TAUBERIAN THEOREMS UNDER STATISTICALLY NÖRLUND–CESÁRO SUMMABILITY METHOD

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Abstract. Let \((p_n)\) and \((q_n)\) be any two non-negative real sequences with

\[ R_n := \sum_{k=0}^{n} p_k q_{n-k} \neq 0 \quad (n \in \mathbb{N}). \]

And \(C^1\)– Cesáro summability method. Let \((x_n)\) be a sequence of real or complex numbers and set

\[ \mathcal{N}^n_{p,q} C^1_n := \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v \]

for \(n \in \mathbb{N}\). In this paper, we present necessary and sufficient conditions under which the existence of the limit \(st \lim_{n \to \infty} x_n = L\) follows from that of \(st \lim_{n \to \infty} \mathcal{N}^n_{p,q} C^1_n = L\). These conditions are one-sided or two-sided if \((x_n)\) is a sequence of real or complex numbers, respectively.

1. Introduction

Let \((p_n)\) and \((q_n)\) be any two non-negative real sequences with

\[ R_n := \sum_{k=0}^{n} p_k q_{n-k} \neq 0 \quad (n \in \mathbb{N}). \]

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\[ \mathcal{N}^n_{p,q} C^1_n := \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v \]

for \(n \in \mathbb{N}\). In this paper, we present necessary and sufficient conditions under which the existence of the limit \(\lim_{n \to \infty} x_n = L\) follows from that of \(\lim_{n \to \infty} \mathcal{N}^n_{p,q} C^1_n = L\). These conditions are one-sided or two-sided if \((x_n)\) is a sequence of real or complex numbers, respectively.


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In what follows we give the concept of the summability method known as the generalized Nörlund summability method \((N, p, q)\) (see \([1, 4]\)). Given two non-negative sequences \((p_n)\) and \((q_n)\), the convolution \((p * q)\) is defined by

\[ R_n := (p * q)_n = \sum_{k=0}^{n} p_k q_{n-k} = \sum_{k=0}^{n} p_{n-k} q_k. \]

With \((C, 1)\) we will denote the Cesàro summability method. Let \((x_n)\) be a sequence. When \((p * q)_n \neq 0\) for all \(n \in \mathbb{N}\), the generalized Nörlund-Cesàro transform of the sequence \((x_n)\) is the sequence \(N^n_{p,q} C^1_n\) obtained by putting

\[ N^n_{p,q} C^1_n = \frac{1}{(p * q)_n} \sum_{k=0}^{n} p_k q_{n-k} \sum_{v=0}^{k} x_v. \] (1.1)

We say that the sequence \((x_n)\) is generalized Nörlund-Cesàro summable to \(L\) determined by the sequences \((p_n)\) and \((q_n)\) or briefly summable \(N^n_{p,q} C^1_n\) to \(L\) if

\[ \lim_{n \to \infty} N^n_{p,q} C^1_n = L. \] (1.2)

Suppose throughout the paper we assume that the sequences \((q_n)\) and \((p_n)\) are satisfying the following conditions:

\[ q_n \geq 1, \sum_{k=0}^{n} p_k \sim n, \quad n \in \mathbb{N}, \] (1.3)

\[ q_{\lambda_{n-k}} \leq 2q_{n-k}, k = 0, 1, 2, 3, \ldots, n; \lambda > 1, \] (1.4)

\[ q_{n-k} \leq 2q_{\lambda_{n-k}}, k = 0, 1, 2, 3, \ldots, \lambda_n; 0 < \lambda < 1, \] (1.5)

where \(\lambda_n = [\lambda \cdot n]\), \(a_n \sim b_n\), means that there are constants \(C, C_1\) such that \(a_n \leq C b_n \leq C_1 a_n\).

If \(\lim_{n \to \infty} x_n = L\) \(\) implies \(1.2\), then the method \(N^n_{p,q} C^1_n\) is called to be regular. However, the converse is not always true. We can show by the following example

**Example 1.1.** Let us consider that \(p_n = q_n = 1\) for all \(n \in \mathbb{N}\). Also we define the following sequence \(x = (x_k) = (-1)^k\), then we have

\[ \frac{1}{n+1} \left| \sum_{k=0}^{n} \frac{1}{k+1} \sum_{v=0}^{k} (-1)^v \right| \leq \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{k+1} \sum_{v=0}^{k} 1 \to 1 \quad \text{as} \quad n \to \infty. \]

And as we know \(x = (x_k)\), is not convergent.
Notice that (1.6) may imply (1.2) under a certain condition, which is called a Tauberian condition. Any theorem which states that convergence of a sequence follows from its $N_{p,q}^n C_n^1$ summability and some Tauberian condition is said to be a Tauberian theorem for the $N_{p,q}^n C_n^1$ summability method. The inclusion and Tauberian type theorems are proved in the papers [4, 5, 2, 3], and some theorems of inclusion, Tauberian and convexity type for certain families of generalized Nörlund methods are obtained in [6].

In this section our aim is to find conditions (so-called Tauberian) under which the converse implication holds, for defined convergence. Exactly, we will prove under which conditions statistical convergence of sequences $(x_n)$ follows from statistically Nörlund-Cesáro summability method.

**Definition 1.2.** A sequence $(x_n)$ is weighted $N_{p,q}^n C_n^1$—statistically convergent to $L$ if for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{(p \ast q)_n} \left\{ k \leq (p \ast q)_n : \left| \frac{1}{(p \ast q)_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L \right| \geq \varepsilon \right\} = 0.$$

And we say that the sequence $(x_n)$ is statistically summable to $L$ by the weighted summability method $N_{p,q}^n C_n^1$, if $\text{st} \lim_n N_{p,q}^n C_n^1 = L$. We denote by $N_{p,q}^n C_n^1(\text{st})$ the set of all sequences which are statistically summable, by the weighted summability method $N_{p,q}^n C_n^1$.

**Theorem 1.3.** If sequence $x = (x_n)$ is $N_{p,q}^n C_n^1$ summable to $L$, then sequence $x = (x_n)$ is $N_{p,q}^n C_n^1$— statistically convergent to $L$. But not conversely.

**Proof.** The first part of the proof is obvious. To prove the second part we will show this example:

**Example 1.4.** We will define

$$x_k = \begin{cases} \sqrt{k}, & \text{for } k = n^2 \\ 0, & \text{otherwise} \end{cases}$$

and $p_n = 1 = q_n$. Under this conditions we get:

$$\frac{1}{n+1} \left\{ k \leq n+1 : \left| \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L \right| \geq \varepsilon \right\} \leq \frac{\sqrt{n+1}}{n+1} \to 0.$$

On the other hand, for $k = n^2$, we have

$$\frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{k+1} \sum_{v=0}^{k} x_v \to \infty, \quad \text{as } n \to \infty.$$

From last relation follows that $x = (x_n)$ is not $N_{p,q}^n C_n^1$ summable to 0. □
Theorem 1.5. Let us suppose that sequence \((x_n)\)-statistically convergent to \(L\), and \(|x_n - L| \leq M\) for every \(n \in \mathbb{N}\). Then it converges \(N_{p,q}^m C_n^1\)-statistically to \(L\). Converse is not true.

Proof. From fact that \((x_n)\) converges statistically to \(L\), we get
\[
\lim_{n \to \infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{n} = 0.
\]
Let us denote by \(B_\varepsilon = \{k \leq n : |x_k - L| \geq \varepsilon\}\) and \(\overline{B_\varepsilon} = \{k \leq n : |x_k - L| \leq \varepsilon\}\). Then
\[
\left| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L \right| \\
\leq \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} |x_v - L| + \frac{1}{R_n} \sum_{k \in B_\varepsilon} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} |x_v - L|
\]
\[
\leq M \frac{1}{R_n} \sum_{k=0}^{n} \frac{1+\varepsilon}{M C_2} \sum_{k \in B_\varepsilon} 1+\varepsilon \to 0 + \varepsilon, \quad \text{as} \quad n \to \infty,
\]
for some constant \(C_2\). To show that converse is not true we will use into consideration this

Example 1.6. Let us consider that \((p_n) = n + 1\), \((q_n) = 1\) for \(n \in \mathbb{N} \cup \{0\}\), and we define the sequence \(x = (x_n)\), as follows:
\[
x_k = \begin{cases} 
1, & \text{for } k = m^2 - m, \ldots, m^2 - 1 \\
-\frac{1}{m}, & \text{for } k = m^2, m = 2, \ldots \\
0, & \text{otherwise}
\end{cases}
\]
Under this conditions, after some calculations we get:
\[
\left| \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} 1 \cdot \sum_{v=0}^{k} x_v - 1 \right| \leq \left| \frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} 1 \cdot \sum_{v=0}^{k} -1 \right| \to 0, \quad \text{as} \quad n \to \infty.
\]
From last relation follows that \(x = (x_n)\) is \(N_{p,q}^m C_n^1\)-summable to 1. Hence from Theorem 1.5, \((x_n)\) is \(N_{p,q}^m C_n^1\)-statistically convergent. On the other hand, the sequence \((m^2; m = 2, 3, \ldots)\) has natural density zero and it is clear that \(st - \liminf_n x_n = 0\) and \(st - \limsup_n x_n = 1\). Thus, \((x_k)\) is not statistically convergent. \(\Box\)

2. Tauberian theorems under statistical Nörlund-Cesáro summability method

In the following theorem we characterize the converse implication when the statistically convergence follows from its \(N_{p,q}^m C_n^1\)-statistically convergence.
THEOREM 2.1. Let \((p_n)\) and \((q_n)\) be two non-negative real sequences, defined as above, and
\[
st - \liminf_{n \to \infty} \frac{R_{\lambda_n}}{R_n} > 1, \quad \text{for every } \lambda > 1,
\] (2.1)
where \(\lambda_n := \lfloor \lambda n \rfloor\) denotes the integral part of \(\lambda n\) for every \(n \in \mathbb{N}\), and let \((x_n)\) be a sequence of real numbers which is \(N_{p,q}^\infty C^1\) statistically convergent to a finite number \(L\). Then \((x_n)\) is \(st\)-convergent to the same number \(L\) if and only if the following two conditions hold:

\[
\inf_{\lambda > 1} \limsup_{n \to \infty} \frac{1}{R_n} \left\{ k \leq R_n \colon \frac{1}{R_{\lambda_k} - R_k} \sum_{j=k+1}^{\lambda_k} p_j q_{\lambda_k - j} \frac{1}{j+1} \sum_{v=0}^{k} (x_j - x_k) \leq -\varepsilon \right\} = 0,
\] (2.2)

and

\[
\inf_{0 < \lambda < 1} \limsup_{n \to \infty} \frac{1}{R_n} \left\{ k \leq R_n \colon \frac{1}{R_k - R_{\lambda_k}} \sum_{j=\lambda_k+1}^{k} p_j q_{k-j} \frac{1}{j+1} \sum_{v=0}^{k} (x_k - x_j) \leq -\varepsilon \right\} = 0.
\] (2.3)

REMARK 2.2. Let us suppose that \(st - \lim_k x_k = L; (x_n)\) is \(N_{p,q}^\infty C^1\) statistically convergent and relation (2.1) satisfies, then for every \(t > 1\), is valid the following relation:

\[
st - \lim_k \frac{1}{R_{\lambda_k} - R_k} \sum_{j=k+1}^{\lambda_k} p_j q_{\lambda_k - j} \frac{1}{j+1} \sum_{v=0}^{k} (x_j - x_k) = 0
\] (2.4)

and in case where \(0 < t < 1\),

\[
st - \lim_k \frac{1}{R_k - R_{\lambda_k}} \sum_{j=\lambda_k+1}^{k} p_j q_{k-j} \frac{1}{j+1} \sum_{v=0}^{k} (x_k - x_j) = 0.
\] (2.5)

In the next result we will consider the case where \(x = (x_n)\) is a sequence of complex numbers.

THEOREM 2.3. Let condition (2.1) be satisfied and let \((x_n)\) be a sequence of complex numbers which is \(N_{p,q}^\infty C^1\) summable to a finite number \(L\). Then \((x_n)\) is convergent to the same number \(L\) if and only if one of the following two conditions holds:

\[
\inf_{\lambda > 1} \limsup_{n \to \infty} \frac{1}{R_n} \left\{ k \leq R_n \colon \left\| \left( \frac{1}{R_{\lambda_k} - R_k} \sum_{j=k+1}^{\lambda_k} p_j q_{\lambda_k - j} \frac{1}{j+1} \sum_{v=0}^{k} (x_j - x_k) \right) \right\| \geq \varepsilon \right\} = 0,
\] (2.6)

and

\[
\inf_{0 < \lambda < 1} \limsup_{n \to \infty} \frac{1}{R_n} \left\{ k \leq R_n \colon \left\| \left( \frac{1}{R_k - R_{\lambda_k}} \sum_{j=\lambda_k+1}^{k} p_j q_{k-j} \frac{1}{j+1} \sum_{v=0}^{k} (x_k - x_j) \right) \right\| \geq \varepsilon \right\} = 0.
\] (2.7)
In what follows we list some auxiliary lemmas which are needful in the sequel.

**Lemma 2.4.** The condition given by relation (2.1) is equivalent to the condition

\[
\text{st} - \liminf_{n \to \infty} \frac{R_n}{R_{\lambda n}} > 1, \quad 0 < \lambda < 1.
\]  

(2.8)

**Proof.** Suppose that relation (2.1) is valid, \(0 < \lambda < 1\) and \(m = \lambda n = \lfloor \lambda n \rfloor, \ n \in \mathbb{N}\). Then it follows that

\[
\frac{1}{\lambda} > 1 \Rightarrow \frac{m}{\lambda} = \frac{\lfloor \lambda n \rfloor}{\lambda} \leq n.
\]

From above relation and definition of the sequences \((p_n)\) and \((q_n)\), we obtain:

\[
\frac{R_n}{R_{\lambda n}} \geq \frac{\lfloor m/\lambda \rfloor}{\lfloor m/\lambda \rfloor} \Rightarrow \text{st} - \liminf_{n \to \infty} \frac{R_n}{R_{\lambda n}} \geq \text{st} - \liminf_{n \to \infty} \frac{\lfloor m/\lambda \rfloor}{\lfloor m/\lambda \rfloor} > 1.
\]

Conversely, suppose that relation (2.8) is valid. Let \(\lambda > 1\) be given number and let \(\lambda_1\) be chosen such that \(1 < \lambda_1 < \lambda\). Set \(m = \lambda n = \lfloor \lambda n \rfloor\). From \(0 < \frac{1}{\lambda} < \frac{1}{\lambda_1} < 1\), it follows that:

\[
n \leq \frac{\lambda n - 1}{\lambda_1} < \frac{\lfloor \lambda n \rfloor}{\lambda_1} = \frac{m}{\lambda_1},
\]

provided \(\lambda_1 \leq \lambda - \frac{1}{n}\), which is a case where if \(n\) is large enough. Under this conditions we have:

\[
\frac{R_{\lambda n}}{R_n} \geq \frac{\lfloor m/\lambda \rfloor}{\lfloor m/\lambda \rfloor} \Rightarrow \text{st} - \liminf_{n \to \infty} \frac{R_{\lambda n}}{R_n} \geq \text{st} - \liminf_{n \to \infty} \frac{\lfloor m/\lambda \rfloor}{\lfloor m/\lambda \rfloor} > 1. \quad \square
\]

**Proposition 2.5.** Let us suppose that relation (2.1) is satisfied and let \(x = (x_k)\) be a sequence of complex numbers which is \(N_{p,q}^{m}C_{n}^{1}\) statistically convergent to \(L\). Then

\[
\text{st} - \lim_{n} \frac{1}{R_{\lambda n} - R_n} \sum_{j=n+1}^{\lambda n} \frac{p_j q_{n-j} 1}{j + 1} \sum_{v=0}^{j} x_v = L, \quad \text{for } \lambda > 1
\]  

(2.9)

and

\[
\text{st} - \lim_{n} \frac{1}{R_{\lambda n} - R_n} \sum_{j=n+1}^{\lambda n} \frac{p_j q_{n-j} 1}{j + 1} \sum_{v=0}^{j} x_v = L, \quad \text{for } 0 < \lambda < 1.
\]

(2.10)

**Proof.** (I) Let us consider the case where \(\lambda > 1\). Then we obtain

\[
\frac{1}{R_{\lambda n} - R_n} \sum_{k=n+1}^{\lambda n} P_k q_{\lambda n-k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - L)
\]

\[
= \frac{R_{\lambda n}}{R_{\lambda n} - R_n} \frac{1}{R_{\lambda n} - R_n} \sum_{k=0}^{n} P_k q_{\lambda n-k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - L) - \frac{R_n}{R_{\lambda n} - R_n} \frac{1}{R_{\lambda n} - R_n} \sum_{k=0}^{n} P_k q_{\lambda n-k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - L)
\]
we get relation (\( \lambda \)) for every \( R = R = R T R R - R R - R N \). From relation (2.11), definition of the sequence \((q_n)\), and relation
\[
\limsup_n \frac{R_{\lambda_n}}{R_{\lambda_n} - R_n} < \infty,
\]
we get relation (2.9).

(II) In this case we have that \( 0 < \lambda < 1 \). Then
\[
\frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - L)
\]
\[
= \frac{R_{\lambda_n}}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - L) - \frac{R_{\lambda_n}}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - L)
\]
\[
= \frac{R_{\lambda_n}}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k (q_{\lambda_n - k} - q_{\lambda_n - k}) \frac{1}{k+1} \sum_{v=0}^{k} (x_v - L).
\]

From relation (2.11), definition of the sequence \((q_n)\), and relation
\[
\limsup_n \frac{R_{\lambda_n}}{R_{\lambda_n} - R_n} < \infty,
\]
we get relation (2.9).

Proof of Theorem 2.1.

Necessity. Suppose that \( \lim_{n \to \infty} x_n = L \), and (2.1) holds. Following Proposition 2.5, we have
\[
\lim_{n \to \infty} \frac{1}{R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - x_n) = \lim_{n \to \infty} \left\{ \left( \frac{1}{R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v \right) - x_n \right\} = 0,
\]
for every \( \lambda > 1 \). In case where \( 0 < \lambda < 1 \), we find that
\[
\lim_{n \to \infty} \frac{1}{R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} (x_n - x_v)
\]
\[ \lim_{n \to \infty} \left\{ x_n - \left( \frac{1}{R_n - R_{n+1}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v \right) \right\} = 0. \]

**Sufficiency.** Assume that conditions (2.2) and (2.3) are satisfied. In what follows we will prove that \( \lim_{n \to \infty} x_n = L \). Given any \( \varepsilon > 0 \), by relation (2.2) we can choose \( \lambda_1 > 0 \) such that

\[ \liminf_{n \to \infty} \frac{1}{R_n - R_{\lambda_n+1}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - x_n) \geq -\varepsilon, \]  \hspace{1cm} (2.12)

where \( \lambda_n = [\lambda_n n] \). By the assumed summability \( N_{p,q}^{n} C_{\lambda_n}^{1} \) of \( (x_n) \), Proposition 2.5 and relation (2.12), we have

\[ \limsup_{n \to \infty} x_n \leq L + \varepsilon, \]  \hspace{1cm} (2.13)

for any \( \lambda > 1 \).

On the other hand, if \( 0 < \lambda < 1 \), for every \( \varepsilon > 0 \), we can choose \( 0 < \lambda_2 < 1 \) such that

\[ \liminf_{n \to \infty} \frac{1}{R_n - R_{\lambda_2 n+1}} \sum_{k=\lambda_2 n+1}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - x_n) \geq -\varepsilon, \]  \hspace{1cm} (2.14)

where \( \lambda_2 = [\lambda_2 n] \). By the assumed summability \( N_{p,q}^{n} C_{\lambda_2}^{1} \) of \( (x_n) \), Proposition 2.5 and (2.14), we have

\[ \liminf_{n \to \infty} x_n \geq L - \varepsilon, \]  \hspace{1cm} (2.15)

for any \( 0 < \lambda < 1 \).

Since \( \varepsilon > 0 \) is arbitrary, combining relations (2.13) and (2.15) we obtain

\[ \lim_{n \to \infty} x_n = L. \] \hspace{1cm} \square

**Proof of Theorem 2.3.**

**Necessity.** If both (1.2) and (1.6) hold, then Proposition 2.5 yields (2.6) for every \( \lambda > 1 \) and (2.7) for every \( 0 < \lambda < 1 \).

**Sufficiency.** Suppose that (1.2), (2.1) and one of the conditions (2.6) and (2.7) are satisfied. For any given \( \varepsilon > 0 \), there exists some \( \lambda_1 > 1 \) such that

\[ \limsup_{n \to \infty} \left| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=\lambda_n+1}^{n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - x_n) \right| \leq \varepsilon, \]

where \( \lambda_n = [\lambda_n n] \). Taking into account fact that \( (x_n) \) is \( N_{p,q} C_{\lambda_n}^{1} \) summable to \( L \) and Proposition 2.5, we get the following estimation

\[ \limsup_{n \to \infty} |L - x_n| \leq \limsup_{n \to \infty} \left| L - \frac{1}{R_{\lambda_n} - R_n} \sum_{k=\lambda_n+1}^{n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v \right|, \]
\[
+ \limsup_{n \to \infty} \left| \frac{1}{R_{\lambda_n1} - R_n} \sum_{k=n+1}^{\lambda_n1} p_k q_{\lambda_n1-k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - x_n) \right| \leq \varepsilon.
\]

For any given \(\varepsilon > 0\), there exists some \(0 < \lambda_2 < 1\) such that

\[
\limsup_{n \to \infty} \left| \frac{1}{R_n - R_{\lambda_n2}} \sum_{k=\lambda_n2+1}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} (x_n - x_v) \right| \leq \varepsilon,
\]

where \(\lambda_{n2} = [\lambda_2 n]\). Taking into account the fact that \((x_n)\) is \(N_{p,q}^n C_1^n\) summable to \(L\) and Proposition 2.5, we obtain the following

\[
\limsup_{n \to \infty} |L - x_n| \leq \limsup_n \left| L - \frac{1}{R_n - R_{\lambda_{n2}}} \sum_{k=\lambda_{n2}+1}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v \right| + \limsup_{n \to \infty} \left| \frac{1}{R_n - R_{\lambda_{n2}}} \sum_{k=\lambda_{n2}+1}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} (x_n - x_v) \right| \leq \varepsilon.
\]

Since \(\varepsilon > 0\) is arbitrary, in either case we get \(\lim_{n \to \infty} x_n = L\). \(\Box\)

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