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BALL–COVERING OF PRODUCT SPACES AND GÂTEAUX DIFFERENTIABILITY OF THE CENTER

AIYU NIE

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Abstract. In this paper, the author proves that if X_1, X_2 are Banach spaces, there exists a real number $\alpha > 0$ and a ball covering $\mathscr{B}_i$ of X_i such that $\mathscr{B}_i$ is α-off the origin and the ball-covering point is a norm Gâteaux differentiability point if and only if there exists a real number $\alpha > 0$ and a ball covering $\mathscr{B}$ of $(X_1 \times X_2, \|\cdot\|_\infty), (X_1 \times X_2, \|\cdot\|_p)$ such that $\mathscr{B}$ is α-off the origin and and the ball-covering point is a norm Gâteaux differentiability point.

1. Introduction and preliminaries

The study of space geometry began in 1932 with the publication of Banach book "Theoriedes operations lineaires", which was an important part of the discipline of generalized functional analysis. The geometric theory of Banach spaces is an integral part of the discipline of generalized functional analysis and has been used as a tool in other fields such as physics and chemistry, playing an important role in the development of other disciplines, intersection properties (MIP), non-tightness problems, use the ball as a direct object of study in [10].

Originating from the study of the coarse embedding problem of Banach spaces, in 2006, Cheng Lixin defined the ball-covering property of Banach spaces in [1]. The ball-covering property of Banach spaces is one of the important components of the geometric theory of Banach spaces. In 2007, Cheng Lixin, Cheng Qingjin and Liu Xiaoyan obtained the ball-covering property of Banach spaces X by constructing l^∞ equivalence parametrizations on spaces is neither linearly homogeneous nor invariant in [2]. In 2006, Cheng Bin proved that every Banach space realization can be made to correspond to a quotient space with the sphere covering property by reassigning a norm. In 2009, Fonf assigned a new norm to X to obtain the equivalence of X with the ball-covering property and X^* being ω^*-divisible in [4]. In 2009, Cheng Lixin gave that any super-self-reflexive space can be inscribed by studying the ball-covering property of its finite-dimensional subspaces in [3]. In 2016, Shang Shaoqiang studied the ball-covering property of product spaces in [8]. In 2017, Liu Jianglai extended this proof by removing the condition, and the conclusion still holds in [6]. In 2021, Shang Shaoqiang studied the ball-covering property of Banach sequence spaces in [9]. In

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this paper, based on the study of the above problems, we continue to explore the ball-covering property of the product space and investigate the Gâteaux differentiability of the ball-covering points.

Let $(X, \|\cdot\|)$ be a real Banach space. $S(X)$ and $B(X)$ denote the unit sphere and the unit ball of X, respectively. We denote the dual space of X by X^*. We denote the open ball centered at x and of radius $r > 0$ by $B(x,r)$. Let N, R and R^+ denote the set of natural numbers, reals and of nonnegative reals, respectively. Let D be a nonempty open convex subset of X and f be a real-valued continuous convex function on D. The continuous convex function f is said to be Gâteaux differentiable at the point x in D if there exists $df(x) \in X^*$ such that the limit

$$\langle df(x), y\rangle = \lim_{t\to 0} \frac{f(x+ty)-f(x)}{t} \tag{$*$}$$

exists for every $y \in X$. First, Let's recall the definitions used in the article.

DEFINITION 1.1. (see [7]) A Banach space X is Gâteaux differentiable if every convex continuous generalisation of it is Gâteaux differentiable at the points of a dense set of X.

DEFINITION 1.2. (see [1]) A Banach space X has the ball-covering property if its unit sphere can be covered by a countable number of balls that do not contain the origin, and we say that a ball covering $\mathscr{B}_i$ is α-off the origin when and only when

$$\inf\{\|x\| :\ x \in \cup\mathscr{B}\} \geqslant \alpha.$$

DEFINITION 1.3. (see [1]) A point $x_0^* \in A^*$ is said to be weak* exposed point of A^* if there exists $x \in S(X)$ such that $x_0^*(x) > x^*(x)$ whenever $x^* \in A^*\backslash\{x_0^*\}$.

DEFINITION 1.4. (see [8]) x is a smooth point if the point $x \in S(X)$ has a unique support generic function f_x. SmoX denotes the set consisting of all smooth points. If every $x \in S(X)$ is a smooth point, then X is smooth.

It is well known that a convex function $f(x) = \|x\|$ is Gâteaux differentiable on x_0 if $x_0 \in S(X)$ is a smooth point.

DEFINITION 1.5. (see [8]) Assuming that X_1, X_2 are Gâteaux differentiability spaces, the following statements are equivalent:

(1) X_1, X_2 have the ball-covering property;

(2) The product space $(X_1 \times X_2, ||\cdot||)$ has the ball- covering property, here

$$\|(x,y)\|_\infty = \max\{\|x\|, \|y\|\},\ \ \|(x,y)\|_p = \left(\|x\|_1^p + \|y\|_2^p\right)^{\frac{1}{p}},\ \ p \in [1,+\infty).$$

DEFINITION 1.6. (see [9]) If X_i is a Gâteaux differentiability space for any $i \in N$, then the following statements are equivalent:

(1) There exists a number $\alpha > 0$ and a ball covering $\mathscr{B}_i$ of X_i, such that $\mathscr{B}_i$ is α-off the origin;

(2) There exists a number $\alpha > 0$ and a ball covering $\mathscr{B}$ of $l^\infty(X_i)$, such that $\mathscr{B}$ is α-off the origin;

(3) There exists a number $\alpha > 0$ and a ball covering $\mathscr{B}$ of $l^1(X_i)$, such that $\mathscr{B}$ is α-off the origin.

LEMMA 1.7. (see [7]) *Suppose p is a Minkowski function defined on a space X. Then p is Gâteaux differentiable at the point x and has Gâteaux derivable x^* when and only when x^* is a weak $*$ of C^* at the point x exposed point, $\mathscr{C}^* = \{y \in X :\ p(y) \leqslant 1\}$.*

LEMMA 1.8. (see [9]) *Assuming that smooth points on Banach spaces are dense on $S(X)$, the following statements are equivalent.*

(1) There exists a number $\alpha > 0$ and a ball covering $\mathscr{B} = \{B(x_i, r_i)\}_{i=1}^\infty$, of X_i, such that $\mathscr{B}$ is α-off the origin and $\mathscr{B} = \{B(x_i, r_i)\}_{i=1}^\infty$, is a smooth point of X.

(2) There exists a number $\alpha > 0$ and a ball covering $B = \{B(x_i, r_i)\}_{i=1}^\infty$ of X_i, such that $\mathscr{B}$ is α-off the origin.

(3) There exists a sequence $\{x_n\}_{n=1}^\infty$ of weak exposure points of $B^*(X)$ such that*

$$\inf_{x \in S(X)} \sup_{n \in N} \langle x_n^*, x \rangle > 0.$$

LEMMA 1.9. (see [7]) *Suppose X_i is a Gâteaux differentiability space for any $i \in \{1,2,3\ldots k\}$, there exists a real number $\alpha > 0$ and a ball covering $\mathscr{B}_i$ of X_i such that $\mathscr{B}_i$ is α-off the origin, then there exists a ball covering $\mathscr{B}$ of $(X_1 + X_2 \ldots X_k, \|\cdot\|)$ such that $\mathscr{B}$ is $\alpha/4$-off the origin, here*

$$\|(x_1, \cdots x_k)\|_1 = (\|x_1\| + \|x_2\| \cdots \|x_k\|).$$

2. Ball-covering of product spaces and gateaux differentiability of the ball-covering point

THEOREM 2.1. *Suppose X_1, X_2 are Banach spaces, there exists a real number $\alpha > 0$ and a ball covering $\mathscr{B}_i$, $i \in \{1,2\}$ of X_i such that $\mathscr{B}_i$ is α-off the origin and the ball-covering point is a norm Gâteaux differentiability point when and only when there exists a real number $\alpha > 0$ and a ball covering $\mathscr{B}$ of $(X_1 \times X_2, \|\cdot\|_\infty)$, $(X_1 \times X_2, \|\cdot\|_p)$ such that $\mathscr{B}$ is α-off the origin and the ball-covering point is a norm Gâteaux differentiability point, here $\|(x,y)\|_\infty = \max\{\|x\|, \|y\|\}, \|(x,y)\|_p = \left(\|x\|_1^p + \|y\|_2^p\right)^{\frac{1}{p}}$, $p \in [1, +\infty)$.*

In order to prove the theorem, we give some lemmas.

LEMMA 2.2. *Assuming that X is a Banach space, the following statements are equivalent*

(1) X has a ball-covering $\{B(x_n, r_n)\}_{n=1}^\infty$ such that the ball-covering point is a norm Gâteaux differentiability point and $\{B(x_n, r_n)\}_{n=1}^\infty$ is α-off the origin;

(2) there exists a sequence $\{B(x_n,r_n)\}_{n=1}^{\infty}$ *such that* x_n *is a norm Gâteaux differentiability point, and*

$$\inf_{x\in S(X)}\left(\sup_{n\in N}\langle d_G\|x_n\|,x\rangle\right)>0.$$

Proof. (2) $\Rightarrow$ (1) For any $\alpha\in\left(0,\inf_{x\in S(X)}\left(\sup_{n\in N}\langle d_G\|x_n\|,x\rangle\right)\right)$, let

$$B_{i,\mu}=B\left((\alpha+\mu)x_i,\mu-\frac{1}{\mu}\right),\ \ i=1,2\ldots,\ \ \mu=1,2\ldots$$

We get every $B_{i,\mu}$ is $\alpha+1/\mu$ away from the origin. It can be asserted that

$$S(X)\subset\cup\left\{B_{i,\mu}:i=1,2\ldots,\mu=1,2\ldots\right\}.$$

Indeed, since $\alpha\in\left(0,\inf_{x\in S(X)}\left(\sup_{n\in N}\langle d_G\|x_n\|,x\rangle\right)\right)$, for any $y\in S(X)$ there exists

$$d_G\|x\|\in\{d_G\|x_n\|\}_{n=1}^{\infty},\ \ r_y>0$$

such that

$$d_G\|x\|(y)\geqslant(\alpha+r_y)\|y\|=\alpha+r_y>0.$$

Without loss of generality, for $1\leqslant j<+\infty$, take $d_G\|x\|=d_G\left\|x_j\right\|$ and let $\gamma=d_G\left\|x_j\right\|(y)$, then

$$\gamma\geqslant\alpha+r_y.$$

Let $Q_j=\left\{x\in X:d_G\left\|x_j\right\|(x)=0\right\}$, Then there exists $q_j\in Q_j$ such that

$$y=\gamma x_j+q_j.$$

First, it is necessary to show that $y\in\cup_{\mu=1}^{\infty}B_{j,\mu}$, otherwise for any $\mu\in N$, we have

$$\mu-\frac{1}{\mu}\leqslant\left\|(\alpha+\mu)x_j-y\right\|=\left\|(\alpha+\mu-\gamma)x_j-q_j\right\|.$$

This shows that

$$\begin{aligned}-\frac{1}{\mu}&\leqslant\left\|(\alpha+\mu-\gamma)x_j-q_j\right\|-\mu\\&=\left\|(\alpha+\mu-\gamma)x_j-q_j\right\|-\mu\left\|x_j\right\|\\&\leqslant\left\|(\mu-\gamma)x_j-q_j\right\|-\mu\left\|x_j\right\|+\alpha\left\|x_j\right\|\\&=(\mu-\gamma)\left[\left\|x_j-\frac{1}{\mu-\gamma}q_j\right\|-\left\|x_j\right\|\right]-\gamma+\alpha\\&=\frac{\left\|x_j-tq_j\right\|-\left\|x_j\right\|}{t}-(\gamma-\alpha).\end{aligned}$$

Here $t=1/\mu-\gamma$, so we get

$$0\leqslant \left\|x_j\right\|'(q_j)=d_G\left\|x_j\right\|(x)-(\gamma-\alpha)=-(\gamma-\alpha)\leqslant -r_y\leqslant 0.$$

Contradicts the fact, it follows that $S(X)\subset\cup\left\{B_{i,\mu}:i=1,2\ldots,\mu=1,2\ldots\right\}$. Hence this shows that for any $\alpha\in\left(0,\inf\limits_{x\in S(X)}\left(\sup\limits_{n\in N}\langle d_G\|x_n\|,x\rangle\right)\right)$, X has a ball covering $\{B(x_n,r_n)\}_{n=1}^{\infty}$ such that the ball-covering point is a norm Gâteaux differentiable point and $\{B(x_n,r_n)\}_{n=1}^{\infty}$ is α-off the origin.

(1) $\Rightarrow$ (2) Let X have a ball-covering $\mathscr{B}=\{B(x_n,r_n)\}_{n=1}^{\infty}$, $\mathscr{B}=\{B(x_n,r_n)\}_{n=1}^{\infty}$ is α-off the origin and $\{x_n\}_{n=1}^{\infty}$ is norm Gâteaux differentiability point then

$$\|x_n-0\|-r_n\geqslant\alpha.$$

If

$$\inf_{x\in S(X)}\left(\sup_{n\in N}\langle d_G\|x_n\|,x\rangle\right)\leqslant\frac{\alpha}{2}.$$

Then there exists $x_0\in S(X)$ such that

$$\sup_{n\in N}\langle d_G\|x_n\|,x_0\rangle\leqslant\frac{\alpha}{2}.$$

Let $y_n\in\{x_n\}_{n=1}^{\infty}$, $\mathscr{B}_1=\{B(y_n,r_n)\}_{n=1}^{\infty}$ is a ball-covering of X and $\mathscr{B}=\{B(x_n,r_n)\}_{n=1}^{\infty}$ is α-off the origin, by Hahn-Banach theorem there exists $d_G\|x_n\|=x_n^*\in S(X^*)$ such that $d_G\|x_n\|(y_n)=\|y_n\|$, and y_n is a norm Gâteaux differentiability point and

$$\|y_n-0\|-r_n\geqslant\alpha.$$

Since $S(X)\subset\bigcup\limits_{n=1}^{\infty}B(y_n,r_n)$, there exists $j\in N$ such that $x_0\in B(y_j,r_j)$. Since

$$d_G\left\|x_j\right\|(y_j)=\left\|y_j\right\|$$

$$\sup_{n\in N}\langle d_G\|x_n\|,x_0\rangle\leqslant\frac{\alpha}{2}$$

we have

$$r_j\geqslant\left\|y_j-x_0\right\|\geqslant d_G\left\|x_j\right\|(y_j-x_0)=\left\|y_j\right\|-d_G\left\|x_j\right\|(x_0)\geqslant\left\|y_j\right\|-\frac{\alpha}{2}.$$

This shows that

$$\left\|y_j\right\|\leqslant r_j+\frac{\alpha}{2}<r_j+\alpha.$$

This contradicts the fact. Hence $\mathscr{B}_1=\{B(y_n,r_n)\}_{n=1}^{\infty}$ is a ball-covering of X and $\mathscr{B}_1=\{B(y_n,r_n)\}_{n=1}^{\infty}$ is α-off the origin, hence

$$\inf_{x\in S(X)}\left(\sup_{n\in N}\langle d_G\|x_n\|,x\rangle\right)>\frac{\alpha}{2}.\quad\square$$

Next prove Theorem 2.1.

Proof. First prove the sufficiency of the theorem. For greater clarity, we divide the proof into two parts.

Firstly, when
$$\|(x_1, x_2)\|_\infty = \max\{\|x_1\|, \|x_2\|\}$$
for any $(x_1, x_2) \in X_1 \times X_2$.

Since there exists a real number $\alpha > 0$ and a ball-covering $\mathscr{B}_i$, $i \in \{1,2\}$ of X_i such that $B_i = \{B(x_n, r_n)\}_{n=1}^\infty$ is α-off the origin and the ball-covering point is a norm Gâteaux differentiability point, by Lemma 2.2, there exists a sequence $\{x_{i,n}\}_{n=1}^\infty$, such that $x_{i,n}$ is a norm Gâteaux differentiability point and
$$\inf_{x_i \in S(X_i)} \left(\sup_{n \in N} \langle d_G \|x_{i,n}\|, x_i \rangle \right) > 0, \;\; i \in \{1,2\}.$$
Then for any $y_1 \in X_1$, $y_2 \in X_2$ such that
$$\lim_{t \to 0} \frac{1}{t}[\|x_{1,n} + ty_1\| + \|x_{1,n} - ty_1\| - 2\|x_{1,n}\|] = 0$$
$$\lim_{t \to 0} \frac{1}{t}[\|x_{2,n} + ty_1\| + \|x_{2,n} - ty_1\| - 2\|x_{2,n}\|] = 0.$$
Hence there exists a sequence $\{(x_{1,n}, x_{2,n})\}_{n=1}^\infty \in X_1 \times X_2$, $(y_1, y_2) \in X_1 \times X_2$, without loss of generality, and assume that $\|x_{1,n}\| > \|x_{2,n}\|$ then there exists $t_0(y_1, y_2) > 0$ such that
$$\|x_{1,n} + ty_1\| > \|x_{2,n} + ty_2\|$$
$$\|x_{1,n} - ty_1\| > \|x_{2,n} - ty_2\|.$$
Whenever $|t| < t_0(y_1, y_2)$, then
$$\begin{aligned}
&\lim_{t \to 0} \frac{1}{t}\Big[\big\|(x_{1,n}, x_{2,n}) + t(y_1, y_2)\big\|_\infty + \big\|(x_{1,n}, x_{2,n}) - t(y_1, y_2)\big\|_\infty - 2\|(x_{1,n}, x_{2,n})\|_\infty\Big] \\
&= \lim_{t \to 0} \frac{1}{t}\Big[\|(x_{1,n} + ty_1, x_{2,n} + ty_2)\|_\infty + \|(x_{1,n} - ty_1, x_{2,n} - ty_2)\|_\infty - 2\|(x_{1,n}, x_{2,n})\|_\infty\Big] \\
&= \lim_{t \to 0} \Big[\max\{\|x_{1,n} + ty_1\|, \|x_{2,n} + ty_2\|\} + \max\{\|x_{1,n} - ty_1\|, \|x_{2,n} - ty_2\| \\
&\qquad - 2\max\{\|x_{1,n}\|, \|x_{2,n}\|\}\Big] \\
&= \lim_{t \to 0} \|x_{1,n} + ty_1\| + \|x_{1,n} - ty_1\| - 2\|x_{1,n}\| = 0.
\end{aligned}$$

Hence there exists a sequence $\{(x_{1,n}, x_{2,n})\}_{n=1}^\infty \in X_1 \times X_2$ and $(x_{1,n}, x_{2,n})$) is a norm Gâteaux differentiability point. Let
$$d_G\|x_{1,n}\|(x_{1,n}) = 1, \;\; \|x_{1,n}\| = 1$$
$$d_G\|x_{2,n}\|(x_{2,n}) = 1, \;\; \|x_{2,n}\| = 1.$$

Then

$$(d_G\|x_{1,n}\|,0),\ \ (0,d_G\|x_{2,n}\|)\in B((X_1\times X_2)^*)$$
$$\langle (d_G\|x_{1,n}\|,0),(x_{1,n},0)\rangle = d_G\|x_{1,n}\|(x_{1,n})=1$$
$$\langle (0,d_G\|x_{2,n}\|),(0,x_{2,n})\rangle = d_G\|x_{2,n}\|(x_{2,n})=1.$$

Then for $(x_1,x_2)\in S(X_1\times X_2)$, let

$$\begin{aligned}&\{d_G\|(x_{1,n},x_{2,n})\|\}_{n=1}^{\infty}\\ =&\{(d_G\|x_{1,n}\|,0)\}_{n=1}^{\infty}\cup\{(-d_G\|x_{1,n}\|,0)\}_{n=1}^{\infty}\cup\{(0,d_G\|x_{2,n}\|)\}_{n=1}^{\infty}\\ &\cup\{(0,-d_G\|x_{2,n}\|)\}_{n=1}^{\infty}\end{aligned}$$

because

$$\inf_{x_i\in S(X_i)}\left(\sup_{n\in N}\langle d_G\|x_{i,n}\|,x_i\rangle\right)>0,\ \ i\in\{1,2\}.$$

Therefore

$$\inf_{(x_1,x_2)\in S(X_1\times X_2)}\left(\sup_{n\varepsilon N}\langle d_G\|(x_{1,n},x_{2,n})\|,(x_1,x_2)\rangle\right)>0.$$

Therefore, it follows from Lemma 2.2, that there exists a real number $\alpha>0$ and a ball-covering $\mathscr{B}$ of $(X_1\times X_2,\|\cdot\|_\infty)$ such that $\mathscr{B}$ is α -off the origin and the ball-covering point is a norm Gâteaux differentiability point.

Secondly, when

$$\|(x_1,x_2)\|_p=\left(\|x_1\|_1^p+\|x_2\|_2^p\right)^{\frac{1}{p}},\ \ p\in[1,+\infty)$$

for any $(x_1,x_2)\in X_1\times X_2$. It is divided into the following two cases.

Case 1. $p=1$

Since there exists a real number $\alpha>0$ and a ball-covering $\mathscr{B}_i$, $i\in\{1,2\}$ of X_i such that $B_i=\{B(x_n,r_n)\}_{n=1}^{\infty}$ is α -off the origin and the ball-covering point is a norm Gâteaux differentiability point, by Lemma 2.2, there exists a sequence $\{x_{i,n}\}_{n=1}^{\infty}$, such that $x_{i,n}$ is a norm Gâteaux differentiability point and

$$\inf_{x_i\in S(X_i)}\left(\sup_{n\in N}\langle d_G\|x_{i,n}\|,x_i\rangle\right)>0,\ \ i\in\{1,2\}.$$

Then for any $y_1\in X_1, y_2\in X_2$ such that

$$\lim_{t\to 0}\frac{1}{t}[\|x_{1,n}+ty_1\|+\|x_{1,n}-ty_1\|-2\|x_{1,n}\|]=0$$
$$\lim_{t\to 0}\frac{1}{t}[\|x_{2,n}+ty_1\|+\|x_{2,n}-ty_1\|-2\|x_{2,n}\|]=0.$$

Thus there exists a sequence $\{(x_{1,n},x_{2,n})\}_{n=1}^{\infty}\in X_1\times X_2$ and $(x_{1,n},x_{2,n})$ is a norm Gâteaux differentiable point. Let

$$d_G\|x_{1,n}\|(x_{1,n})=1,\ \ \|x_{1,n}\|=1$$
$$d_G\|x_{2,n}\|(x_{2,n})=1,\ \ \|x_{2,n}\|=1.$$

Then

$$\lim_{t\to 0}\frac{1}{t}[\|(x_{1,n},x_{2,n})+t(y_1,y_2)\|_1+\|(x_{1,n},x_{2,n})-t(y_1,y_2)\|_1-2\|(x_{1,n},x_{2,n})\|_1]$$
$$=\lim_{t\to 0}\frac{1}{t}\left[\|(x_{1,n}+ty_1,x_{2,n}+ty_2)\|_1+\|(x_{1,n}-ty_1,x_{2,n}-ty_2)\|_1\,\|(x_{1,n},x_{2,n})\|_1\right]$$
$$=\lim_{t\to 0}\frac{1}{t}[\|x_{1,n}+ty_1\|+\|x_{2,n}+ty_2\|+\|x_{1,n}-ty_1\|+\|x_{2,n}-ty_2\|-2\|x_{1,n}\|-2\|x_{2,n}\|]$$
$$=\lim_{t\to 0}\frac{1}{t}[\|x_{1,n}+ty_1\|+\|x_{1,n}-ty_1\|-2\|x_{1,n}\|]$$
$$+\lim_{t\to 0}\frac{1}{t}[\|x_{2,n}+ty_1\|+\|x_{2,n}-ty_1\|-2\|x_{2,n}\|]$$
$$=0+0=0.$$

Then $(d_G\|x_{1,n}\|,d_G\|x_{2,n}\|)\in B\left((X_1\times X_2)^*\right)$. And

$$\begin{aligned}&(d_G\|x_{1,n}\|,d_G\|x_{2,n}\|)(x_{1,n},x_{2,n})\\&=d_G\|x_{1,n}\|(x_{1,n})+d_G\|x_{2,n}\|(x_{2,n})\\&=\|x_{1,n}\|+\|x_{2,n}\|\\&=\|(x_{1,n},x_{2,n})\|_p.\end{aligned}$$

Then for $(x_1,x_2)\in S(X_1\times X_2)$, let

$$\begin{aligned}&\{d_G\|(x_{1,n},x_{2,n})\|\}_{n=1}^{\infty}\\&=\{(d_G\|x_{1,n}\|,d_G\|x_{2,n}\|)\}_{n=1}^{\infty}\cup\{(-d_G\|x_{1,n}\|,d_G\|x_{2,n}\|)\}_{n=1}^{\infty}\\&\cup\{(d_G\|x_{1,n}\|,d_G\|x_{2,n}\|)\}_{n=1}^{\infty}\cup\{(-d_G\|x_{1,n}\|,-d_G\|x_{2,n}\|)\}_{n=1}^{\infty}.\end{aligned}$$

Thus it is easy to obtain that when $(x_1,x_2)\in S(X_1\times X_2)$

$$\inf_{(x_1,x_2)\in S(X_1\times X_2)}\left(\sup_{n\in N}\langle d_G\|(x_{1,n},x_{2,n})\|,(x_1,x_2)\rangle\right)>0.$$

Therefore, by Lemma 2.2 we get that there exists a real number $\alpha>0$ and a ball-covering $\mathscr{B}$ of $(X_1\times X_2,\|\cdot\|_1)$ such that $\mathscr{B}$ is α-off the origin and the ball-covering point is a norm Gâteaux differentiability point.

Case 2. $p\in(1,+\infty)$

Since there exists a real number $\alpha>0$ and a ball-covering $\mathscr{B}_i$, $i\in\{1,2\}$ of X_i such that $B_i=\{B(x_n,r_n)\}_{n=1}^{\infty}$ is α-off the origin and the ball-covering point is a norm Gâteaux differentiability point, by Lemma 2.2 there exists a sequence $\{x_{i,n}\}_{n=1}^{\infty}$, such that $x_{i,n}$ is a norm Gâteaux differentiability point and

$$\inf_{x_i\in S(X_i)}\left(\sup_{n\in N}\langle d_G\|x_{i,n}\|,x_i\rangle\right)>0,\quad i\in\{1,2\}.$$

We assert that the convex function $f(x,y)=\|x\|$ is Gâteaux differentiable at $\{(x_{1,n},x_{2,n})\}_{n=1}^{\infty}$, and in fact it is Gâteaux differentiable on the set $D_1\times X_2$, where

$D_1 = \left\{x \in \{x_{1,n}\}_{i=1}^{\infty}, \|x\|_1 \text{ is Gâteaux differentiable at } x\right\}$. Similarly, there exists D_2 where $f(x,y) = \|y\|$ is Gâteaux differentiable at $X_1 \times D_2$, with the chain rule for differentiation, and we get

$$\|(x,y)\|_p = \left(\|x\|_1^p + \|y\|_2^p\right)^{\frac{1}{p}}, \ \ p \in (1,+\infty).$$

Every point on $D_1 \times D_2$ is Gâteaux differentiable. Therefore there exists a sequence $\{(x_{1,n}, x_{2,n})\}_{n=1}^{\infty} \in X_1 \times X_2$ is norm Gâteaux differentiable. Let

$$d_G\|x_{1,n}\|(x_{1,n}) = 1, \ \ \|x_{1,n}\| = 1$$
$$d_G\|x_{2,n}\|(x_{2,n}) = 1, \ \ \|x_{2,n}\| = 1$$

then

$$\left(\frac{1}{2^{1/q}} d_G\|x_{1,n}\|, \frac{1}{2^{1/q}} d_G\|x_{2,n}\|\right) \in S\left((X_1 \times X_2)^*\right)$$

and

$$\begin{aligned}
&\left(\frac{1}{2^{1/q}} d_G\|x_{1,n}\|, \frac{1}{2^{1/q}} d_G\|x_{2,n}\|\right)(x_{1,n}, x_{2,n}) \\
&= \frac{1}{2^{1/q}} d_G\|x_{1,n}\|\left(x_{1,n}\right) + \frac{1}{2^{1/q}} d_G\|x_{2,n}\|\left(x_{2,n}\right) \\
&= \frac{1}{2^{1/q}}\left(\|d_G\|x_{1,n}\|\|^q + \|d_G\|x_{2,n}\|\|^q\right)^{1/q}\left(\|x_{1,n}\|^p + \|x_{2,n}\|^p\right)^{1/p} \\
&= \frac{1}{2^{1/q}} \cdot \frac{1}{2^{1/p}} \|(x_{1,n}, x_{2,n})\|_p \\
&= \|(x_{1,n}, x_{2,n})\|_p.
\end{aligned}$$

Here $1/p + 1/q = 1$. Take

$$\begin{aligned}
&\{d_G\|(x_{1,n}, x_{2,n})\|\}_{n=1}^{\infty} \\
= \ &\left\{\left(\frac{1}{2^{1/q}} d_G\|x_{1,n}\|, \frac{1}{2^{1/q}} d_G\|x_{2,n}\|\right)\right\}_{n=1}^{\infty} \\
&\cup \left\{\left(-\frac{1}{2^{1/q}} d_G\|x_{1,n}\|, \frac{1}{2^{1/q}} d_G\|x_{2,n}\|\right)\right\}_{n=1}^{\infty} \\
&\cup \left\{\left(\frac{1}{2^{1/q}} d_G\|x_{1,n}\|, \frac{1}{2^{1/q}} d_G\|x_{2,n}\|\right)\right\}_{n=1}^{\infty} \\
&\cup \left\{\left(-\frac{1}{2^{1/q}} d_G\|x_{1,n}\|, -\frac{1}{2^{1/q}} d_G\|x_{2,n}\|\right)\right\}_{n=1}^{\infty}.
\end{aligned}$$

Thus it is easy to obtain that when $(x_1, x_2) \in S(X_1 \times X_2)$

$$\inf_{(x_1,x_2) \in S(X_1 \times X_2)} \left(\sup_{n \in N} \langle d_G\|(x_{1,n}, x_{2,n})\|, (x_1, x_2)\rangle\right) > 0.$$

Therefore, by Lemma 2.2 we get that there exists a real number $\alpha > 0$ and a ball covering $\mathscr{B}$ of $(X_1 \times X_2, \|\cdot\|_p)$ such that $\mathscr{B}$ is α-off the origin and the ball covering point is a norm Gâteaux differentiability point.

Next, prove the necessity of the theorem.

There exists a real number $\alpha > 0$ and a ball covering $\mathscr{B}$ of $X_1 \times X_2$ such that $\mathscr{B}$ is α-off the origin and the ball covering point is a norm Gâteaux differentiability point, and by Lemma 2.2, there exists a sequence $\{(x_{1,n},x_{2,n})\}_{n=1}^{\infty} X_1 \times X_2$ and $(x_{1,n},x_{2,n})$ is a norm Gâteaux differentiability point and

$$\inf_{(x_1,x_2)\in S(X_1\times X_2)} \left(\sup_{n\in N} \langle d_G \|(x_{1,n},x_{2,n})\|, (x_1,x_2)\rangle \right) > 0$$

Firstly, when

$$\|(x_1,x_2)\|_{\infty} = \max\{\|x_1\|, \|x_2\|\}$$

for any $(x_1,x_2) \in X_1 \times X_2$.

Since, there exists a sequence $\{(x_{1,n},x_{2,n})\}_{n=1}^{\infty}$ such that $\{(x_{1,n},x_{2,n})\}_{n=1}^{\infty}$ is norm Gâteaux differentiable, for any $(x,y) \in X_1 \times X_2$, we have

$$\lim_{t\to 0} \frac{1}{t} [\|(x_{1,n},x_{2,n}) + t(x,y)\| + \|(x_{1,n},x_{2,n}) - t(x,y)\| - 2\|(x_{1,n},x_{2,n})\|] = 0.$$

Without loss of generality we assume that $\|x_{1,n}\| > \|x_{2,n}\|$, then there exists a sequence $\{x_{1,n}\}_{n=1}^{\infty}$ of X_1, when

$$\begin{aligned}
&\frac{1}{t}[\|x_{1,n}+tx\| + \|x_{1,n}-tx\| - 2\|x_{1,n}\|] \\
&= \frac{1}{t}\big[\max\{\|x_{1,n}+tx\|, \|x_{2,n}+ty\|\} + \max\{\|x_{1,n}-tx\|, \|x_{2,n}-ty\|\} \\
&\quad - 2\max\{\|x_{1,n}\|, \|x_{2,n}\|\}\big] \\
&= \frac{1}{t}\Big[\|(x_{1,n},x_{2,n}) + t\big(x,y\big)\|_{\infty} + \|(x_{1,n},x_{2,n}) - t\big(x,y\big)\|_{\infty} - 2\|(x_{1,n}, x_{2,n})\|_{\infty}\Big] = 0.
\end{aligned}$$

At the same time there exists a sequence $\{x_{2,n}\}_{n=1}^{\infty}$ of X_2,we have

$$\begin{aligned}
&\frac{1}{t}[\|x_{2,n}+ty\| + \|x_{2,n}-ty\| - 2\|x_{2,n}\|] \\
&= \frac{1}{t}[\|(0,x_{2,n}) + t(0,y)\|_{\infty} + \|(0,x_{2,n}) - t(0,y)\|_{\infty} - 2\|(0,x_{2,n})\|_{\infty}] \\
&= \frac{1}{t}\Big[\big\|\big(x_{1,n}-x_{1,n},x_{2,n}\big) + t\big(x-x,y\big)\big\|_{\infty} + \|(x_{1,n}-x_{1,n},x_{2,n}) - t(x-x,y)\|_{\infty} \\
&\quad - 2\|(x_{1,n}-x_{1,n},x_{2,n})\|_{\infty}\Big] \\
&= \frac{1}{t}\Big[\|(x_{1,n},x_{2,n}) - (x_{1,n},0) + t(x,y) - t(x,0)\|_{\infty} \\
&\quad + \|(x_{1,n},x_{2,n}) - (x_{1,n},0) - t(x,y) + t(x,0)\|_{\infty} - 2\|(x_{1,n},x_{2,n}) - (x_{1,n},0)\|_{\infty}\Big] \\
&\leqslant \frac{1}{t}\Big[\|(x_{1,n},x_{2,n}) + t(x,y)\|_{\infty} + \|(x_{1,n},x_{2,n}) - t(x,y)\|_{\infty} \\
&\quad - 2\|(x_{1,n},x_{2,n})\|_{\infty} + \|(x_{1,n},0) + t(x,0)\|_{\infty} \\
&\quad + \|(x_{1,n},0) + t(x,0)\|_{\infty} - 2\|(x_{1,n},0)\|_{\infty}\Big]
\end{aligned}$$

$$\leqslant \frac{1}{t}\Big[\|(x_{1,n},x_{2,n})+t(x,y)\|_\infty+\|(x_{1,n},x_{2,n})-t(x,y)\|_\infty$$
$$-2\|(x_{1,n},x_{2,n})\|_\infty+\|x_{1,n}+tx\|+\|x_{1,n}-tx\|-2\|x_{1,n}\|\Big]$$
$$\leqslant \frac{1}{t}[\|(x_{1,n},x_{2,n})+t(x,y)\|_\infty+\|(x_{1,n},x_{2,n})-t(x,y)\|_\infty-2\|(x_{1,n},x_{2,n})\|_\infty]$$
$$+\frac{1}{t}[\|x_{1,n}+tx\|+\|x_{1,n}-tx\|-2\|x_{1,n}\|]$$
$$=0+0=0.$$

Thus there exists a sequence $\{x_{1,n}\}_{n=1}^\infty$ of X_1, a sequence $\{x_{2,n}\}_{n=1}^\infty$ of X_2 with $\{x_{1,n}\}_{n=1}^\infty$ and $\{x_{2,n}\}_{n=1}^\infty$ is a norm Gâteaux differentiability point.

For any $x_1 \in S(X_1)$, we have

$$\begin{aligned}
&\inf_{x_1\in S(X_1)}\sup_{n\in N} d_G\|x_{1,n}\|(x_1)\\
&=\inf_{x_1\in S(X_1)}\sup_{n\in N}\big(d_G\big\|x_{1,n}\big\|,0\big)(x_1,0)\\
&=\inf_{x_1\in S(X_1)}\sup_{n\in N}\langle d_G\|\big(x_{1,n},x_{2,n}\big)\|,(x_1,0)\rangle\\
&>0.
\end{aligned}$$

Similarly, for an arbitrary $x_2 \in S(X_2)$, we have

$$\begin{aligned}
&\inf_{x_2\in S(X_2)_{n\in N}}\sup_{G}\|x_{2,n}\|(x_2)\\
&=\inf_{x_2\in S(X_2)}\sup_{n\in N}\big(0,d_G\big\|x_{2,n}\big\|\big)(0,x_2)\\
&=\inf_{x_2\in S(X_2)}\sup_{n\in N}\langle d_G\|\big(x_{1,n},x_{2,n}\big)\|,(0,x_2)\rangle\\
&>0.
\end{aligned}$$

Therefore, by Lemma 2.2 we get that there exists a real number $\alpha>0$ and a ball covering $\mathscr{B}$ of X_1,X_2 such that $\mathscr{B}$ is α-off the origin and the ball covering point is a norm Gâteaux differentiability point.

Secondly, when

$$\big(\|x\|_1^p+\|y\|_2^p\big)^{\frac{1}{p}},\quad p\in(1,+\infty).$$

Case 1. $p=1$

Since there exists a ball-covering $\mathscr{B}=\{B(x_{1,n},x_{2,n}),r_n\}_{n=1}^\infty$ of $(X_1\times X_2,\|\cdot\|_p)$ such that the ball-covering point is a norm Gâteaux differentiability point, for an arbitrary $(x,y)\in X_1\times X_2$

$$\lim_{t\to 0}\frac{1}{t}\left[\left\|(x_{1,n},x_{2,n})+t(x,y)\right\|_1+\left\|(x_{1,n},x_{2,n})-t(x,y)\right\|_1-2\left\|(x_{1,n},x_{2,n})\right\|_1\right]=0$$
$$=\lim_{t\to 0}\frac{1}{t}[\|(x_{1,n}+tx,x_{2,n}+ty)\|_1+\|(x_{1,n}-tx,x_{2,n}-ty)\|_1-2\|(x_{1,n},x_{2,n})\|_1]$$
$$=\lim_{t\to 0}\frac{1}{t}[\|x_{1,n}+tx\|+\|x_{2,n}+ty\|+\|x_{1,n}-tx\|+\|x_{2,n}-ty\|-2\|x_{1,n}\|-2\|x_{2,n}\|]$$
$$=\lim_{t\to 0}\frac{1}{t}[\|x_{1,n}+tx\|+\|x_{1,n}-tx\|-2\|x_{1,n}\|]$$
$$+\lim_{t\to 0}\frac{1}{t}[\|x_{2,n}+ty\|+\|x_{2,n}-ty\|-2\|x_{2,n}\|]$$

hence

$$\lim_{t\to 0}\frac{1}{t}[\|x_{1,n}+tx\|+\|x_{1,n}-tx\|-2\|x_{1,n}\|]$$
$$=\lim_{t\to 0}\frac{1}{t}[\|x_{2,n}+ty\|+\|x_{2,n}-ty\|-2\|x_{2,n}\|]=0.$$

Thus there exists a sequence $\{x_{1,n}\}_{n=1}^{\infty}$ of X_1, a sequence $\{x_{2,n}\}_{n=1}^{\infty}$ of X_2 with $\{x_{1,n}\}_{n=1}^{\infty}$, $\{x_{2,n}\}_{n=1}^{\infty}$ are norm Gâteaux differentiability points.

Thus for any $x_1\in S(X_1)$, we have

$$\inf_{x_1\in S(X_1)}\sup_{n\in N} d_G\|x_{1,n}\|(x_1)$$
$$=\inf_{x_1\in S(X_1)}\sup_{n\in N}\left(d_G\left\|x_{1,n}\right\|,0\right)(x_1,0)$$
$$=\inf_{x_1\in S(X_1)}\sup_{n\in N}\langle d_G\|\left(x_{1,n},x_{2,n}\right)\|,(x_1,0)\rangle$$
$$>0.$$

Similarly, for an arbitrary $x_2\in S(X_2)$, we have

$$\inf_{x_2\in S(X_2)_{n\in N}}\sup_{G}\|x_{2,n}\|(x_2)$$
$$=\inf_{x_2\in S(X_2)}\sup_{n\in N}\left(0,d_G\left\|x_{2,n}\right\|\right)(0,x_2)$$
$$=\inf_{x_2\in S(X_2)}\sup_{n\in N}\langle d_G\|\left(x_{1,n},x_{2,n}\right)\|,(0,x_2)\rangle$$
$$>0.$$

Hence, by Lemma 2.2, when $p=1$ there exists a real number $\alpha>0$ and X_1, X_2 of a ball covering $\mathscr{B}$ such that $\mathscr{B}$ is α-off the origin and the ball covering point is a norm Gâteaux differentiability point.

Case 2. $p\in(1,+\infty)$

Since there exists a ball covering $\mathscr{B}=\{B(x_{1,n},x_{2,n}),r_n\}_{n=1}^{\infty}$ of $(X_1\times X_2,\|\cdot\|_p)$ such that the ball covering point is a norm Gâteaux differentiable point, for

$$f(x,y)=\|(x,y)\|_p=\left(\|x\|_1^p+\|y\|_2^p\right)^{\frac{1}{p}},\ \ p\in(1,+\infty)$$

Every point on $D_1 \times D_2$ is a norm Gâteaux differentiable point, here

$$D_1 \times D_2 = \left\{(x,y) \in \{(x_{1,n}, x_{2,n})\}_{i=1}^{\infty} : \|(x,y)\|_p \text{ is Gâteaux differentiable at } (x,y)\right\}$$

then

$$f(x,y) = \|(x,y)\|_p = \left(\|x\|_1^p + \|y\|_2^p\right)^{\frac{1}{p}}, \quad p \in (1,+\infty)$$

is Gâteaux differentiable at $(x_{1,n}, 0)$, and for all $(y_1, y_2) \in X_1 \times X_2$ has

$$\lim_{t\to 0} \frac{1}{t}\left[\|(x_{1,n},0)+t(y_1,y_2)\|_p + \|(x_{1,n},0)-t(y_1,y_2)\|_p - 2\|(x_{1,n},0)\|_p\right] = 0$$

then

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}[\|x_{1,n}+ty_1\| + \|x_{1,n}-ty_1\| - 2\|x_{1,n}\|] \\
&= \lim_{t\to 0}\frac{1}{t}\left[\|(x_{1,n}+ty_1,0)\|_p + \|(x_{1,n}-ty_1,0)\|_p - 2\|(x_{1,n},0)\|_p\right] \\
&= \lim_{t\to 0}\frac{1}{t}\left[\|(x_{1,n},0)+t(y_1,0)\|_p + \|(x_{1,n},0)-t(y_1,0)\|_p - 2\|(x_{1,n},0)\|_p\right] \\
&= 0
\end{aligned}$$

Thus there exists a sequence $\{x_{1,n}\}_{n=1}^{\infty}$ of X_1 that is a norm Gâteaux differentiability point, and similarly there exists a sequence $\{x_{2,n}\}_{n=1}^{\infty}$ of X_2 that is a norm Gâteaux differentiability point. Thus for any $x_1 \in S(X_1)$, we have

$$\begin{aligned}
&\inf_{x_1\in S(X_1)}\sup_{n\in N} d_G\|x_{1,n}\|(x_1) \\
&= \inf_{x_1\in S(X_1)}\sup_{n\in N}\left(d_G\left\|x_{1,n}\right\|,0\right)(x_1,0) \\
&= \inf_{x_1\in S(X_1)}\sup_{n\in N}\langle d_G\|\left(x_{1,n},x_{2,n}\right)\|,(x_1,0)\rangle \\
&> 0.
\end{aligned}$$

Similarly, for an arbitrary $x_2 \in S(X_2)$, we have

$$\begin{aligned}
&\inf_{x_2\in S(X_2)_{n\in N}}\sup_G \|x_{2,n}\|(x_2) \\
&= \inf_{x_2\in S(X_2)}\sup_{n\in N}\left(0,d_G\left\|x_{2,n}\right\|\right)(0,x_2) \\
&= \inf_{x_2\in S(X_2)}\sup_{n\in N}\langle d_G\|\left(x_{1,n},x_{2,n}\right)\|,(0,x_2)\rangle \\
&> 0.
\end{aligned}$$

Therefore, by Lemma 2.2 we get that when $p \in (1,+\infty)$ there exists a real number $\alpha > 0$ and a ball covering $\mathscr{B}$ of X_1, X_2 such that $\mathscr{B}$ is α-off the origin and the ball covering point is a norm Gâteaux differentiability point. Proof complete. $\square$

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Aiyu Nie
College of mathematics and science
Harbin Engineering University
Harbin 150001, P. R. China
e-mail: nieaiyu@hrbeu.edu.cn
nieaiyu2021@163.com