and we say that B is a positive if $B(f;x) \geqslant 0$, whenever $f(x) \geqslant 0$.

The Korovkin type theorems are investigated by several mathematicians in generalization of them in many ways and several settings such as function spaces, abstract Banach latices, Banach algebras, and so on. This theory is useful in real analysis, functional analysis, harmonic analysis, and so on. For more results related to the Korovkin type theorems see ([8, 15, 16, 10, 11, 3, 14, 6, 7, 5, 9]). In this section, we prove Korovkin type theorem for the weighted $\alpha\beta-$statistically convergent defined as in definition 1.1.

Let $C[a,b]$ be the Banach space with the uniform norm $||.||_\infty$ of all real continuous functions on $[a,b]$. Suppose that $B_n : C[a,b] \to [a,b]$. We write $B_n(f;x)$ for $B_n(f(s);x)$.

THEOREM 3.2. *Let (B_k) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then for all $f \in C[a,b]$*

$$st_{(N^\gamma_{n,p,q})} - \lim_n ||B_n(f) - f|| \to 0, \quad on \quad [a,b] \tag{3.1}$$

if and only if

$$st_{(N^\gamma_{n,p,q})} - \lim_n ||B_n(f_i) - f_i|| \to 0, \quad on \quad [a,b], \quad with \quad f_i(x) = x^i; i \in \{0,1,2\}. \tag{3.2}$$

Proof. Let us suppose that relation (3.1) is true, since functions $1, x, x^2$ are continuous, then relations (3.2) follow immediately from (3.1). Now we will prove the conversely, that relations (3.2) are valid, and we will prove that relation (3.1) is valid, too. Let $f \in C[a,b]$, then there exist a constant $K > 0$ such that $|f(x)| \leqslant K$ for all $x \in [a,b]$. Therefore

$$|f(t) - f(x)| \leqslant 2K, x \in [a,b]. \tag{3.3}$$

For every given $\varepsilon > 0$ there exist a $\delta > 0$ such that

$$|f(t) - f(x)| \leqslant \varepsilon \tag{3.4}$$

whenever $|t - x| < \delta$ for all $x \in [a,b]$. Let us denote by $\psi \equiv \psi(t,x) = (t-x)^2$. If $|t - x| \geqslant \delta$, then we have:

$$|f(t) - f(x)| \leqslant \frac{2K}{\delta^2}\psi(t,x). \tag{3.5}$$

Now from relations (3.3)-(3.5), we get

$$|f(t) - f(x)| < \varepsilon + \frac{2K}{\delta^2}\psi(t,x).$$

Respectively,

$$-\varepsilon - \frac{2K}{\delta^2}\psi(t,x) < f(t) - f(x) < \frac{2K}{\delta^2}\psi(t,x) + \varepsilon.$$

Applying the operator $B_k(1,x)$ in this inequality, since $B_k(1,x)$ is monotone and linear, we obtain:

$$B_k(1,x)\left(-\varepsilon-\frac{2K}{\delta^2}\psi\right)<B_k(1,x)\left(f(t)-f(x)\right)<B_k(1,x)\left(\frac{2K}{\delta^2}\psi+\varepsilon\right)\Rightarrow$$

$$-\varepsilon B_k(1,x)-\frac{2K}{\delta^2}B_k(\psi(t),x)<B_k(f,x)-f(x)B_k(1,x)<\frac{2K}{\delta^2}B_k(\psi(t),x)+\varepsilon B_k(1,x). \tag{3.6}$$

On the other hand

$$B_k(f,x)-f(x)=B_k(f,x)-f(x)B_k(1,x)+f(x)[B_k(1,x)-1]. \tag{3.7}$$

From relations (3.6) and (3.7) we have:

$$B_k(f,x)-f(x)<\frac{2K}{\delta^2}B_k(\psi(t),x)+\varepsilon B_k(1,x)+f(x)[B_k(1,x)-1]. \tag{3.8}$$

Let us now estimate the following expression:

$$\begin{aligned}B_k(\psi(t),x)&=B_k\left((x-t)^2,x\right)=B_k\left(\left(x^2-2xt+t^2\right),x\right)\\&=x^2B_k(1,x)-2xB_k(t,x)+B_k\left(t^2,x\right)\end{aligned}$$

Now, from the last relation and (3.8), we obtain that

$$\begin{aligned}B_k(f,x)-f(x)&<\frac{2K}{\delta^2}\Big\{x^2[B_k(1,x)-1]-2x[B_k(t,x)-x]\\&\qquad+\left[B_k(t^2,x)-x^2\right]\Big\}+\varepsilon B_k(1,x)+f(x)[B_k(1,x)-1]\\&=\varepsilon+\varepsilon[B_k(1,x)-1]+f(x)[B_k(1,x)-1]\\&\qquad+\frac{2K}{\delta^2}\Big\{x^2B_k(1,x)-1]-2x[B_k(t,x)-x]+\left[B_k(t^2,x)-x^2\right]\Big\}.\end{aligned}$$

Therefore,

$$\begin{aligned}|B_k(f,x)-f(x)|\leqslant{}&\varepsilon+\left(\varepsilon+K+\frac{2Kb^2}{\delta^2}\right)|B_k(1,x)-1|+\frac{4Kb}{\delta^2}|B_k(t,x)-x|\\&+\frac{2K}{\delta^2}\left|B_k(t^2,x)-x^2\right|.\end{aligned}$$

Now taking the $\sup_{x\in[a,b]}$ in the above relation, we get:

$$\begin{aligned}\|B_k(f,x)-f(x)\|_{C[a,b]}\leqslant{}&\varepsilon+K\Big(||B_k(1,x)-1||_{C[a,b]}+||B_k(t,x)-x||_{C[a,b]}\\&+\left\|B_k(t^2,x)-x^2\right\|_{C[a,b]}\Big),\end{aligned}$$

where $M=\max\left\{\varepsilon+K+\frac{2Kb^2}{\delta^2},\frac{4Kb}{\delta^2},\frac{2K}{\delta^2}\right\}$.

For a given $r>0$, we can choose ε_1 such that $\varepsilon_1<r$. Now we will define the following sets:

$$D=\left\{k\leqslant n: ||B_k(f,x)-f(x)||_{C[a,b]}\geqslant r\right\},$$
$$D_i=\left\{k\leqslant n: ||B_k(f_i,x)-f_i(x)||_{C[a,b]}\geqslant \frac{r-\varepsilon_1}{3M}\right\},\quad i=0,1,2.$$

Then $D\subset\cup_{i=0}^{2}D_i$ and for their densities is satisfied relation:

$$\delta(D)\leqslant\delta(D_0)+\delta(D_1)+\delta(D_2).$$

Finally, from relations (3.2) and the above estimation we get

$$st_{(N^{\gamma}_{n,p,q})}-\lim_n||B_n(f)-f||\to 0,\quad \text{on}\quad [a,b],$$

for every $r>0$, which completes the proof.

REMARK 3.3. Those results are generalization of the known results given in [16]. For $p_n=1$ and $q_n=\lambda_n$, where λ_n- is given as in [16], then we obtain results from [16].

In what follows we will give example with which prove that our result is extension of the classical Korovkin approximation theorem.

EXAMPLE 3.4. We will consider this type of modified Bernstein type operators

$$B_n^{p,q}(f,x)=\sum_{k=0}^{n}p_{n-k}q_kf\left(\frac{k}{n}\right)\binom{n}{k}x^k\cdot(1-x)^{n-k},$$

where $x\in[0,1]$.

Let $p_n=q_n=1$, and $L_n: C(D)\to C(D)$ be sequence of operators defined as follows:

$$L_n(f,x)=(1+x_n)B_n(f,x),$$

where (x_n) is defined by Example 2.2.

$$B_n(1,x)=1,$$
$$B_n(t,x)=x,$$
$$B_n(t^2,x)=x^2+\frac{x(1-x)}{n},$$

and sequence of operators (L_n) satisfies conditions (3.2). Hence,

$$st_{(N^{\gamma}_{n,p,q})}-\lim_n||L_n(f,x)-f(x)||\to 0,\quad \text{on}\quad [a,b].$$

On the other hand, $L_n(f,0)=(1+x_n)B_nf(0)=(1+x_n)f(0)$, which obtains that

$$||L_n(f,x)-f(x)||_\infty\geqslant|T_n(f,0)-f(0)|\geqslant x_n|f(0)|.$$

Last relation shows that sequence of operators (L_n) does not satisfies the classical Korovkin type theorem, because sequence (x_n) is not convergent.

4. Rate of convergences

In this section, we study the rate of the weighted $\alpha\beta-$ statistically convergent of order γ, for a sequence of positive linear operators T_n defined on $C[a,b]$. We begin by presenting the following definition.

DEFINITION 4.1. Let (a_n) be any positive, nondecreasing sequence of positive numbers. We say that sequence (x_n) is a weighted $\alpha\beta-$ statistically convergent of order γ to x with rate of convergence $o(a_n)$, if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{a_n R_n^\gamma} \left|\left\{k \leqslant R_n : p_k q_{\beta(n)-k} |x_k - L| \geqslant \varepsilon\right\}\right| = 0.$$

In this case, we write $x_k - x = o(a_n)$(weighted $\alpha\beta-$ st. convergent of order γ), on $[a,b]$.

LEMMA 4.2. *Let (a_n) and (b_n) be two positive nondecreasing numeric sequences. Let (x_n) and (y_n) be two sequences such that $x_n - x = o(a_n)$ (weighted $\alpha\beta-$ st. convergent of order γ) and $y_n - y = o(b_n)$(weighted $\alpha\beta-$ st. convergent of order γ). Then:*

1. $(x_n - x) \pm (y_n - y) = o(c_n)$*(weighted $\alpha\beta-$ st. convergent of order γ),*

2. $\alpha(x_n - x) = o(a_n)$*(weighted $\alpha\beta-$ st. convergent of order γ), for any scalar α,*

3. $(x_n - x)(y_n - y) = o(a_n b_n)$*(weighted $\alpha\beta-$ st. convergent of order γ),*

where $c_n = \max\{a_n, b_n\}$.

Proof. In what follows we will prove just the first statement, the others we can prove in similar way. For $\varepsilon > 0$, let us denote by

$$\begin{aligned}
A_1 &= \{k \leqslant R_n : p_k q_{\beta(n)-k} |f_k + g_k - (f+g)| \geqslant \varepsilon\}, \\
A_2 &= \left\{k \leqslant R_n : p_k q_{\beta(n)-k} |f_k - f| \geqslant \frac{\varepsilon}{2}\right\}, \\
A_3 &= \left\{k \leqslant R_n : p_k q_{\beta(n)-k} |g_k - g| \geqslant \frac{\varepsilon}{2}\right\}.
\end{aligned}$$

Then observe that $A_1 \subset A_2 \cup A_3$. Moreover, since

$$c_n = \max\{a_n, b_n\},$$

we get:

$$\frac{|A_1|}{c_n R_n^\gamma} \leqslant \frac{|A_2|}{a_n R_n^\gamma} + \frac{|A_3|}{b_n R_n^\gamma}. \tag{4.1}$$

Now by taking limit as $n\to\infty$ in (4.1) and using the hypothesis, we conclude that

$$\lim_{n\to\infty}\frac{|A_1|}{c_nR_n^{\gamma}}=0,$$

which proves that first statement of the lemma.

Now let us recall the notion of the modules of continuity. The modulus of continuity for function $f(x)\in C[a,b]$, is defined as follows:

$$\omega(f,\delta)=\sup_{\substack{|h|<\delta\\ x,x+h\in[a,b]}}|f(x+h)-f(x)|.$$

It is known that

$$|f(x)-f(y)|\leqslant\omega(f,\delta)\left(\frac{|x-y|}{\delta}+1\right),\tag{4.2}$$

for all $x,y\in[a,b]$.

We have the following result:

THEOREM 4.3. *Let (B_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Suppose that*

1. $||B_n(1,x)-1||_\infty=o(a_n)$ *(weighted $\alpha\beta-$ st. convergent of order γ),*

2. $\omega(f,\lambda_k)=o(b_n)$ *(weighted $\alpha\beta-$ st. convergent of order γ), where $\lambda_n=\sqrt{B_n(\psi,x)}$ and $\psi\equiv\psi(t,x)=(t-x)^2$.*

Then for all $f\in C[a,b]$ and $x\in[a,b]$, we have:

$$||B_n(f,x)-f(x)||_\infty=o(c_n)(\textit{weighted } \alpha\beta-\textit{st. convergent of order } \gamma),$$

where $c_n=\max\{a_n,b_n\}$.

Proof. Let $f\in C[a,b]$, $||f||_\infty=K$ and $x\in[a,b]$. From relations (3.7) and (4.2) we get this estimation:

$$\begin{aligned}
|B_n(f,x)-f(x)| &\leqslant |B_n(|f(y)-f(x)|,x)|+|f(x)|\cdot|B_n(1,x)-1|\\
&\leqslant B_n\left(\frac{|x-y|}{\delta}+1,x\right)\omega(f,\delta)+|f(x)|\cdot|B_n(1,x)-1|\\
\text{(by Cauchy-Schwartz inequality)} &\leqslant \frac{1}{\delta}B_n\left((x-y)^2,x\right)^{\frac{1}{2}}B_n\left(f_0(x),x\right)^{\frac{1}{2}}\omega(f,\delta)\\
&\quad +B_n(1,x)\omega(f,\delta)+|f(x)|\cdot|B_n(1,x)-1|\\
\text{(for } \delta=\lambda_n\text{, we get)} &\leqslant K|B_n(1,x)-1|+2\omega(f,\delta)+\omega(f,\delta)|B_n(1,x)-1|\\
&\quad +\omega(f,\delta)\sqrt{|B_n(1,x)-1|}.
\end{aligned}$$

Now, by using relations (1) and (2) in the theorem and Lemma 4.2, we complete proof of theorem.

5. Voronovskaya type theorem

In this section, we will show that the positive linear operators L_n defined as follows:

$$L_n(f,x) = \frac{(1+x_n)}{n^2} B_n(f,x),$$

where sequences (x_n) are defined in Example 1.2, satisfy a Voronovskaja type property in the weighted $\alpha\beta-$ st. convergent of order $\gamma-$ sense. We first prove the following lemma.

LEMMA 5.1. *For $x \in [a,b]$, $\Phi(y) = y - x$ then*

$$n^2 L_n(\Phi^4) \sim x^2(2x^2+1)(x-1) \ (\textit{weighted } \alpha\beta - st.\ \textit{convergent of order } \gamma),\ \textit{on } [a,b].$$

Proof. After some calculations we get:

$$\begin{aligned} n^2 L_n(\Phi^4) =& (1+x_n)\left[\left(2 - \frac{5}{n} + \frac{8}{n^2} - \frac{11}{n^3} + \frac{6}{n^4}\right)x^5 + \left(-2 + \frac{4}{n} - \frac{5}{n^2} + \frac{9}{n^3} - \frac{6}{n^4}\right)x^4 \right. \\ & \left. + \left(1 - \frac{2}{n} + \frac{1}{n^3}\right)x^3 - \left(1 - \frac{2}{n} + \frac{3}{n^3} - \frac{2}{n^4}\right)x^2 + \left(\frac{1}{n^2} - \frac{3}{n^3} + \frac{2}{n^4}\right)x \right]. \end{aligned}$$

Thus we obtain:

$$\begin{aligned} & \left| n^2 L_n(\Phi^4) - x^2(2x^2+1)(x-1) \right| \\ \leqslant & |(1+x_n)-1| \left| (2x^5 - 2x^4 + x^3 - x^2) \right| + \left| \left(-\frac{5}{n} + \frac{8}{n^2} - \frac{11}{n^3} + \frac{6}{n^4} \right) x^5 \right| \\ & + \left| \left(\frac{4}{n} - \frac{5}{n^2} + \frac{9}{n^3} - \frac{6}{n^4} \right) x^4 \right| + \left| \left(-\frac{2}{n} + \frac{1}{n^3} \right) x^3 \right| + \left| \left(-\frac{2}{n} + \frac{3}{n^3} - \frac{2}{n^4} \right) x^2 \right| \\ & + \left| \left(\frac{1}{n^2} - \frac{3}{n^3} + \frac{2}{n^4} \right) x \right| \to 0 \ (\text{weighted } \alpha\beta - st.\ \text{convergent of order } \gamma), \end{aligned}$$

as $n \to \infty$, on $[0,1]$. This completes proof of the Lemma.

In what follows we establish the following Voronovskaya type theorem for operators L_n, defined as in above Lemma.

THEOREM 5.2. *For every $f \in [a,b]$ such that $f^{'}, f^{''} \in [a,b]$, then*

$$n\left[n^2 L_n(f) - f(x)\right] \sim \frac{1}{2}\left(x - x^2\right) f^{''}(x) (\textit{weighted } \alpha\beta - st.\ \textit{convergent of order } \gamma),$$

on $[a,b]$.

Proof. Let us suppose that $f^{'}, f^{''} \in C[a,b]$ and $x \in [a,b]$. Define

$$\psi_x(y) = \begin{cases} \frac{f(y) - f(x) - (y-x)f^{'}(x) - \frac{1}{2}(y-x)^2 f^{''}(x)}{(y-x)^2} & \text{for} \quad x \neq y \\ o & \quad x = y. \end{cases}$$

Then $\psi_x(x)=0$ and $\psi_x\in C[a,b]$. By Taylor's formula, we get

$$f(y)=f(x)+(y-x)f^{'}(x)+\frac{1}{2}(y-x)^2f^{''}(x)+(y-x)^2\psi_x(y). \tag{5.1}$$

Knowing that

$$L_n(1,x)=\frac{(1+x_n)}{n^2};L_n((y-x),x)=0 \quad \text{and} \quad L_n((y-x)^2,x)=(1+x_n)\frac{x-x^2}{n^3},$$

and after operating in the both sides of relation (5.1) by operator L_n, we obtain:

$$n^2L_n(f)=f(x)+x_nf(x)+\frac{f^{''}(x)}{2}\frac{x-x^2}{n}(1+x_n)+(1+x_n)L_n(\Phi^2\psi_x,x),$$

which yields

$$\left|n\left[n^2L_n(f)-f(x)\right]-\frac{1}{2}(x-x^2)f^{''}(x)\right|\leqslant nx_n|f(x)|+x_n\left|f^{''}(x)\right|+n(1+x_n)\left|L_n(\Phi^2\psi_x,x)\right|,$$

receptively

$$\left|n\left[n^2L_n(f)-f(x)\right]-\frac{1}{2}(x-x^2)f^{''}(x)\right|\leqslant nx_nM+n(1+x_n)\left|L_n(\Phi^2\psi_x,x)\right|, \tag{5.2}$$

where $\Phi(y)=y-x$ and $M=||f||_{C[a,b]}+||f^{''}||_{C[a,b]}$. After application of the Cauchy-Schwartz inequality in the terms of the right side of the relation (5.2), we obtain:

$$n\left|L_n(\Phi^2\psi_x,x)\right|\leqslant\left[n^2L_n(\Phi^4,x)\right]^{\frac{1}{2}}\cdot\left[L_n(\psi_x^2,x)\right]^{\frac{1}{2}}. \tag{5.3}$$

Putting $\eta_x(y)=(\psi_x(y))^2$, we get that $\eta_x(x)=0$ and $\eta_x(\cdot)\in C[a,b]$. Also

$$nx_n\left|L_n(\Phi^2\psi_x,x)\right|\leqslant x_n\left[n^2L_n(\Phi^4,x)\right]^{\frac{1}{2}}\cdot\left[L_n(\psi_x^2,x)\right]^{\frac{1}{2}}, \tag{5.4}$$

where $x_n\to 0$(weighted $\alpha\beta-st$. convergent of order γ).

Now from Theorem 3.2, it follows that

$$L_n(\eta_x)\to 0(\text{weighted } \alpha\beta-st.\text{ convergent of order } \gamma) \text{ on } [a,b]. \tag{5.5}$$

Now, from relations (5.3), (5.4), (5.5) and Lemma 5.1, we have

$$n(1+x_n)L_n(\Phi^2\psi_x,x)\to 0(\text{weighted } \alpha\beta-st.\text{ convergent of order } \gamma) \text{ on } [a,b]. \tag{5.6}$$

For a given $\varepsilon>0$, we define the following sets:

$$A_n(x,\varepsilon)=\left|\left\{k:k\leqslant R_n:p_kq_{\beta(n)-k}\left|k\left[k^2L_k(f)-f(x)\right]-\frac{1}{2}(x-x^2)f^{''}(x)\right|\geqslant\varepsilon\right\}\right|,$$

$$A_{1,n}(x,\varepsilon) = \left|\left\{k : k \leqslant R_n : p_k q_{\beta(n)-k}|kx_k| \geqslant \frac{\varepsilon}{2M}\right\}\right|,$$

and

$$A_{2,n}(x,\varepsilon) = \left|\left\{k : k \leqslant R_n : p_k q_{\beta(n)-k}|k(1+x_k)L_k(\Phi^2\psi_x,x)| \geqslant \frac{\varepsilon}{2}\right\}\right|.$$

From last relation we have

$$\frac{A_n(x,\varepsilon)}{a_n R_n^\gamma} \leqslant \frac{A_{1,n}(x,\varepsilon)}{a_n R_n^\gamma} + \frac{A_{2,n}(x,\varepsilon)}{a_n R_n^\gamma}. \tag{5.7}$$

From definition of the sequence (x_n), we get

$$nx_n \to 0(\text{weighted } \alpha\beta - st.\text{ convergent of order } \gamma) \text{ on } \quad [a,b]. \tag{5.8}$$

Now from relations (5.6) and (5.8), the right hand side of the relation (5.7), tends to zero as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty} \frac{A_n(x,\varepsilon)}{a_n R_n^\gamma} = 0,$$

which proves that

$$n\left[n^2 L_n(f) - f(x)\right] \sim \frac{1}{2}\left(x - x^2\right) f^{''}(x)(\text{weighted } \alpha\beta - st.\text{ convergent of order } \gamma)$$

on $[a,b]$.

6. Concluding remarks

In this section we will give some remarks related to the results obtain in this paper and their relationship with other results.

REMARK 6.1. Suppose that we replace the conditions (1) and (2) in Theorem 4.3 by the following condition:

$$B_n(x_i) - x_i = o(a_{n_i})(\text{weighted } \alpha\beta - st.\text{ convergent of order } \gamma) \text{ on } [a,b](i=0,1,2). \tag{6.1}$$

Then, since

$$B_n(\psi^2;x) = B_n(t^2,x) - 2xB_n(t^1,x) + x^2 B_n(1,x),$$

we may write

$$B_n(\psi^2,x) \leqslant K[|B_n(1,x)-1| + |B_n(t,x)-t| + |B_n(t^2,x)-t^2|],$$

where

$$K = 1 + 2||t|| + ||t^2||.$$

Now it follows from above relations and Lemma 4.2 that

$$\delta_n = \sqrt{B_n(\psi^2)} = o(d_n)(\text{weighted } \alpha\beta - st. \text{ convergent of order } \gamma)$$

on $[a,b]$, where $d_n = \min\{a_{n_0}, a_{n_1}, a_{n_2}\}$. Hence

$$\omega(f, d_n) = o(d_n)(\text{weighted } \alpha\beta - st. \text{ convergent of order } \gamma)$$

on $[a,b]$. If those conditions which are given here we can use in Theorem 3.2, we can thus see that, for all $f \in C[a,b]$,

$$B_n(f) - f = o(d_n)(\text{weighted } \alpha\beta - st. \text{ convergent of order } \gamma)$$

on $[a,b]$. Therefore, if we use the condition (6.1) in Theorem 4.3 instead of the conditions (1) and (2), then we obtain the rates of the weighted $\alpha\beta-$st. convergent of order γ of the sequence of positive linear operators in Theorem 3.2.

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(Received November 11, 2019)

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