

VARIOUS INEQUALITIES RELATED TO THE ADAMS INEQUALITY ON WEIGHTED MORREY SPACES

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Abstract. We consider various inequalities related to the Adams inequality for the fractional integral operators on weighted Morrey spaces. In 2014, Izumi, Komori-Furuya and Sato proved an inequality which is the type of the Adams inequality on weighted Morrey spaces. Firstly, we investigate another proof of their result. Secondly, we investigate various inequalities related to their result for higher order commutators generated by BMO-functions and the fractional integral operator on weighted Morrey spaces. One of the main results in this paper recovers the result due to Cruz-Uribe and Fiorenza in 2003. Thirdly, we extend the fractional integrals to the multilinear fractional integrals. The result of the multilinear fractional integrals partially recovers the Moen result.

1. Introduction

In this paper, we study the boundedness the fractional integral operator, higher order commutators generated by BMO-functions (see Definition 4 below for the definition of BMO) and the fractional integral operator, fractional integral operator with homogeneous kernel and multilinear fractional integral operator on weighted Morrey spaces. In this paper, the main results are Theorems 1, 3 and 4. Theorem 2 is investigated by Izumi, Komori-Furuya and Sato [17] and plays a central role in this paper.

We list the definitions to state the one of main result. Let $m \in \mathbb{Z}_+$. The m -fold commutator $[b, I_\alpha]^{(m)}$ is defined as the following:

DEFINITION 1. Given $0 < \alpha < n$ and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, define

$$[b, I_\alpha]^{(m)} f(x) := \int_{\mathbb{R}^n} \frac{(b(x) - b(y))^m}{|x - y|^{n-\alpha}} f(y) dy,$$

as long as the integral makes sense. Write $[b, I_\alpha]f := [b, I_\alpha]^{(1)}f$.

We recall the definition of weighted Lebesgue spaces. By a ‘weight’ we will mean a non-negative function w that is positive measure a.e. on \mathbb{R}^n .

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DEFINITION 2. Let $0 < p < \infty$ and w be a weight function; $w(x) \geq 0$ and $w \in L^1_{\text{loc}}(\mathbb{R}^n)$. One says that $f \in L^p(w)$, if the weighted Lebesgue norm is defined as the following is finite:

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}}.$$

Komori and Shirai introduced the weighted Morrey spaces (see [19]):

DEFINITION 3. Let $0 < p < \infty$ and $0 \leq \lambda < 1$. Let u and v be weights. One says that $f \in L^{p,\lambda}(u, v)$, if the weighted Morrey norm which is defined as the following is finite for $f \in L^p_{\text{loc}}(u)$:

$$\|f\|_{L^{p,\lambda}(u,v)} := \sup_{\substack{Q \subset \mathbb{R}^n, \\ Q: \text{cube}}} \left(\frac{1}{v(Q)^\lambda} \int_Q |f(x)|^p u(x) dx \right)^{\frac{1}{p}},$$

where $v(Q) := \int_Q v(x) dx$.

The following result is one of the most important theorems in this paper:

THEOREM 1. Let $0 < \alpha < n$, $0 \leq \lambda < 1 - \frac{\alpha}{n}$, $1 < p < \frac{n}{\alpha}(1 - \lambda)$, $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$. If $w \in A_{p,q_1}(\mathbb{R}^n)$ and $b \in \text{BMO}(\mathbb{R}^n)$, then we have

$$\| [b, I_\alpha]^{(m)} f \|_{L^{q_2, \lambda}(w^{q_1}, w^{q_1})} \leq C \|b\|_{\text{BMO}}^m \|f\|_{L^{p, \lambda}(w^p, w^{q_1})}.$$

As far as we know, Theorem 1 is a new result. In Theorem 1, taking $\lambda = 0$ and $m = 1$, we obtain the following corollary which is due to Cruz-Uribe and Fiorenza (see [5, Theorem 1.6]).

COROLLARY 1. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $w \in A_{p,q}(\mathbb{R}^n)$ and $b \in \text{BMO}(\mathbb{R}^n)$, then we have

$$\| [b, I_\alpha] f \|_{L^q(w^q)} \leq C \|b\|_{\text{BMO}} \|f\|_{L^p(w^p)}.$$

We recall the definition of the BMO space.

DEFINITION 4. For an $L^1_{\text{loc}}(\mathbb{R}^n)$ -function b , define

$$\|b\|_{\text{BMO}} := \sup_{\substack{Q \subset \mathbb{R}^n, \\ Q: \text{cube}}} \int_Q |b(x) - m_Q(b)| dx,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and the integral average of a measurable function f over Q is written as

$$m_Q(f) := \int_Q f(x) dx := \frac{1}{|Q|} \int_Q f(x) dx.$$

Define

$$BMO(\mathbb{R}^n) := \{b \in L^1_{loc}(\mathbb{R}^n) : \|b\|_{BMO} < \infty\}.$$

REMARK 1.

1. To prove of Theorem 1, we can use the Calderón-Zygmund decomposition.
2. To prove of Theorem 2, we can use the idea which is due to Hedberg [11].
3. The following inequality holds:

$$\left| [b, I_\alpha]^{(m)} f(x) \right| \leq \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f(y)| dy. \tag{1}$$

As shall be verified in the proof of Theorem 1, we consider the operator

$$x \mapsto \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} f(y) dy$$

and hence it will turn out that the integral defining $[b, I_\alpha]^{(m)} f(x)$ converges for a.e. $x \in \mathbb{R}^n$.

4. When $u = v = 1$, then the weighted Morrey spaces is reduced to the ordinary Morrey spaces:

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \|f\|_{L^{p,\lambda}(1,1)}.$$

5. If $\lambda = 0$, then the weighted Morrey spaces is reduced to the weighted Lebesgue spaces:

$$\|f\|_{L^{p,0}(u,v)} = \|f\|_{L^p(u)}.$$

We recall the definitions of the fractional integral operator I_α and the fractional maximal operator M_α .

DEFINITION 5. Given $0 < \alpha < n$, define

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy.$$

Given $0 \leq \alpha < n$, define

$$M_\alpha f(x) := \sup_{\substack{Q \ni x, \\ Q: \text{cube}}} I(Q)^\alpha \int_Q |f(y)| dy.$$

We list the results of the boundedness of the fractional integral operator on Morrey spaces.

Spanne investigated the boundedness of the fractional integral operator on Morrey spaces. Theorem A is in [25].

THEOREM A. Let $0 < \alpha < n$, $0 \leq \lambda < 1 - \frac{\alpha}{n}$, $1 < p < \frac{n}{\alpha}(1 - \lambda)$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{\mu}{q} = \frac{\lambda}{p}$. Then we have

$$\|I_\alpha f\|_{L^{q,\mu}} \leq C \|f\|_{L^{p,\lambda}}.$$

Adams showed the following inequality (see [1]):

THEOREM B. Let $0 < \alpha < n$, $0 < \lambda < 1 - \frac{\alpha}{n}$, $1 < p < \frac{n}{\alpha}(1 - \lambda)$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$. Then we have

$$\|I_\alpha f\|_{L^{q,\lambda}} \leq C \|f\|_{L^{p,\lambda}}.$$

REMARK 2. Let

$$\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)} \quad \text{and} \quad \frac{\mu}{q_1} = \frac{\lambda}{p}.$$

Since $q_1 < q_2$, by Hölder’s inequality, we have

$$\|F\|_{L^{q_1,\mu}} \leq \|F\|_{L^{q_2,\lambda}}.$$

Hence, Theorem B is a sharper result than Theorem A:

$$\|I_\alpha f\|_{L^{q_1,\mu}} \leq \|I_\alpha f\|_{L^{q_2,\lambda}} \leq C \|f\|_{L^{p,\lambda}}.$$

We list the results of the fractional integral operator on weighted Lebesgue spaces. Firstly, we recall the definition of A_p -weights:

DEFINITION 6. Let $1 < p < \infty$. One says that $w \in A_p(\mathbb{R}^n)$, if the following condition holds:

$$[w]_{A_p} := \sup_{\substack{Q \subset \mathbb{R}^n, \\ Q: \text{cube}}} \left(\int_Q w(x) dx \right) \left(\int_Q w(x)^{-\frac{p'}{p}} dx \right)^{p-1} < \infty, \tag{2}$$

where $p' := \frac{p}{p-1}$. The condition (2) is called the A_p -condition, and the weights which satisfy it are called A_p -weights. The case of $p = \infty$ is defined by the following:

$$A_\infty(\mathbb{R}^n) := \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n).$$

REMARK 3. The property of the A_p -weights is well known (for example, see [8, 10, 20]). This implies that generally speaking, we should check whether a weight u satisfies an A_p -condition or not.

Next, we recall the definition of $A_{p,q}$ -weights:

DEFINITION 7. Let $1 < p < \infty$ and $0 < q < \infty$. One says that $w \in A_{p,q}(\mathbb{R}^n)$, if the following condition holds:

$$[w]_{A_{p,q}} := \sup_{\substack{Q \subset \mathbb{R}^n, \\ Q: \text{cube}}} \left(\int_Q w(x)^q dx \right)^{\frac{1}{q}} \left(\int_Q w(x)^{-p'} dx \right)^{\frac{1}{p'}} < \infty.$$

REMARK 4. We can characterize the class $A_{p,q}(\mathbb{R}^n)$ in terms of A_p -weight: $w \in A_{p,q}(\mathbb{R}^n)$ if and only if $w^q \in A_{1+\frac{q}{p'}}(\mathbb{R}^n)$ if and only if $w^{-p'} \in A_{1+\frac{p'}{q}}(\mathbb{R}^n)$.

Muckenhoupt and Wheeden [23] showed the following inequality:

THEOREM C. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $w \in A_{p,q}(\mathbb{R}^n)$, then we have

$$\|I_{\alpha}f\|_{L^q(w^q)} \leq C \|f\|_{L^p(w^p)}.$$

Next, we recall the fractional integral operator with homogeneous kernels $I_{\Omega,\alpha}$. The definition of $I_{\Omega,\alpha}$ dates back to [7]:

DEFINITION 8. Given $0 < \alpha < n$,

$$I_{\Omega,\alpha}f(x) := \int_{\mathbb{R}^n} \frac{\Omega(x-y)f(y)}{|x-y|^{n-\alpha}} dy.$$

Ding and Lu [7] generalized Theorem C (see [20, Theorem 1]):

THEOREM D. Let $0 < \alpha < n$, $1 < s' < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\Omega \in L^s(\mathbb{S}^{n-1})$. Assume that $\Omega(\lambda x) = \Omega(x)$ for $\lambda > 0$ and $w^{s'} \in A_{\frac{p}{s'}, \frac{q}{s'}}(\mathbb{R}^n)$, then we have

$$\left(\int_{\mathbb{R}^n} |I_{\Omega,\alpha}f(x)|^q w(x)^q dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)^p dx \right)^{\frac{1}{p}}.$$

We list the results of the fractional integral operator on weighted Morrey spaces. Komori and Shirai proved the following theorem (see [19, Theorem 3.6]):

THEOREM E. Let $0 < \alpha < n$, $0 < \lambda < 1 - \frac{\alpha}{n}$, $1 < p < \frac{n}{\alpha}(1 - \lambda)$, $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{\lambda}{q_1} = \frac{\lambda}{p}$. If $w \in A_{p,q_1}(\mathbb{R}^n)$, then we have

$$\|I_{\alpha}f\|_{L^{q_1,\mu}(w^{q_1},w^{q_1})} \leq C \|f\|_{L^{p,\lambda}(w^p,w^{q_1})}.$$

In 2014, Izumi, Komori-Furuya and Sato proved the following theorem (see [17, Theorem 2.1]):

THEOREM 2. Let $0 < \alpha < n$, $0 < \lambda < 1 - \frac{\alpha}{n}$, $1 < p < \frac{n}{\alpha}(1 - \lambda)$, $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$. If $w \in A_{p,q_1}(\mathbb{R}^n)$, then we have

$$\|I_{\alpha}f\|_{L^{q_2,\lambda}(w^{q_1},w^{q_1})} \leq C \|f\|_{L^{p,\lambda}(w^p,w^{q_1})}.$$

REMARK 5. Theorem 2 is a sharper result than Theorem E. Indeed, Since $q_1 < q_2$, by Hölder’s inequality, we have

$$\|F\|_{L^{q_1,\mu}(w^{q_1},w^{q_1})} \leq \|F\|_{L^{q_2,\lambda}(w^{q_1},w^{q_1})}.$$

Hence, we have

$$\|I_\alpha f\|_{L^{q_1,\mu}(w^{q_1},w^{q_1})} \leq \|I_\alpha f\|_{L^{q_2,\lambda}(w^{q_1},w^{q_1})} \leq C \|f\|_{L^{p,\lambda}(w^p,w^{q_1})}.$$

Therefore Theorem 2 recovers Theorems B, C and E.

Moreover, there is a close connection between Theorems 1, 2, 3 and 4. Therefore, it is worth considering an another proof of Theorem 2.

Next, we consider the boundedness of the higher order commutators generated by BMO-functions and the fractional integral operator (see [29, 30]). The study of commutators with the fractional integral operator was initiated by Chanillo [2] on Lebesgue spaces.

Di Fazio and Ragusa [6] obtained the next theorem on Morrey spaces:

THEOREM F. *Let $0 < \alpha < n$, $0 \leq \lambda < 1$, $1 < p < \infty$, $1 < q_2 < \infty$ and $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$. If $b \in BMO(\mathbb{R}^n)$, then we have*

$$\|[b, I_\alpha]f\|_{L^{q_2,\lambda}(\mathbb{R}^n)} \leq C \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Conversely if $n - \alpha$ is an even integer and

$$\|[b, I_\alpha]f\|_{L^{q_2,\lambda}(\mathbb{R}^n)} \leq C \|f\|_{L^{p,\lambda}(\mathbb{R}^n)},$$

then $b \in BMO(\mathbb{R}^n)$.

Komori and Mizuhara [18] removed the restriction $n - \alpha$ is an even integer.

Moreover, Komori and Shirai [19] proved the following (see [19, Theorem 3.7]):

THEOREM G. *Let $0 < \alpha < n$, $0 < \lambda < 1 - \frac{\alpha}{n}$, $1 < p < \frac{n}{\alpha}(1 - \lambda)$, $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{\mu}{q_1} = \frac{\lambda}{p}$.*

If $w \in A_{p,q_1}(\mathbb{R}^n)$ and $b \in BMO(\mathbb{R}^n)$, then we have

$$\|[b, I_\alpha]f\|_{L^{q_1,\mu}(w^{q_1},w^{q_1})} \leq C \|f\|_{L^{p,\lambda}(w^p,w^{q_1})}.$$

REMARK 6. By Remark 5, Theorem 1 is a sharper result than Theorem G.

REMARK 7. In [14], the author investigated the weighted inequalities for $[b, I_\alpha]^{(m)}$ which are not a directly relevant to Theorem 2. In this paper, we shall investigate the weighted inequalities which is a directly relevant to Theorem 2.

Theorem 2 gives us crucial applications the following result:

THEOREM 3. *Let $0 < \alpha < n$, $0 \leq \lambda < 1 - \frac{\alpha}{n}$, $1 < s \leq \infty$, $1 \leq s' < p < \frac{n}{\alpha}(1 - \lambda)$, $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$, $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n} \cdot \frac{1}{1-\lambda}$ and $\Omega \in L^s(\mathbb{S}^{n-1})$. Assume that, $\Omega(\mu x) = \Omega(x)$ for every $\mu > 0$. If $w^{s'} \in A_{\vec{s}, \frac{q_1}{s'}}(\mathbb{R}^n)$, then we have*

$$\|I_{\Omega, \alpha}(f)\|_{L^{q_2, \lambda}(w^{q_1}, w^{q_1})} \leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|f\|_{L^{p, \lambda}(w^p, w^{q_1})}. \tag{3}$$

REMARK 8.

1. In [13], the author investigated the weighted inequalities for $I_{\Omega, \alpha}$ which are not a directly relevant to Theorem 2. In this paper, we shall investigate the weighted inequalities which is a directly relevant to Theorem 2. As far as we know, Theorem 3 is also a new result.
2. Taking $\lambda = 0$, Theorem 3 is reduced to Theorem D.
3. Since the operator $I_{\Omega, \alpha}$ is controlled by the fractional maximal operator M_{α} on weighted Morrey spaces, the proof of Theorem 3 is not difficult.

Next, we list the result of the multilinear fractional integral operator. We recall the multilinear fractional integral operator and the multilinear fractional maximal operator.

DEFINITION 9. Given $0 < \alpha < mn$, define

$$I_{\alpha, m}(\vec{f})(x) := \int_{\mathbb{R}^{mm}} \frac{f_1(y_1) \cdots f_m(y_m)}{|(x - y_1, \dots, x - y_m)|^{mn - \alpha}} d\vec{y}.$$

Given $0 \leq \alpha < mn$, define

$$\mathcal{M}_{\alpha, m}(\vec{f})(x) := \sup_{\substack{Q \ni x, \\ Q: \text{cube}}} l(Q)^\alpha \prod_{j=1}^m \int_Q |f_j(y_j)| dy_j.$$

Moen [22] introduced the following class of weights:

DEFINITION 10. Let $1 < p_1, \dots, p_m < \infty$ and $0 < q < \infty$. For a multiple weight $\vec{w} := (w_1, \dots, w_m)$, one says that $\vec{w} \in A_{\vec{p}, q}(\mathbb{R}^n)$, if the following condition holds:

$$[\vec{w}]_{A_{\vec{p}, q}} := \sup_{\substack{Q \subset \mathbb{R}^n, \\ Q: \text{cube}}} \left(\int_Q (w_1 \cdots w_m)(x)^q dx \right)^{\frac{1}{q}} \prod_{j=1}^m \left(\int_Q w_j(x)^{-p'_j} dx \right)^{\frac{1}{p'_j}} < \infty.$$

Moen proved the following result which is a multilinear version of Theorem C:

THEOREM H. *Let $0 < \alpha < mn$, $1 < p_1, \dots, p_m < \infty$, $0 < q < \infty$ and $\frac{1}{q} = \frac{1}{p_1} + \dots + \frac{1}{p_m} - \frac{\alpha}{n} > 0$. If $\vec{w} \in A_{\vec{p}, q}(\mathbb{R}^n)$, then we have*

$$\|I_{\alpha, m}(\vec{f})\|_{L^q((w_1 \cdots w_m)^q)} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w^{p_j})}.$$

Chen and Xue [4] and the author [12] completely characterized the $A_{\vec{p},q}$ -multiple weights in terms of A_p -weights:

PROPOSITION 1. *Let $1 < p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ and $0 < q < \infty$. A vector \vec{w} of weights satisfies $\vec{w} \in A_{\vec{p},q}(\mathbb{R}^n)$ if and only if*

$$\begin{cases} (w_1 \cdots w_m)^q & \in A_{1+q(m-\frac{1}{p})}(\mathbb{R}^n) \\ w_j^{-p'_j} & \in A_{1+p'_j(\frac{1}{q}+m-\frac{1}{p}-\frac{1}{p'_j})}(\mathbb{R}^n) \quad (j = 1, 2, \dots, m). \end{cases}$$

REMARK 9. In Proposition 1, if taking $m = 1$ we learn, Proposition 1 is reduced to Remark 4.

On the other hand, in 2014, Chen, Wu and Xue [3] investigated the fundamental properties of multiple weights $A_{\vec{p},q}$ and characterized the $A_{\vec{p},q}$ -multiple weights in terms of A_p -weights, under the natural restricted condition $p \leq q$.

PROPOSITION 2. *Let $1 < p_1, \dots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ and q with $\frac{1}{m} < p \leq q < \infty$. Then $\vec{w} \in A_{\vec{p},q}(\mathbb{R}^n)$ if and only if*

$$\begin{cases} (w_1 \cdots w_m)^q & \in A_{mq}(\mathbb{R}^n) \\ w_j^{-p'_j} & \in A_{mp'_j}(\mathbb{R}^n) \quad (j = 1, \dots, m) \end{cases}$$

REMARK 10. In the ordinary case, we may assume that $p \leq q$. Hence, Proposition 2 is worth enough using practically. However, in Proposition 2, we can not remove the condition $p \leq q$. If $q < p$, then Proposition 2 fails even for $n = 1$ and $m = 1$ (linear and 1-dimension case). Notice that in Proposition 1, the condition $p \leq q$ is not exist.

In Proposition 2, when $q < p$, the counterexample (see Section 5). Lastly, we investigate a multilinear version of Theorem 2.

THEOREM 4. *Let $0 < \alpha < mn$, $0 \leq \lambda < 1$, $1 < p_1, \dots, p_m < \infty$ and $0 < p, q_1, q_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n} \cdot \frac{1}{1-\lambda}$. If $(v^{\frac{p}{p_1}}, \dots, v^{\frac{p}{p_m}}) \in A_{\vec{p},q_1}(\mathbb{R}^n)$, then, we have*

$$\|I\alpha_m(\vec{f})\|_{L^{q_2,\lambda}(v^{q_1},v^{q_1})} \leq C \sup_{\substack{Q \subset \mathbb{R}^n \\ Q:\text{cube}}} \prod_{j=1}^m \left(\frac{1}{v^{q_1}(Q)^\lambda} \int_Q |f_j(y_j)|^{p_j} v(y_j)^p dy_j \right)^{\frac{1}{p_j}}. \quad (4)$$

REMARK 11.

1. If $m = 1$ in Theorem 4, Theorem 4 is reduced to Theorem 2.
2. If $v = 1$ in Theorem 4, Theorem 4 recovers [15, Theorem 1.2].

3. Theorem 4 partially recovers Theorem H.

4. By Theorems 1 and 2, the weights v^{q_1} and $v^{-\frac{p_j}{p}p}$ ($j = 1, \dots, m$) in Theorem 4 satisfy the condition of the A_p -weights for some $1 < p < \infty$.

The remainder of this paper is organized as follows. In Section 2, we list lemmas in this paper. In Section 3, we prove the lemmas which are not known. In Section 4, we prove the main results. In Section 5, we give the appendices.

Throughout this paper all notation is standard or will be defined as needed. Let $\mathcal{D}(\mathbb{R}^n)$ be the collection of all dyadic cubes on \mathbb{R}^n . All cubes are assumed to have their sides parallel to the coordinate axes. For a cube $Q \subset \mathbb{R}^n$, we use $l(Q)$ to denote the side-length $l(Q)$ and cQ to denote the cube with the same center as Q but with side-length $cl(Q)$. Given a measurable set of E , $|E|$ will denote the measure of E .

The integral average of a vector valued function $\vec{f} = (f_1, \dots, f_m)$ over Q is written

$$m_Q(\vec{f}) := \prod_{j=1}^m \int_Q f_j(y_j) dy_j.$$

In this paper, we may assume that $f \geq 0$ without loss of generality. A symbol C denotes a positive constant which may change from line to line.

2. Some Lemmas

Firstly, we use the reverse Hölder inequality (see [8], [10] and [20]):

LEMMA 1. *Let $w \in A_p(\mathbb{R}^n)$ for some $1 \leq p < \infty$. Then there exists constants $C > 0$ and $\varepsilon > 0$ that depend only on the dimension n , on p , and on $[w]_{A_p}$ such that for every cube Q we have*

$$\left(\int_Q w(x)^{1+\varepsilon} dx \right)^{\frac{1}{1+\varepsilon}} \leq C \int_Q w(x) dx.$$

We divide Lemmas to follow into two parts. We list Lemmas the 1-linear version.

2.1. 1-linear version

To prove Theorem 2, we use the following inequalities:

LEMMA 2. *$0 < \alpha < n$, $0 \leq \lambda < 1$ and $0 < q < \infty$. If $v \in A_\infty(\mathbb{R}^n)$ and $f \in L_c^\infty(\mathbb{R}^n)$, then we have*

$$\|I_\alpha f\|_{L^{q,\lambda}(v,v)} \leq C \|M_\alpha f\|_{L^{q,\lambda}(v,v)}.$$

Lemma 2 implies that we may concentrate on proving the boundedness of M_α on weighted Morrey spaces.

We define the following operator. Let V be weight. Given $0 < \alpha < n$, define

$$(M_{V,\alpha}F)(x) := \sup_{\substack{Q \ni x, \\ Q: \text{cube}}} V(Q)^{\frac{\alpha}{n}} \left(\frac{1}{V(Q)} \int_Q |F(y)|V(y)dy \right).$$

To prove Theorem 2, we use the following pointwise inequality:

LEMMA 3. *Under the condition of Theorem 2, assume that for a small number $a > 1$ and $\theta := q_1 \left(\frac{a}{p} - \frac{\alpha}{n} \right)$. Then we have,*

$$M_\alpha f(x) \leq C[w]_{A_{p,q_1}}^\theta M_{w^{q_1}, \frac{\alpha p}{a}} \left(\frac{(fw^\theta)^{\frac{p}{a}}}{w^{q_1}} \right) (x)^{\frac{a}{p}}. \tag{5}$$

REMARK 12.

1. In Lemma 3, we can choose $a > 1$ be small so as to be able to apply Lemma 1 to follow:

$$\left(\int_Q w(y)^{-\theta(\frac{p}{a})'} dy \right)^{\frac{1}{(\frac{p}{a})'}} \leq C \left(\int_Q w(y)^{-p'} dy \right)^{\frac{\theta}{p'}}.$$

2. Komori and Shirai [19] and Lerner [21] give the similar argument of Lemma 3 for the Hardy-Littlewood maximal function M .

Next, we recall the definition of the doubling condition. One says that $V \in \Delta_2$, if there exists constant $C > 0$ such that $V(2Q) \leq CV(Q)$ for every cube Q . If $V \in \Delta_2$, then V satisfies the reverse doubling condition (see [17, 19]): There exists $D > 1$ such that $V(2Q) \geq DV(Q)$ for every cube Q . To prove Theorem 2, we use the following norm inequality ([27, Theorem 4.3]):

LEMMA 4. *Let $0 \leq \alpha < n$, $0 \leq \lambda < 1 - \frac{\alpha}{n}$, $1 < p < \frac{n}{\alpha}(1 - \lambda)$ and $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n(1-\lambda)}$.
If $V \in \Delta_2$, then we have*

$$\|M_{V,\alpha}F\|_{L^{q_2,\lambda}(V,V)} \leq C \|F\|_{L^{p,\lambda}(V,V)}.$$

To prove Theorem 3, we use the following lemma:

LEMMA 5. *Let $0 < \alpha < n$, $0 \leq \lambda < 1$, $(\frac{n}{\alpha})' < s \leq \infty$, $0 < q < \infty$, $v \in A_\infty(\mathbb{R}^n)$ and $\Omega \in L^s(\mathbb{S}^{n-1})$. Assume that $\Omega(\mu x) = \Omega(x)$ for every $\mu > 0$. If $v \in A_\infty(\mathbb{R}^n)$, then we have*

$$\|I_{\Omega,\alpha}f\|_{L^{q,\lambda}(v,v)} \leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \left\| M_{\alpha s'} \left(|f|^{s'} \right)^{\frac{1}{s'}} \right\|_{L^{q,\lambda}(v,v)}.$$

To prove Lemma 5, we use the following Lemma:

LEMMA 6. *Under the condition of Lemma 5,*

$$\sum_{k=0}^{\infty} \inf_{z \in 2^{k+3}\sqrt{n}Q_0} M_{\alpha s'} \left(|f|^{s'} \right) (z)^{\frac{1}{s'}} \leq C v(Q_0)^{\frac{\lambda-1}{q}} \left\| M_{\alpha s'} \left(|f|^{s'} \right)^{\frac{1}{s'}} \right\|_{L^{q,\lambda}(v,v)}.$$

The following inequality is a well known result (see [9]):

LEMMA 7. *Let $1 < p < \infty$. Suppose that $v \in \Delta_2$, then we have*

$$\|M_v f\|_{L^p(v)} \leq C \|f\|_{L^p(v)},$$

where

$$M_v f(x) := \sup_{\substack{Q \ni x, \\ Q: \text{cube}}} \frac{1}{v(Q)} \int_Q |f(y)| v(y) dy.$$

We use the following property (see [16, 28]).

LEMMA 8. *For $\beta \geq 1$ and $f \geq 0$, let $\gamma(\beta) := m_{3Q_0} (f^\beta)^{\frac{1}{\beta}}$ and $A(\beta) > (2 \cdot 18^n)^{\frac{1}{\beta}}$. For $k = 1, 2, \dots$ we set*

$$D_k(\beta) := \bigcup \left\{ Q \in \mathcal{D}(Q_0) : m_{3Q} \left(f^\beta \right)^{\frac{1}{\beta}} > \gamma(\beta) A(\beta)^k \right\}.$$

Considering the maximal cubes, we have

$$D_k(\beta) = \bigcup_j Q_{k,j}(\beta).$$

Then we have

$$\gamma(\beta) A(\beta)^k < m_{3Q_{k,j}(\beta)} \left(f^\beta \right)^{\frac{1}{\beta}} \leq 2^{\frac{n}{\beta}} \gamma(\beta) A(\beta)^k.$$

Let $E_0(\beta) := Q_0 \setminus D_1(\beta)$ and $E_{k,j}(\beta) := Q_{k,j}(\beta) \setminus D_{k+1}(\beta)$. Moreover we obtain

$$|Q_{k,j}(\beta) \setminus D_{k+1}(\beta)| \leq \frac{1}{2} |Q_{k,j}(\beta)|.$$

Moreover, suppose that $W \in A_\infty(\mathbb{R}^n)$, then there exists $C > 1$ such that

$$W(Q_0(\beta)) \leq CW(E_0(\beta)) \quad \text{and} \quad W(Q_{k,j}(\beta)) \leq CW(E_{k,j}(\beta)).$$

Moreover, $\{E_0(\beta)\} \cup \{E_{k,j}(\beta)\}$ is a disjoint family of sets, which decomposes Q_0 .

$$Q_0 = E_0(\beta) \cup \left(\bigcup_{k,j} E_{k,j}(\beta) \right).$$

REMARK 13. In Lemma 8, for the sake of simplicity, when $\beta = 1$, we omit the index β :

$$\gamma := \gamma(1), A := A(1), D_k := D_k(1), Q_{k,j} = Q_{k,j}(1), E_0 := E_0(1) \text{ and } E_{k,j} := E_{k,j}(1).$$

By the similar argument in the text [10, pp.124-126], we can prove the following inequality:

LEMMA 9. *If $b \in \text{BMO}(\mathbb{R}^n)$ and $W \in A_\infty(\mathbb{R}^n)$, then there exist $C_1 > 0$ and $C_2 > 0$ such that for every $Q \subset \mathbb{R}^n$ and for every $\lambda > 0$*

$$W(\{x \in Q : |b(x) - m_{W,Q}(b)| > \lambda\}) \leq C_1 W(Q) e^{-\frac{C_2 \lambda}{\|b\|_{\text{BMO}}}},$$

where

$$m_{W,Q}(b) = \frac{1}{W(Q)} \int_Q b(x)W(x)dx.$$

Lemma 9 gives us the following inequality which we use to prove Theorem 1,

LEMMA 10. *Let $1 < p < \infty$. If $b \in \text{BMO}(\mathbb{R}^n)$ and $W \in A_\infty(\mathbb{R}^n)$, then we obtain the following inequality:*

$$\sup_{\substack{Q \subset \mathbb{R}^n, \\ Q; \text{cube}}} \left(\frac{1}{W(Q)} \int_Q |b(x) - m_{W,Q}(b)|^p W(x)dx \right)^{\frac{1}{p}} \cong \|b\|_{\text{BMO}(\mathbb{R}^n)}. \tag{6}$$

$$\sup_{\substack{Q \subset \mathbb{R}^n, \\ Q; \text{cube}}} \left(\int_Q |b(x) - m_{W,Q}(b)|^p dx \right)^{\frac{1}{p}} \leq C \|b\|_{\text{BMO}(\mathbb{R}^n)}. \tag{7}$$

We list Lemmas the multilinear version.

2.2. Multilinear version

To prove Theorem 4, we use the following inequalities:

LEMMA 11. *$0 < \alpha < mn$ and $0 < q < \infty$. Suppose that $v \in A_\infty(\mathbb{R}^n)$, then we have*

$$\left\| I_{\alpha,m}(\vec{f}) \right\|_{L^{q,\lambda}(v,v)} \leq C \left\| \mathcal{M}_{\alpha,m}(\vec{f}) \right\|_{L^{q,\lambda}(v,v)}.$$

Lemma 11 implies that we may concentrate on proving the boundedness of $\mathcal{M}_{\alpha,m}$ on weighted Morrey spaces. To prove Lemma 11, we use the following lemma:

LEMMA 12. *Under the condition of Lemma 11, for every cube $Q_0 \subset \mathbb{R}^n$, suppose that $f_j^0 = f_j \chi_{3Q_0}$, $f_j^\infty = f_j \chi_{(3Q_0)^c}$ and $\vec{f}_l = (f_1^{l_1}, \dots, f_m^{l_m})$, where $\vec{l} = (l_1, \dots, l_m) \in \{0, \infty\}^m$ and $\vec{l} \neq \vec{0}$. Then for $x \in Q_0$, we have*

$$\left| I_{\alpha,m}(\vec{f}_l)(x) \right| \leq C \sum_{k=0}^{\infty} \inf_{z \in 2^{k+3}\sqrt{n}Q_0} \mathcal{M}_{\alpha,m}(\vec{f})(z).$$

For a vector valued function $\vec{F} = (F_1, \dots, F_m)$, let

$$\mathcal{M}_{V, \vec{R}, \alpha}(\vec{F})(x) := \sup_{\substack{Q \ni x, \\ Q: \text{cube}}} V(Q)^{\frac{\alpha}{n}} \prod_{j=1}^m \left(\frac{1}{V(Q)} \int_Q |F_j(y)|^{r_j} V(y) dy \right)^{\frac{1}{r_j}}.$$

We have several inequalities for the preceding operator:

LEMMA 13. Let $0 < \alpha < mn$, $0 \leq \lambda < 1$, $1 < a < p_j < \infty$ ($j = 1, 2, \dots, m$), $0 < p, q_1, q_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$ and $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{n} \frac{1}{1-\lambda}$. For a weight v , suppose that $(v^{\frac{p}{p_1}}, \dots, v^{\frac{p}{p_m}}) \in A_{\vec{p}, q_1}(\mathbb{R}^n)$. Then we have

$$\mathcal{M}_{\alpha, m}(\vec{f})(x) \leq \left[(v^{\frac{p}{p_1}}, \dots, v^{\frac{p}{p_m}}) \right]_{A_{\vec{p}, q_1}}^{\theta} \mathcal{M}_{v^{q_1}, \frac{\vec{p}}{a}, \alpha}(\vec{F})(x),$$

where $\frac{\vec{p}}{a} := (\frac{p_1}{a}, \dots, \frac{p_m}{a})$, $\theta := q_1 \left(\frac{a}{p} - \frac{\alpha}{n} \right) > 1$, $F_j(x) := \frac{|f_j(x)| v(x)^{\theta \cdot \frac{p}{p_j}}}{v(x)^{q_1 \cdot \frac{a}{p_j}}}$ and $\vec{F} := (F_1, \dots, F_m)$.

REMARK 14. In Lemma 13, we can choose $a > 1$ and $\theta > 1$ be small so as to be able to apply Lemma 1 to the following situation. For every cube $Q \subset \mathbb{R}^n$,

$$\left(\int_Q v(y_j)^{-\frac{p}{p_j} \theta \left(\frac{p_j}{a} \right)'} dy_j \right)^{\frac{1}{\left(\frac{p_j}{a} \right)'}} \leq C \left(\int_Q v(y_j)^{-\frac{p}{p_j} p_j'} dy_j \right)^{\frac{\theta}{p_j}}.$$

LEMMA 14. Under the condition of Lemma 13, we have

$$\begin{aligned} \mathcal{M}_{V, \frac{\vec{p}}{a}, \alpha}(\vec{F})(x) &\leq \left(\sup_{\substack{Q \subset \mathbb{R}^n, \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{1}{p_j}} \right)^{1 - \frac{p}{q_2}} \\ &\quad \times \left(\mathcal{M}_V \left(\vec{F}^{\frac{\vec{p}}{a}} \right) (x)^{\frac{a}{\vec{p}}} \right)^{\frac{p}{q_2}}, \end{aligned}$$

where

$$\mathcal{M}_V \left(\vec{F}^{\frac{\vec{p}}{a}} \right) (x)^{\frac{a}{\vec{p}}} := \sup_{\substack{Q \ni x, \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)} \int_Q |F_j(y_j)|^{\frac{p_j}{a}} V(y_j) dy_j \right)^{\frac{a}{p_j}}.$$

LEMMA 15. Under the condition of Lemma 13, we have

$$\left\| \left(\mathcal{M}_V \left(\vec{F}^{\frac{\vec{p}}{a}} \right) (\cdot)^{\frac{a}{\vec{p}}} \right)^{\frac{p}{q_2}} \right\|_{L^{q_2, \lambda}(V, V)} \leq C \left(\sup_{\substack{Q \subset \mathbb{R}^n, \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{1}{p_j}} \right)^{\frac{p}{q_2}}.$$

LEMMA 16. *Under the condition of Lemma 13, we have*

$$\begin{aligned} & \sup_{\substack{Q \subset \mathbb{R}^n \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{1}{p_j}} \\ &= \sup_{\substack{Q \subset \mathbb{R}^n \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{v^{q_1}(Q)^\lambda} \int_Q |f_j(y_j)|^{p_j} v(y_j)^p dy_j \right)^{\frac{1}{p_j}}. \end{aligned}$$

We use the following property which is a multilinear version of Lemma 8 (see [16, 28]).

LEMMA 17. *Let $\gamma := m_3 Q_0(\vec{f})$ and $A' > (2m)^m \cdot 18^{mn}$. Let*

$$D'_k := \bigcup \left\{ Q \in \mathcal{D}(Q_0), m_{3Q}(\vec{f}) > \gamma(A')^k \right\}.$$

Considering the maximal cubes with respect to inclusion, we can write

$$D'_k = \bigcup_j Q'_{k,j},$$

where the cubes $\{Q'_{k,j}\} \subset \mathbb{R}^n$ are non-overlapping. By the maximality of $Q'_{k,j}$, we see that

$$\gamma(A')^k < m_{3Q'_{k,j}}(\vec{f}) \leq 2^{mn} \gamma(A')^k. \tag{8}$$

Moreover, suppose that $W \in A_\infty(\mathbb{R}^n)$, then we have

$$W(Q_0) \leq CW(E'_0) \quad \text{and} \quad W(Q'_{k,j}) \leq CW(E'_{k,j}), \tag{9}$$

where $E'_0 := Q_0 \setminus D'_1$ and $E'_{k,j} := Q'_{k,j} \setminus D'_{k+1}$. On the other hand, $\{E'_0\} \cup \{E'_{k,j}\}$ is a disjoint family of sets, which decomposes Q_0 :

$$Q_0 = E'_0 \cup \left(\bigcup_{k,j} E'_{k,j} \right). \tag{10}$$

3. Proofs of Lemmas

We omit the proofs of Lemmas 1, 2, 4, 7, 8, 9 and 17. That is, We prove Lemmas 3, 5, 6, 10, 11, 12, 13, 14, 15 and 16.

Lemma 11 gives us the proof of Lemma 2. Lemma 1, 4, 7, 8 and 17 are known results. The proof of Lemma 9 is obtained by the similar argument in the text [10, pp. 124–126],

Proof of Lemma 3. Since $f(x) = f(x)w(x)^\theta w(x)^{-\theta}$, by Hölder's inequality for $\frac{p}{a} > 1$, we have

$$\begin{aligned} M_\alpha f(x) &= \sup_{\substack{Q \ni x, \\ Q: \text{cube}}} l(Q)^\alpha \int_Q |f(y)| dy \\ &\leq \sup_{\substack{Q \ni x, \\ Q: \text{cube}}} l(Q)^\alpha \left(\int_Q (f(y)w(y)^\theta)^{\frac{p}{a}} dy \right)^{\frac{a}{p}} \left(\int_Q w(y)^{-\theta(\frac{p}{a})'} dy \right)^{\frac{1}{(\frac{p}{a})'}}. \end{aligned}$$

Note that for every cube $Q \subset \mathbb{R}^n$,

$$\begin{aligned} &l(Q)^\alpha \left(\int_Q (f(y)w(y)^\theta)^{\frac{p}{a}} dy \right)^{\frac{a}{p}} \\ &\leq |Q|^{\frac{\alpha}{n} - \frac{\alpha}{p}} w^{q_1}(Q)^{\frac{\alpha}{p} - \frac{\alpha}{n}} w^{q_1}(Q)^{\frac{\alpha}{n}} \left(\frac{1}{w^{q_1}(Q)} \int_Q (f(y)w(y)^\theta)^{\frac{p}{a}} dy \right)^{\frac{a}{p}}. \end{aligned}$$

Since $\frac{a}{p} - \frac{\alpha}{n} = \frac{\theta}{q_1}$, we obtain

$$\begin{aligned} &|Q|^{\frac{\alpha}{n} - \frac{\alpha}{p}} w^{q_1}(Q)^{\frac{\alpha}{p} - \frac{\alpha}{n}} w^{q_1}(Q)^{\frac{\alpha}{n}} \left(\frac{1}{w^{q_1}(Q)} \int_Q (f(y)w(y)^\theta)^{\frac{p}{a}} dy \right)^{\frac{a}{p}} \\ &= \left(\int_Q w(x)^{q_1} dx \right)^{\frac{\theta}{q_1}} \left(w^{q_1}(Q)^{\frac{\alpha p}{na}} \frac{1}{w^{q_1}(Q)} \int_Q \frac{(f(y)w(y)^\theta)^{\frac{p}{a}}}{w(y)^{q_1}} w(y)^{q_1} dy \right)^{\frac{a}{p}}. \end{aligned}$$

If $x \in Q$, then we have

$$\left(w^{q_1}(Q)^{\frac{\alpha p}{na}} \frac{1}{w^{q_1}(Q)} \int_Q \frac{(f(y)w(y)^\theta)^{\frac{p}{a}}}{w(y)^{q_1}} w(y)^{q_1} dy \right)^{\frac{a}{p}} \leq M_{w^{q_1}, \frac{\alpha p}{a}} \left(\frac{(fw^\theta)^{\frac{p}{a}}}{w^{q_1}} \right) (x)^{\frac{a}{p}}.$$

This implies that

$$M_\alpha f(x) \leq \left(\int_Q w(x)^{q_1} dx \right)^{\frac{\theta}{q_1}} \left(\int_Q w(y)^{-\theta(\frac{p}{a})'} dy \right)^{\frac{1}{(\frac{p}{a})'}} M_{w^{q_1}, \frac{\alpha p}{a}} \left(\frac{(fw^\theta)^{\frac{p}{a}}}{w^{q_1}} \right) (x)^{\frac{a}{p}}.$$

Applying Lemma 1 to $\theta(\frac{p}{a})' > p'$, we learn

$$\left(\int_Q w(y)^{-\theta(\frac{p}{a})'} dy \right)^{\frac{1}{(\frac{p}{a})'}} \leq C \left(\int_Q w(y)^{-p'} dy \right)^{\frac{\theta}{p'}}.$$

By $w \in A_{p, q_1}(\mathbb{R}^n)$, we have

$$\left(\int_Q w(x)^{q_1} dx \right)^{\frac{\theta}{q_1}} \left(\int_Q w(y)^{-\theta(\frac{p}{a})'} dy \right)^{\frac{1}{(\frac{p}{a})'}} \leq C[w]_{A_{p, q_1}}^\theta. \quad \square$$

Proof of Lemma 5. For every cube $Q_0 \subset \mathbb{R}^n$, we have

$$|I_{\Omega,\alpha}f(x)| \leq |I_{\Omega,\alpha}f_0(x)| + |I_{\Omega,\alpha}f_\infty(x)|,$$

where $f_0 := f\chi_{3Q_0}$ and $f_\infty := f\chi_{(3Q_0)^c}$. We evaluate $|I_{\Omega,\alpha}f_\infty(x)|$. We use a routine estimate (see [28]):

$$\begin{aligned} |I_{\Omega,\alpha}f_\infty(x)| &\leq \int_{(3Q_0)^c} \frac{|\Omega(x-y)|f(y)}{|x-y|^{n-\alpha}} dy \leq \int_{|x-y|>l(Q_0)} \frac{|\Omega(x-y)|f(y)}{|x-y|^{n-\alpha}} dy \\ &= \sum_{k=0}^{\infty} \int_{2^k l(Q_0) < |x-y| \leq 2^{k+1} l(Q_0)} \frac{|\Omega(x-y)|f(y)}{|x-y|^{n-\alpha}} dy \\ &\leq \sum_{k=0}^{\infty} (2^k l(Q_0))^{\alpha-n} \int_{|x-y| \leq 2^{k+1} l(Q_0)} |\Omega(x-y)||f(y)| dy. \end{aligned}$$

By Hölder’s inequality, we have

$$\begin{aligned} |I_{\Omega,\alpha}f_\infty(x)| &\leq \sum_{k=0}^{\infty} (2^k l(Q_0))^{\alpha-n} \left(\int_{|x-y| \leq 2^{k+1} l(Q_0)} |\Omega(x-y)|^s dy \right)^{\frac{1}{s}} \\ &\quad \times \left(\int_{|x-y| \leq 2^{k+1} l(Q_0)} f(y)^{s'} dy \right)^{\frac{1}{s'}} \\ &= \sum_{k=0}^{\infty} (2^k l(Q_0))^{\alpha-n} \left(\int_{\mathbb{S}^{n-1}} \int_0^{2^{k+1} l(Q_0)} |\Omega(l \cdot \xi)|^s l^{n-1} dl d\xi \right)^{\frac{1}{s}} \\ &\quad \times \left(\int_{|x-y| \leq 2^{k+1} l(Q_0)} f(y)^{s'} dy \right)^{\frac{1}{s'}}. \end{aligned}$$

By the homogeneity of Ω , we obtain

$$\begin{aligned} |I_{\Omega,\alpha}f_\infty(x)| &\leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{k=0}^{\infty} (2^k l(Q_0))^{\alpha-n} (2^{k+1} l(Q_0))^{\frac{n}{s}} \left(\int_{|x-y| \leq 2^{k+1} l(Q_0)} f(y)^{s'} dy \right)^{\frac{1}{s'}} \\ &\leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{k=0}^{\infty} \left| 2^{k+3} \sqrt{n} Q_0 \right|^{\frac{\alpha}{n}} \left(\int_{2^{k+3} \sqrt{n} Q_0} f(y)^{s'} dy \right)^{\frac{1}{s'}} \\ &\leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{k=0}^{\infty} \inf_{z \in 2^{k+3} \sqrt{n} Q_0} M_{\alpha s'}(f^{s'})(z) \frac{1}{s'}. \end{aligned}$$

By Lemma 6, we have

$$\left(\frac{1}{v(Q_0)^\lambda} \int_{Q_0} |I_{\Omega,\alpha}f_\infty(x)|^q v(x) dx \right)^{\frac{1}{q}} \leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \left\| M_{\alpha s'}(f^{s'}) \right\|_{L^{q,\lambda}(v,v)}^{\frac{1}{s'}}.$$

Next, we evaluate $I_{\Omega,\alpha}f_0(x)$. By Hölder's inequality, we have

$$|I_{\Omega,\alpha}f_0(x)| \leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^\alpha m_{3Q}(f^{s'})^{\frac{1}{s'}} \chi_Q(x).$$

We apply Lemma 8 to $\beta = s'$ and $W = v$.

Let

$$\mathcal{D}_0(Q_0; s') := \left\{ Q \in \mathcal{D}(Q_0); \left(\int_{3Q} f(y)^{s'} dy \right)^{\frac{1}{s'}} \leq \gamma(\beta)A(\beta) \right\}$$

and

$$\mathcal{D}_{k,j}(Q_0; s') := \left\{ Q \in \mathcal{D}(Q_0); Q \subset Q_{k,j}(s'), \right. \\ \left. \gamma(s')A(s')^k < \left(\int_{3Q} f(y)^{s'} dy \right)^{\frac{1}{s'}} \leq \gamma(s')A(s')^{k+1} \right\}.$$

Then we have

$$\mathcal{D}(Q_0) = \mathcal{D}_0(Q_0; s') \cup \left(\bigcup_{k,j} \mathcal{D}_{k,j}(Q_0; s') \right).$$

Case 1. $0 < q \leq 1$. Since

$$|I_{\Omega,\alpha}f_0(x)|^q \leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})}^q \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^{\alpha q} m_{3Q}(f^{s'})^{\frac{q}{s'}} \chi_Q(x),$$

we obtain

$$\left(\int_{Q_0} |I_{\Omega,\alpha}f_0(x)|^q v(x) dx \right) \\ \leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})}^q \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^{\alpha q} m_{3Q}(f^{s'})^{\frac{q}{s'}} v(Q) \\ = C \|\Omega\|_{L^s(\mathbb{S}^{n-1})}^q \left(\sum_{Q \in \mathcal{D}_0(Q_0; s')} + \sum_{k,j} \sum_{Q \in \mathcal{D}_{k,j}(Q_0; s')} \right) l(Q)^{\alpha q} m_{3Q}(f^{s'})^{\frac{q}{s'}} v(Q) \\ = C \|\Omega\|_{L^s(\mathbb{S}^{n-1})}^q \left(I(s') + \sum_{k,j} II_{k,j}(s') \right).$$

We evaluate $II_{k,j}(s')$. By the definition of the $\mathcal{D}_{k,j}(Q_0; s')$,

$$II_{k,j}(s') < \left(\gamma(s')A(s')^{k+1} \right)^q \sum_{Q \in \mathcal{D}_{k,j}(Q_0; s')} \int_Q v(x) dx \\ \leq A(s')^q l(Q_{k,j}(s'))^{\alpha q} m_{3Q_{k,j}(s')}(f^{s'})^{\frac{q}{s'}} v(Q_{k,j}(s')).$$

Since $v(Q_{k,j}(s')) \leq Cv(E_{k,j}(s'))$, we have

$$\begin{aligned} II_{k,j}(s') &\leq CA(s')^q l(Q_{k,j}(s'))^{\alpha q} m_{3Q_{k,j}(s')}(f^{s'})^{\frac{q}{s'}} v(E_{k,j}(s')) \\ &= CA(s')^q \int_{E_{k,j}(s')} l(Q)^{\alpha q} m_{3Q}(f^{s'})^{\frac{q}{s'}} v(x) dx \\ &\leq CA(s')^q \int_{E_{k,j}(s')} M_{\alpha s'}(f^{s'})(x)^{\frac{q}{s'}} v(x) dx. \end{aligned}$$

A similar argument gives us the following:

$$I(s') \leq CA(s')^q \int_{E_0(s')} M_{\alpha s'}(f^{s'})(x)^{\frac{q}{s'}} v(x) dx.$$

Therefore we have

$$I(s') + \sum_{k,j} II_{k,j}(s') \leq CA(s')^q \int_{Q_0} M_{\alpha s'}(f^{s'})(x)^{\frac{q}{s'}} v(x) dx.$$

This implies that

$$\left(\frac{1}{v(Q_0)^\lambda} \int_{Q_0} |I_{\Omega,\alpha} f_0(x)|^q v(x) dx \right)^{\frac{1}{q}} \leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \left(\frac{1}{v(Q_0)^\lambda} \int_{Q_0} M_{\alpha s'}(f^{s'})(x)^{\frac{q}{s'}} v(x) dx \right)^{\frac{1}{q}}.$$

Case 2. $1 < q < \infty$. By the duality, we have

$$\begin{aligned} &\left(\int_{Q_0} |I_{\Omega,\alpha} f_0(x)|^q v(x) dx \right)^{\frac{1}{q}} \\ &= \sup \left\{ \int_{Q_0} |I_{\Omega,\alpha} f_0(x)| |g(x)| dx : g \text{ satisfies that } \|g\|_{L^{q'}\left(v^{-\frac{q'}{q}}\right)} = 1 \right\}. \end{aligned}$$

Let $g \geq 0$, $\text{supp}(g) \subset Q_0$ and

$$\|g\|_{L^{q'}\left(v^{-\frac{q'}{q}}\right)} = \left(\int_{Q_0} g(x)^{q'} v(x)^{-\frac{q'}{q}} dx \right)^{\frac{1}{q'}} = 1.$$

Then, we obtain

$$\begin{aligned} \int_{Q_0} |I_{\Omega,\alpha} f_0(x)| |g(x)| dx &\leq C \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^\alpha m_{3Q}(f^{s'})^{\frac{1}{s'}} g(Q) \\ &\leq C \left(\sum_{Q \in \mathcal{D}_0(Q_0; s')} + \sum_{k,j} \sum_{Q \in \mathcal{D}_{k,j}(Q_0; s')} \right) l(Q)^\alpha m_{3Q}(f^{s'})^{\frac{1}{s'}} g(Q) \\ &= C \left(I(s') + \sum_{k,j} II_{k,j}(s') \right). \end{aligned}$$

We evaluate $II_{k,j}(s')$. By the definition of the $\mathcal{D}_{k,j}(Q_0; s')$, we have

$$\begin{aligned} II_{k,j}(s') &< \sum_{Q \in \mathcal{D}_{k,j}(Q_0; s')} \left(\gamma(s') A(s')^{k+1} \right) \int_Q g(x) dx \\ &\leq A(s') l(Q_{k,j}(s'))^\alpha m_{3Q_{k,j}(s')}(f^{s'})^{\frac{1}{s'}} \sum_{Q \in \mathcal{D}_{k,j}(Q_0; s')} g(Q). \end{aligned}$$

Since $\bigcup_{Q \in \mathcal{D}_{k,j}(Q_0; s')} Q \subset Q_{k,j}(s')$,

$$\begin{aligned} II_{k,j}(s') &\leq CA(s') l(Q_{k,j}(s'))^\alpha m_{3Q_{k,j}(s')}(f^{s'})^{\frac{1}{s'}} g(Q_{k,j}(s')) \\ &= CA(s') l(Q_{k,j}(s'))^\alpha m_{3Q_{k,j}(s')}(f^{s'})^{\frac{1}{s'}} \frac{1}{v(Q_{k,j}(s'))} g(Q_{k,j}(s')) \cdot v(Q_{k,j}(s')). \end{aligned}$$

Since $v(Q_{k,j}(s')) \leq Cv(E_{k,j}(s'))$, we have

$$\begin{aligned} II_{k,j}(s') &\leq CA(s') l(Q_{k,j}(s'))^\alpha m_{3Q_{k,j}(s')}(f^{s'})^{\frac{1}{s'}} \frac{1}{v(Q_{k,j}(s'))} \int_{Q_{k,j}(s')} g(x) dx \cdot v(E_{k,j}(s')) \\ &= CA(s') \int_{E_{k,j}(s')} l(Q_{k,j}(s'))^\alpha m_{3Q_{k,j}(s')}(f^{s'})^{\frac{1}{s'}} \left(\frac{1}{v(Q_{k,j}(s'))} \int_{Q_{k,j}(s')} g(x) dx \right) v(y) dy \\ &\leq CA(s') \int_{E_{k,j}(s')} M_{\alpha s'}(f^{s'})(y)^{\frac{1}{s'}} \cdot M_v\left(\frac{g}{v}\right)(y) v(y) dy. \end{aligned}$$

A similar argument gives us the following:

$$I(s') \leq CA(s') \int_{E_0(s')} M_{\alpha s'}(f^{s'})(y)^{\frac{1}{s'}} \cdot M_v\left(\frac{g}{v}\right)(y) v(y) dy.$$

Hence, we have

$$I(s') + \sum_{k,j} II_{k,j}(s') \leq CA(s') \int_{Q_0} M_{\alpha s'}(f^{s'})(y)^{\frac{1}{s'}} \cdot M_v\left(\frac{g}{v}\right)(y) v(y) dy.$$

By Hölder's inequality, we obtain the following inequality:

$$I(s') + \sum_{k,j} II_{k,j}(s') \leq CA(s') \left(\int_{Q_0} M_{\alpha s'}(f^{s'})(y)^{\frac{q}{s'}} v(y) dy \right)^{\frac{1}{q}} \left(\int_{Q_0} M_v\left(\frac{g}{v}\right)(y)^{q'} v(y) dy \right)^{\frac{1}{q'}}.$$

By Lemma 7, we obtain

$$\begin{aligned} \left(\int_{Q_0} M_v\left(\frac{g}{v}\right)(y)^{q'} v(y) dy \right)^{\frac{1}{q'}} &\leq C \left(\int_{Q_0} \left(\frac{g(x)}{v(x)} \right)^{q'} v(x) dx \right)^{\frac{1}{q'}} \\ &= C \left(\int_{Q_0} g(x)^{q'} v(x)^{-\frac{q'}{q}} dx \right)^{\frac{1}{q'}} = C. \end{aligned}$$

Therefore we obtain

$$I(s') + \sum_{k,j} II_{k,j}(s') \leq CA(s') \left(\int_{Q_0} M_{\alpha s'}(f^{s'})(x)^{\frac{q}{s'}} v(x) dx \right)^{\frac{1}{q}}.$$

This implies that

$$\left(\frac{1}{v(Q_0)^\lambda} \int_{Q_0} |I_{\Omega,\alpha} f(x)|^q v(x) dx \right)^{\frac{1}{q}} \leq C \left(\frac{1}{v(Q_0)^\lambda} \int_{Q_0} M_{\alpha s'}(f^{s'})(x)^{\frac{q}{s'}} v(x) dx \right)^{\frac{1}{q}}.$$

Therefore we get the desired inequality. \square

Proof of Lemma 6. We evaluate $\inf_{z \in 2^{k+3}\sqrt{n}Q_0} M_{\alpha s'}(f^{s'})(z)^{\frac{1}{s'}}$. By the definition of $v(2^{k+3}\sqrt{n}Q_0)$, we have

$$\begin{aligned} & \inf_{z \in 2^{k+3}\sqrt{n}Q_0} M_{\alpha s'}(f^{s'})(z)^{\frac{1}{s'}} \\ &= \left(\frac{1}{v(2^{k+3}\sqrt{n}Q_0)^\lambda} \int_{2^{k+3}\sqrt{n}Q_0} \inf_{z \in 2^{k+3}\sqrt{n}Q_0} M_{\alpha s'}(f^{s'})(z)^{\frac{q}{s'}} v(x) dx \right)^{\frac{1}{q}} v(2^{k+3}\sqrt{n}Q_0)^{\frac{\lambda-1}{q}} \\ &\leq \left(\frac{1}{v(2^{k+3}\sqrt{n}Q_0)^\lambda} \int_{2^{k+3}\sqrt{n}Q_0} M_{\alpha s'}(f^{s'})(x)^{\frac{q}{s'}} v(x) dx \right)^{\frac{1}{q}} v(2^{k+3}\sqrt{n}Q_0)^{\frac{\lambda-1}{q}} \\ &\leq \left\| M_{\alpha s'}(f^{s'})^{\frac{1}{s'}} \right\|_{L^{q,\lambda}(v,v)} v(2^{k+3}\sqrt{n}Q_0)^{\frac{\lambda-1}{q}}. \end{aligned}$$

Since $v \in A_\infty(\mathbb{R}^n)$, there exists $D > 1$ such that $v(2Q_0) \geq Dv(Q_0)$. Therefore we have

$$v(2^{k+3}\sqrt{n}Q_0)^{\frac{\lambda-1}{q}} \leq \left(\frac{1}{D^{\frac{1-\lambda}{q}}} \right)^k v(Q_0)^{\frac{\lambda-1}{q}}.$$

This implies that

$$\sum_{k=0}^{\infty} \inf_{z \in 2^{k+3}\sqrt{n}Q_0} M_{\alpha s'}(f^{s'})(z)^{\frac{1}{s'}} \leq Cv(Q_0)^{\frac{\lambda-1}{q}} \left\| M_{\alpha s'}(f^{s'})^{\frac{1}{s'}} \right\|_{L^{q,\lambda}(v,v)}. \quad \square$$

Proof of Lemma 10. For every cube Q , we have

$$\begin{aligned} \int_Q |b(x) - m_{W,Q}(b)|^p dx &= \frac{p}{|Q|} \int_0^\infty \lambda^{p-1} |\{x \in Q : |b(x) - m_{W,Q}(b)| > \lambda\}| d\lambda \\ &= p \int_0^\infty \lambda^{p-1} \frac{|\{x \in Q : |b(x) - m_{W,Q}(b)| > \lambda\}|}{|Q|} d\lambda. \end{aligned}$$

If $W \in A_\infty(\mathbb{R}^n)$, then there exists $1 < r < \infty$ such that $W \in A_r(\mathbb{R}^n)$. Therefore we invoke the property (9.2.1) in text [10, pp.293]. We have for every $A \subset Q$

$$\frac{|A|}{|Q|} \leq C \left(\frac{W(A)}{W(Q)} \right)^{\frac{1}{r}}.$$

This implies that

$$\begin{aligned} & \int_0^\infty \lambda^{p-1} \frac{|\{x \in Q : |b(x) - m_{W,Q}(b)| > \lambda\}|}{|Q|} d\lambda \\ & \leq C \int_0^\infty \lambda^{p-1} \left(\frac{W(\{x \in Q : |b(x) - m_{W,Q}(b)| > \lambda\})}{W(Q)} \right)^{\frac{1}{r}} d\lambda. \end{aligned}$$

By Lemma 9, we have

$$\begin{aligned} \int_0^\infty \lambda^{p-1} \left(\frac{W(\{x \in Q : |b(x) - m_{W,Q}(b)| > \lambda\})}{W(Q)} \right)^{\frac{1}{r}} d\lambda & \leq C_1 \int_0^\infty \lambda^{p-1} e^{-\frac{C_2 \lambda}{r \|b\|_{\text{BMO}}}} d\lambda \\ & \leq C^p \|b\|_{\text{BMO}}^p. \end{aligned}$$

This implies that

$$\left(\int_Q |b(x) - m_{W,Q}(b)|^p dx \right)^{\frac{1}{p}} \leq C \|b\|_{\text{BMO}}. \quad \square$$

Proof of Lemma 11. For $x \in Q_0$, we have

$$\left| I_{\alpha,m}(\vec{f})(x) \right| \leq |I_{\alpha,m}(\vec{f}_0)(x)| + \sum_{\substack{\vec{l} \neq \vec{0}, \\ \vec{l} \in \{0,\infty\}^m}} |I_{\alpha,m}(\vec{f}_l)(x)|,$$

where $f_j^0(y_j) = f_j(y_j)\chi_{3Q_0}(y_j)$, $f_j^\infty(y_j) = f_j(y_j)\chi_{(3Q_0)^c}(y_j)$, $\vec{f}_0 = (f_1^0, \dots, f_m^0)$ and $\vec{f}_l = (f_1^{l_1}, \dots, f_m^{l_m})$. We prove the following inequality. We have

$$\left(\int_{Q_0} |I_{\alpha,m}(\vec{f}_0)(x)|^q v(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_{Q_0} \mathcal{M}_{\alpha,m}(\vec{f}_0)(x)^q v(x) dx \right)^{\frac{1}{q}}.$$

We use the following in [26]:

$$\left| I_{\alpha,m}(\vec{f}_0)(x) \right| \leq C \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^\alpha m_{3Q}(\vec{f}) \chi_Q(x).$$

Let

$$\mathcal{D}'_0(Q_0) := \left\{ Q \in \mathcal{D}(Q_0) : m_{3Q}(\vec{f}) \leq \gamma A' \right\}$$

and

$$\mathcal{D}'_{k,j}(Q_0) := \left\{ Q \in \mathcal{D}(Q_0), Q \subset Q'_{k,j}, \gamma'(A')^k < m_{3Q}(\vec{f}) \leq \gamma'(A')^{k+1} \right\}.$$

Then we have

$$\mathcal{D}(Q_0) = \mathcal{D}'_0(Q_0) \cup \left(\bigcup_{k,j} \mathcal{D}'_{k,j}(Q_0) \right).$$

Case 1. $0 < q \leq 1$. Since $0 < q \leq 1$, we have

$$\left| I_{\alpha,m}(\vec{f}_0)(x) \right|^q \leq C^q \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^{\alpha q} m_{3Q}(\vec{f})^q \chi_Q(x).$$

Integrating over the cube Q_0 , we obtain

$$\begin{aligned} & \int_{Q_0} \left| I_{\alpha,m}(\vec{f}_0)(x) \right|^q v(x) dx \\ & \leq C^q \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^{\alpha q} m_{3Q}(\vec{f})^q \left(\int_Q v(x) dx \right) \\ & = C^q \left(\sum_{Q \in \mathcal{D}'_0(Q_0)} + \sum_{k,j} \sum_{Q \in \mathcal{D}'_{k,j}(Q_0)} \right) l(Q)^{\alpha q} m_{3Q}(\vec{f})^q \left(\int_Q v(x) dx \right) = I_0 + \sum_{k,j} I_{k,j}. \end{aligned}$$

We evaluate $I_{k,j}$.

$$\begin{aligned} I_{k,j} & \leq \sum_{Q \in \mathcal{D}'_{k,j}(Q_0)} l(Q)^{\alpha q} m_{3Q}(\vec{f})^q \left(\int_Q v(x) dx \right) \\ & \leq A' l(Q'_{k,j})^{\alpha q} m_{3Q'_{k,j}}(\vec{f})^q v(Q'_{k,j}). \end{aligned}$$

By $v(Q'_{k,j}) \leq C v(E'_{k,j})$, we have

$$I_{k,j} \leq CA' \int_{E'_{k,j}} \mathcal{M}_{\alpha,m}(\vec{f}_0)(x)^q v(x) dx.$$

A similar argument gives us the following:

$$I_0 \leq CA' \int_{E'_0} \mathcal{M}_{\alpha,m}(\vec{f}_0)(x)^q v(x) dx.$$

We sum up I_0 and $I_{k,j}$. The result is:

$$I_0 + \sum_{k,j} I_{k,j} \leq C \int_{Q_0} \mathcal{M}_{\alpha,m}(\vec{f}_0)(x)^q v(x) dx.$$

Case 2. $1 < q < \infty$. By the duality, we have

$$\left(\int_{Q_0} \left| I_{\alpha,m}(\vec{f}_0)(x) \right|^q v(x) dx \right)^{\frac{1}{q}} = \sup \left\{ \int_{Q_0} \left| I_{\alpha,m}(\vec{f}_0)(x) \right| |g(x)| dx : g \text{ satisfies that } \|g\|_{L^{q'}\left(v^{-\frac{q'}{q}}\right)} = 1 \right\}.$$

Let $g \geq 0$, $\text{supp}(g) \subset Q_0$ and $\|g\|_{L^{q'}\left(v^{-\frac{q'}{q}}\right)} = 1$. We evaluate $\int_{Q_0} \left| I_{\alpha,m}(\vec{f}_0)(x) \right| g(x) dx$.

$$\begin{aligned} \int_{Q_0} \left| I_{\alpha,m}(\vec{f}_0)(x) \right| g(x) dx &\leq C \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^\alpha m_{3Q}(\vec{f}) \int_Q g(x) dx \\ &\leq C \left(\sum_{Q \in \mathcal{D}'_0(Q_0)} + \sum_{k,j} \sum_{Q \in \mathcal{D}'_{k,j}(Q_0)} \right) l(Q)^\alpha m_{3Q}(\vec{f}) \int_Q g(x) dx \\ &= II_0 + \sum_{k,j} II_{k,j}. \end{aligned}$$

We evaluate $II_{k,j}$.

$$II_{k,j} \leq A' l(Q'_{k,j})^\alpha \cdot m_{3Q'_{k,j}}(\vec{f}) \left(\frac{1}{v(Q'_{k,j})} \int_{Q'_{k,j}} g(x) dx \right) v(Q'_{k,j}). \tag{11}$$

By $v(Q'_{k,j}) \leq C v(E'_{k,j})$, we have

$$II_{k,j} \leq A' \int_{E'_{k,j}} \mathcal{M}_{\alpha,m}(\vec{f}_0)(y) \cdot M_v\left(\frac{g}{v}\right)(y) \cdot v(y) dy. \tag{12}$$

A similar argument gives us the following:

$$II_0 \leq A' \int_{E'_0} \mathcal{M}_{\alpha,m}(\vec{f}_0)(y) \cdot M_v\left(\frac{g}{v}\right)(y) \cdot v(y) dy. \tag{13}$$

We sum up II_0 and $II_{k,j}$. By (12) and (13), the result is:

$$II_0 + \sum_{k,j} II_{k,j} \leq A' \int_{Q_0} \mathcal{M}_{\alpha,m}(\vec{f}_0)(y) \cdot M_v\left(\frac{g}{v}\right)(y) \cdot v(y) dy.$$

By Hölder's inequality for $q > 1$, we have

$$II_0 + \sum_{k,j} II_{k,j} \leq A' \left(\int_{Q_0} \mathcal{M}_{\alpha,m}(\vec{f}_0)(y)^q v(y) dy \right)^{\frac{1}{q}} \cdot \left(\int_{Q_0} M_v\left(\frac{g}{v}\right)(y)^{q'} v(y) dy \right)^{\frac{1}{q'}}.$$

By Lemma 7, we have

$$\left(\int_{Q_0} M_v \left(\frac{g}{v} \right) (y)^{q'} v(y) dy \right)^{\frac{1}{q'}} \leq C \left(\int_{Q_0} \left(\frac{g(y)}{v(y)} \right)^{q'} v(y) dy \right)^{\frac{1}{q'}} = C \|g\|_{L^{q'} \left(v^{-\frac{q'}{q}} \right)} = C.$$

Hence, we have

$$II_0 + \sum_{k,j} II_{k,j} \leq A'' \left(\int_{Q_0} \mathcal{M}_{\alpha,m}(\vec{f}_0) (y)^q v(y) dy \right)^{\frac{1}{q}}.$$

The Case 1 and the Case 2 imply that the following inequality holds for $0 < q < \infty$:

$$\|I_{\alpha,m}(\vec{f}_0)\|_{L^{q,\lambda}(v,v)} \leq C \|\mathcal{M}_{\alpha,m}(\vec{f}_0)\|_{L^{q,\lambda}(v,v)}.$$

Next, we prove for $\vec{l} \neq \vec{0}$,

$$\|I_{\alpha,m}(\vec{f}_l)\|_{L^{q,\lambda}(v,v)} \leq C \|\mathcal{M}_{\alpha,m}(\vec{f}_l)\|_{L^{q,\lambda}(v,v)}.$$

By Lemma 12, we have

$$|I_{\alpha,m}(\vec{f}_l)(x)| \leq C \sum_{k=0}^{\infty} \inf_{z \in 2^{k+3}\sqrt{n}Q_0} \mathcal{M}_{\alpha,m}(\vec{f})(z).$$

On the other hand, for every $Q' \subset \mathbb{R}^n$ the following holds:

$$\begin{aligned} \inf_{z \in Q'} \mathcal{M}_{\alpha,m}(\vec{f})(z) &\leq \left(\frac{1}{v(Q')^\lambda} \int_{Q'} \mathcal{M}_{\alpha,m}(\vec{f})(x)^q v(x) dx \right)^{\frac{1}{q}} v(Q')^{\frac{\lambda-1}{q}} \\ &\leq \|\mathcal{M}_{\alpha,m}(\vec{f})\|_{L^{q,\lambda}(v,v)} \cdot v(Q')^{\frac{\lambda-1}{q}}. \end{aligned}$$

Therefore we have

$$|I_{\alpha,m}(\vec{f}_l)(x)| \leq C \|\mathcal{M}_{\alpha,m}(\vec{f})\|_{L^{q,\lambda}(v,v)} \cdot \sum_{k=0}^{\infty} v(2^{k+3}\sqrt{n}Q_0)^{\frac{\lambda-1}{q}}.$$

On the other hand, by the reverse doubling condition of v , there exists $D > 1$ such that

$$v(2^{k+3}\sqrt{n}Q_0) \geq Dv(2^{k+2}\sqrt{n}Q_0) \geq \dots \geq D^{k+3}v(\sqrt{n}Q_0) \geq D^{k+3}v(Q_0).$$

Therefore we have

$$|I_{\alpha,m}(\vec{f}_l)(x)| \leq C \|\mathcal{M}_{\alpha,m}(\vec{f})\|_{L^{q,\lambda}(v,v)} \cdot v(Q_0)^{\frac{\lambda-1}{q}}.$$

This implies that the following holds for $0 < q < \infty$:

$$\|I_{\alpha,m}(\vec{f}_l)\|_{L^{q,\lambda}(v,v)} \leq C \|\mathcal{M}_{\alpha,m}(\vec{f})\|_{L^{q,\lambda}(v,v)}.$$

Therefore we get the desired result. \square

Proof of Lemma 12. Fix $x \in Q_0$. Because $\vec{l} \neq \vec{0}$, there exists $1 \leq j \leq m$ such that $y_j \in (3Q_0)^C$. Therefore $|x - y_j| > l(Q_0)$. Hence $|(x - y_1, \dots, x - y_m)| > l(Q_0)$. This implies that

$$\begin{aligned} |I_{\alpha,m}(\vec{f})(x)| &\leq \int_{|(x-y_1, \dots, x-y_m)| > l(Q_0)} \frac{|f_1(y_1)| \cdots |f_m(y_m)|}{|(x-y_1, \dots, x-y_m)|^{mn-\alpha}} d\vec{y} \\ &= \sum_{k=0}^{\infty} \int_{2^k l(Q_0) < |(x-y_1, \dots, x-y_m)| \leq 2^{k+1} l(Q_0)} \frac{|f_1(y_1)| \cdots |f_m(y_m)|}{|(x-y_1, \dots, x-y_m)|^{mn-\alpha}} d\vec{y} \\ &\leq \sum_{k=0}^{\infty} \frac{1}{2^k l(Q_0)^{mn-\alpha}} \prod_{j=1}^m \int_{|x-y_j| \leq 2^{k+1} l(Q_0)} |f_j(y_j)| dy_j \end{aligned}$$

Let x_0 be a center point of Q_0 . By the triangle inequality, we have

$$\begin{aligned} |x_0 - y_j| &\leq |x_0 - x| + |x - y_j| \leq \sqrt{n}l(Q_0) + 2^{k+1}l(Q_0) \\ &\leq 2^{k+1}\sqrt{n}l(Q_0) + 2^{k+1}\sqrt{n}l(Q_0) = 2^{k+2}\sqrt{n}l(Q_0). \end{aligned}$$

Since $B(x_0, 2^{k+2}\sqrt{n}l(Q_0)) \subset Q(x_0, 2^{k+3}\sqrt{n}l(Q_0)) = 2^{k+3}\sqrt{n}Q_0$, we obtain the following:

$$\{y_j : |x - y_j| \leq 2^{k+1}l(Q_0)\} \subset 2^{k+3}\sqrt{n}Q_0.$$

This implies that

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1}{2^k l(Q_0)^{mn-\alpha}} \prod_{j=1}^m \int_{|x-y_j| \leq 2^{k+1}l(Q_0)} |f_j(y_j)| dy_j \\ &\leq C \sum_{k=0}^{\infty} \left| 2^{k+3}\sqrt{n}Q_0 \right|^{\frac{\alpha}{n}} \prod_{j=1}^m \left(\int_{2^{k+3}\sqrt{n}Q_0} |f_j(y_j)| dy_j \right) \\ &\leq C \sum_{k=0}^{\infty} \inf_{z \in 2^{k+3}\sqrt{n}Q_0} \mathcal{M}_{\alpha,m}(\vec{f})(z). \quad \square \end{aligned}$$

Proof of Lemma 13. Note that

$$\begin{aligned} \mathcal{M}_{\alpha,m}(\vec{f})(x) &= \sup_{Q \ni x} l(Q)^\alpha \prod_{j=1}^m \int_Q |f_j(y_j)| dy_j \\ &= \sup_{Q \ni x} l(Q)^\alpha \prod_{j=1}^m \int_Q |f_j(y_j)| v(y_j)^{\frac{p_j}{a}\theta} v(y_j)^{-\frac{p_j}{a}\theta} dy_j. \end{aligned}$$

By Hölder’s inequality for $\frac{p_j}{a} > 1$,

$$\begin{aligned} \mathcal{M}_{\alpha,m}(\vec{f})(x) &\leq \sup_{Q \ni x} l(Q)^\alpha \prod_{j=1}^m \left(\int_Q |f_j(y_j)|^{\frac{p_j}{a}} v(y_j)^{\frac{p_j}{a}\theta} dy_j \right)^{\frac{a}{p_j}} \\ &\quad \times \left(\int_Q v(y_j)^{\frac{p_j}{a}\theta \left(\frac{p_j}{a}\right)'} dy_j \right)^{\frac{1}{\left(\frac{p_j}{a}\right)'}}. \end{aligned}$$

Calculating the right-hand side, we have

$$\begin{aligned} & \sup_{Q \ni x} l(Q)^\alpha \prod_{j=1}^m \left(\int_Q |f_j(y_j)|^{\frac{p_j}{a}} v(y_j)^{\frac{p_j}{a} \theta} dy_j \right)^{\frac{a}{p_j}} \cdot \left(\int_Q v(y_j)^{\frac{p_j}{p_j} \theta \left(\frac{p_j}{a} \right)'} dy_j \right)^{\frac{1}{\left(\frac{p_j}{a} \right)'}} \\ &= \sup_{Q \ni x} v^{q_1}(Q)^{\frac{\alpha}{n}} \prod_{j=1}^m \left(\frac{1}{v^{q_1}(Q)} \int_Q |f_j(y_j)|^{\frac{p_j}{a}} v(y_j)^{\frac{p_j}{a} \theta} dy_j \right)^{\frac{a}{p_j}} \\ & \quad \times \left(\frac{v^{q_1}(Q)}{|Q|} \right)^{\frac{a}{p} - \frac{\alpha}{n}} \prod_{j=1}^m \left(\int_Q v(y_j)^{\frac{p_j}{p_j} \theta \left(\frac{p_j}{a} \right)'} dy_j \right)^{\frac{1}{\left(\frac{p_j}{a} \right)'}}. \end{aligned}$$

Since $\frac{a}{p} - \frac{\alpha}{n} = \frac{\theta}{q_1}$, we have

$$\left(\frac{v^{q_1}(Q)}{|Q|} \right)^{\frac{a}{p} - \frac{\alpha}{n}} = \left(\int_Q v(x)^{q_1} dx \right)^{\frac{\theta}{q_1}}.$$

Applying Lemma 1 to $\frac{p}{p_j} \theta \left(\frac{p_j}{a} \right)' > \frac{p}{p_j} p'_j$, we have

$$\left(\int_Q v(y_j)^{\frac{p}{p_j} \theta \left(\frac{p_j}{a} \right)'} dy_j \right)^{\frac{1}{\left(\frac{p_j}{a} \right)'}} \leq C \left(\int_Q v(y_j)^{-\frac{p}{p_j} p'_j} dy_j \right)^{\frac{\theta}{p_j}} \quad (j = 1, 2, \dots, m).$$

Hence

$$\begin{aligned} & \left(\frac{v^{q_1}(Q)}{|Q|} \right)^{\frac{a}{p} - \frac{\alpha}{n}} \prod_{j=1}^m \left(\int_Q v(y_j)^{\frac{p}{p_j} \theta \left(\frac{p_j}{a} \right)'} dy_j \right)^{\frac{1}{\left(\frac{p_j}{a} \right)'}} \\ & \leq C \left(\int_Q v(x)^{q_1} dx \right)^{\frac{\theta}{q_1}} \prod_{j=1}^m \left(\int_Q v(y_j)^{-\frac{p}{p_j} p'_j} dy_j \right)^{\frac{\theta}{p_j}}. \end{aligned}$$

Since $\frac{p}{p_1} + \dots + \frac{p}{p_m} = 1$, we decompose

$$\left(\int_Q v(x)^{q_1} dx \right)^{\frac{\theta}{q_1}} = \left(\int_Q \left(v(x)^{\frac{p}{p_1}} \dots v(x)^{\frac{p}{p_m}} \right)^{q_1} dx \right)^{\frac{\theta}{q_1}}.$$

This implies that

$$\left(\int_Q v(x)^{q_1} dx \right)^{\frac{\theta}{q_1}} \prod_{j=1}^m \left(\int_Q v(y_j)^{-\frac{p}{p_j} p'_j} dy_j \right)^{\frac{\theta}{p_j}} \leq \left[\left(v^{\frac{p}{p_1}}, \dots, v^{\frac{p}{p_m}} \right) \right]_{A_{\vec{p}, q_1}}^\theta.$$

Therefore we have

$$\mathcal{M}_{\alpha, m}(\vec{f})(x) \leq C \left[\left(v^{\frac{p}{p_1}}, \dots, v^{\frac{p}{p_m}} \right) \right]_{A_{\vec{p}, q_1}}^\theta \cdot \mathcal{M}_{v^{q_1}, \frac{\vec{p}}{a}, \alpha} \left(\frac{f_1 v^{\theta \cdot \frac{p}{p_1}}}{(v^{q_1})^{\frac{a}{p_1}}}, \dots, \frac{f_m v^{\theta \cdot \frac{p}{p_m}}}{(v^{q_1})^{\frac{a}{p_m}}} \right)(x).$$

By the definition of F_j , we get the desired inequality. \square

Proof of Lemma 14. We follow the idea of in [28, Lemma 2.6] and [11].

$$\begin{aligned} \mathcal{M}_{V, \vec{p}, \alpha}(\vec{F})(x) &= \inf_{\varepsilon > 0} \max \left\{ \sup_{\substack{Q \ni x, \\ V(Q) < \varepsilon, Q: \text{cube}}} , \sup_{\substack{Q \ni x, \\ V(Q) \geq \varepsilon, Q: \text{cube}}} \right\} V(Q)^{\frac{\alpha}{n}} \\ &\quad \times \prod_{j=1}^m \left(\frac{1}{V(Q)} \int_Q |F_j(y_j)|^{\frac{p_j}{a}} V(y_j) dy_j \right)^{\frac{a}{p_j}} \\ &= \inf_{\varepsilon > 0} \max \{I_\varepsilon, II_\varepsilon\}. \end{aligned}$$

For $\varepsilon > 0$, we evaluate I_ε . We have

$$I_\varepsilon \leq \sup_{\substack{Q \ni x, \\ V(Q) < \varepsilon, Q: \text{cube}}} \varepsilon^{\frac{\alpha}{n}} \prod_{j=1}^m \left(\frac{1}{V(Q)} \int_Q |F_j(y_j)|^{\frac{p_j}{a}} V(y_j) dy_j \right)^{\frac{a}{p_j}} \leq \varepsilon^{\frac{\alpha}{n}} \mathcal{M}_V(\vec{F}^{\frac{p}{a}})(x)^{\frac{a}{p}}.$$

Next, we evaluate II_ε . By Hölder’s inequality for $a > 1$, we have

$$\begin{aligned} II_\varepsilon &\leq \sup_{\substack{Q \ni x, \\ V(Q) \geq \varepsilon, Q: \text{cube}}} V(Q)^{\frac{\alpha}{n} - \frac{1}{p} + \frac{\lambda}{p}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{1}{p_j}} \\ &\leq \varepsilon^{\frac{\alpha}{n} - \frac{1}{p} + \frac{\lambda}{p}} \sup_{\substack{Q \ni x, \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{1}{p_j}}. \end{aligned}$$

We choose $\varepsilon > 0$ such that the following equality holds:

$$\varepsilon^{\frac{\alpha}{n}} \mathcal{M}_V(\vec{F}^{\frac{p}{a}})(x)^{\frac{a}{p}} = \varepsilon^{\frac{\alpha}{n} - \frac{1}{p} + \frac{\lambda}{p}} \sup_{\substack{Q \ni x, \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{1}{p_j}}.$$

This implies that

$$\varepsilon = \left(\frac{\sup_{\substack{Q \subset \mathbb{R}^n, \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{1}{p_j}}}{\mathcal{M}_V(\vec{F}^{\frac{p}{a}})(x)^{\frac{a}{p}}} \right)^{\frac{p}{1-\lambda}}.$$

Then we have

$$\begin{aligned} \mathcal{M}_{V, \frac{\vec{p}}{a}, \alpha}(\vec{F})(x) &\leq \left(\sup_{\substack{Q \subset \mathbb{R}^n \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)| V(y_j) dy_j \right)^{\frac{1}{p_j}} \right)^{\frac{p}{1-\lambda} \cdot \frac{\alpha}{n}} \\ &\quad \times \left(\mathcal{M}_V \left(\vec{F} \frac{\vec{p}}{a} \right) (x)^{\frac{a}{\vec{p}}} \right)^{1 - \frac{p}{1-\lambda} \cdot \frac{\alpha}{n}}. \end{aligned}$$

Since $\frac{p}{1-\lambda} \cdot \frac{\alpha}{n} = 1 - \frac{p}{q_2}$ and $1 - \frac{p}{1-\lambda} \cdot \frac{\alpha}{n} = \frac{p}{q_2}$, we get the desired inequality.

$$\begin{aligned} \mathcal{M}_{V, \frac{\vec{p}}{a}, \alpha}(\vec{F})(x) &\leq \left(\sup_{\substack{Q \subset \mathbb{R}^n \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)| V(y_j) dy_j \right)^{\frac{1}{p_j}} \right)^{1 - \frac{p}{q_2}} \\ &\quad \times \left(\mathcal{M}_V \left(\vec{F} \frac{\vec{p}}{a} \right) (x)^{\frac{a}{\vec{p}}} \right)^{\frac{p}{q_2}}. \quad \square \end{aligned}$$

Proof of Lemma 15. Writing out the norm in fully, we have

$$\left\| \left(\mathcal{M}_V \left(\vec{F} \frac{\vec{p}}{a} \right) (\cdot)^{\frac{a}{\vec{p}P}} \right) \right\|_{L^{q_2, \lambda}(V, V)}^{\frac{p}{q_2}} = \sup_{\substack{Q \subset \mathbb{R}^n \\ Q: \text{cube}}} \left(\frac{1}{V(Q)^\lambda} \int_Q \mathcal{M}_V \left(\vec{F} \frac{\vec{p}}{a} \right) (x)^{\frac{a}{\vec{p}P}} V(x) dx \right)^{\frac{1}{q_2}}.$$

For every cube $Q_0 \subset \mathbb{R}^n$, let $F_j(y_j) = F_j(y_j)\chi_{3Q_0}(y_j) + F_j(y_j)\chi_{(3Q_0)^c}(y_j) = F_j^0(y_j) + F_j^\infty(y_j)$ ($j = 1, 2, \dots, m$), where $F_j^0 = F_j\chi_{3Q_0}$ and $F_j^\infty = F_j\chi_{(3Q_0)^c}$. Then we have

$$\begin{aligned} \mathcal{M}_V \left(\vec{F} \frac{\vec{p}}{a} \right) (x)^{\frac{a}{\vec{p}P}} &\leq \mathcal{M}_V \left((F_1^0)^{\frac{p_1}{a}}, \dots, (F_m^0)^{\frac{p_m}{a}} \right) (x)^{\frac{a}{\vec{p}P}} \\ &\quad + \sum_{\substack{\vec{l} \neq \vec{0}, \\ \vec{l} \in \{0, \infty\}^m}} \mathcal{M}_V \left((F_1^{l_1})^{\frac{p_1}{a}}, \dots, (F_m^{l_m})^{\frac{p_m}{a}} \right) (x)^{\frac{a}{\vec{p}P}}. \end{aligned}$$

We evaluate

$$\left(\int_{Q_0} \mathcal{M}_V \left((F_1^0)^{\frac{p_1}{a}}, \dots, (F_m^0)^{\frac{p_m}{a}} \right) (x)^{\frac{a}{\vec{p}P}} V(x) dx \right).$$

By change ‘sup’ and ‘ \prod ’, we obtain

$$\mathcal{M}_V \left((F_1^0)^{\frac{p_1}{a}}, \dots, (F_m^0)^{\frac{p_m}{a}} \right) (x)^{\frac{a}{\vec{p}P}} \leq \prod_{j=1}^m \mathcal{M}_V \left((F_j^0)^{\frac{p_j}{a}} \right) (x)^{\frac{a}{p_j P}}.$$

Therefore we have

$$\begin{aligned} &\left(\int_{Q_0} \mathcal{M}_V \left((F_1^0)^{\frac{p_1}{a}}, \dots, (F_m^0)^{\frac{p_m}{a}} \right) (x)^{\frac{a}{\vec{p}P}} V(x) dx \right) \\ &\leq \left(\int_{Q_0} \prod_{j=1}^m \mathcal{M}_V \left((F_j^0)^{\frac{p_j}{a}} \right) (x)^{\frac{a}{p_j P}} V(x) dx \right). \end{aligned}$$

By Hölder’s inequality for $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, we have

$$\left(\int_{Q_0} \prod_{j=1}^m M_V \left((F_j^0)^{\frac{p_j}{a}} \right) (x)^{\frac{a}{p_j} p} V(x) dx \right) \leq \prod_{j=1}^m \left(\int_{Q_0} M_V \left((F_j^0)^{\frac{p_j}{a}} \right) (x)^a V(x) dx \right)^{\frac{p}{p_j}}.$$

By Lemma 7, we obtain

$$\prod_{j=1}^m \left(\int_{Q_0} M_V \left((F_j^0)^{\frac{p_j}{a}} \right) (x)^a V(x) dx \right)^{\frac{p}{p_j}} \leq C \prod_{j=1}^m \left(\int_{3Q_0} |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{p}{p_j}}.$$

Taking supremum for every cube, we have

$$\begin{aligned} & \prod_{j=1}^m \left(\int_{3Q_0} |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{p}{p_j}} \\ & \leq CV(3Q_0)^\lambda \left(\sup_{\substack{Q \subset \mathbb{R}^n \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{1}{p_j}} \right)^P. \end{aligned}$$

Since $V \in \Delta_2$, we have $V(3Q_0) \leq CV(Q_0)$. Therefore we have

$$\begin{aligned} & \frac{1}{v(Q_0)^\lambda} \int_{Q_0} \mathcal{M}_V \left((F_1^0)^{\frac{p_1}{a}}, \dots, (F_m^0)^{\frac{p_m}{a}} \right) (x)^{\frac{a}{p} p} V(x) dx \\ & \leq C \left(\sup_{\substack{Q \subset \mathbb{R}^n \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{1}{p_j}} \right)^P. \end{aligned}$$

Next, for $\vec{l} \neq \vec{0}$, we evaluate

$$\mathcal{M}_V \left((F_1^l)^{\frac{p_1}{a}}, \dots, (F_m^l)^{\frac{p_m}{a}} \right) (x)^{\frac{a}{p} p}.$$

Since $\vec{l} \neq \vec{0}$, there exists $i \in \{1, 2, \dots, m\}$ such that $y_i \in (3Q_0)^C$. If $x \in Q_0$, $y_i \in (3Q_0)^C$, $x \in Q$ and $y_i \in Q$, then we have $Q_0 \subset 3Q$. This implies that

$$\begin{aligned} & \mathcal{M}_V \left((F_1^l)^{\frac{p_1}{a}}, \dots, (F_m^l)^{\frac{p_m}{a}} \right) (x)^{\frac{a}{p} p} \\ & \leq C \sup_{\substack{Q_0 \subset Q \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{a}{p_j} p}. \end{aligned}$$

By Hölder’s inequality for $a > 1$, we have

$$\left(\frac{1}{V(Q)} \int_Q |F_j(y_j)|^{\frac{p_j}{a} p} V(y_j) dy_j \right)^{\frac{a}{p_j} p} \leq \left(\frac{1}{V(Q)} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{p}{p_j}} \quad (j = 1, \dots, m).$$

This implies that

$$\begin{aligned} & \mathcal{M}_V \left(\left(F_1^{l_1} \right)^{\frac{p_1}{a}}, \dots, \left(F_m^{l_m} \right)^{\frac{p_m}{a}} \right) (x)^{\frac{a}{p}P} \\ & \leq C \sup_{\substack{Q_0 \subset Q, \\ Q_0: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{p}{p_j}} V(Q)^{\lambda-1}. \end{aligned}$$

Since $V(Q)^{\lambda-1} \leq V(Q_0)^{\lambda-1}$ for $Q_0 \subset Q$, we obtain

$$\begin{aligned} & \mathcal{M}_V \left(\left(F_1^{l_1} \right)^{\frac{p_1}{a}}, \dots, \left(F_m^{l_m} \right)^{\frac{p_m}{a}} \right) (x)^{\frac{a}{p}P} \\ & \leq CV(Q_0)^{\lambda-1} \sup_{\substack{Q_0 \subset Q, \\ Q_0: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{p}{p_j}}. \end{aligned}$$

Therefore we have

$$\begin{aligned} & \frac{1}{V(Q)^\lambda} \int_{Q_0} \mathcal{M}_V \left(\left(F_1^{l_1} \right)^{\frac{p_1}{a}}, \dots, \left(F_m^{l_m} \right)^{\frac{p_m}{a}} \right) (x)^{\frac{a}{p}P} V(x) dx \\ & \leq CV(Q_0)^\lambda \sup_{\substack{Q_0 \subset Q, \\ Q_0: \text{cube}}} \prod_{j=1}^m \left(\int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{p}{p_j}}. \end{aligned}$$

Hence we get the desired inequality. \square

Proof of Lemma 16. Note that $F_j(y_j) = \frac{|f_j(y_j)|v(y_j)^{\frac{\theta}{p_j}}}{v(y_j)^{q_1 \cdot \frac{a}{p_j}}}$ and $V(y_j) = v(y_j)^{q_1}$.

For every cube $Q \subset \mathbb{R}^n$, we have

$$\begin{aligned} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j &= \int_Q \left(\frac{|f_j(y_j)|v(y_j)^{\frac{\theta}{p_j}}}{v(y_j)^{q_1 \cdot \frac{a}{p_j}}} \right)^{p_j} v(y_j)^{q_1} dy_j \\ &= \int_Q |f_j(y_j)|^{p_j} v(y_j)^{\theta p - q_1 a + q_1} dy_j. \end{aligned}$$

Since $\frac{\theta}{q_1} = \frac{a}{p} - \frac{\alpha}{n}$, we have

$$\theta p - q_1 a + q_1 = p q_1 \left(\frac{\theta}{q_1} - \frac{a}{p} + \frac{1}{p} \right) = p q_1 \left(\frac{1}{p} - \frac{\alpha}{n} \right) = p.$$

Hence we have

$$\int_Q |f_j(y_j)|^{p_j} v(y_j)^{\theta p - q_1 a + q_1} dy_j = \int_Q |f_j(y_j)|^{p_j} v(y_j)^p dy_j.$$

Therefore we get the following equality:

$$\begin{aligned} & \sup_{\substack{Q \subset \mathbb{R}^n \\ Q; \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)| V(y_j) dy_j \right)^{\frac{1}{p_j}} \\ &= \sup_{\substack{Q \subset \mathbb{R}^n \\ Q; \text{cube}}} \prod_{j=1}^m \left(\frac{1}{v^{q_1}(Q)^\lambda} \int_Q |f_j(y_j)|^{p_j} v(y_j)^p dy_j \right)^{\frac{1}{p_j}}. \quad \square \end{aligned}$$

4. Proofs of Theorems

Firstly, we prove Theorem 1.

Proof of Theorem 1. For every dyadic cube $Q_0 \in \mathcal{D}(\mathbb{R}^n)$, for $x \in Q_0$, we have

$$\begin{aligned} & \left| [b, I_\alpha]^{(m)} f(x) \right| \\ & \leq \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f(y)| dy \\ & \leq \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f(y)| \chi_{3Q_0}(y) dy + \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f(y)| \chi_{(3Q_0)^c}(y) dy \\ & = \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f_0(y)| dy + \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f_\infty(y)| dy. \end{aligned}$$

We evaluate $\int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f_\infty(y)| dy$. From the definition of f_∞ , we have

$$\int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f_\infty(y)| dy = \int_{(3Q_0)^c} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f(y)| dy.$$

If $x \in Q_0$ and $y \in (3Q_0)^c$, then we have $|x - y| > l(Q_0)$. This implies that

$$\int_{(3Q_0)^c} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f(y)| dy \leq \int_{|x-y|>l(Q_0)} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f(y)| dy.$$

Since $\{y : |x - y| > l(Q_0)\} = \bigcup_{k=0}^\infty \{y : 2^k l(Q_0) < |x - y| \leq 2^{k+1} l(Q_0)\}$, we obtain

$$\int_{|x-y|>l(Q_0)} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f(y)| dy = \sum_{k=0}^\infty \int_{2^k l(Q_0) < |x-y| < 2^{k+1} l(Q_0)} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f(y)| dy.$$

If $2^k l(Q_0) < |x - y|$, then we have

$$\frac{1}{|x - y|^{n-\alpha}} < \frac{1}{(2^k l(Q_0))^{n-\alpha}}.$$

This implies that

$$\begin{aligned} & \sum_{k=0}^{\infty} \int_{2^k l(Q_0) < |x-y| < 2^{k+1} l(Q_0)} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f(y)| dy \\ & \leq \sum_{k=0}^{\infty} \frac{1}{(2^k l(Q_0))^{n-\alpha}} \int_{|x-y| < 2^{k+1} l(Q_0)} |b(x) - b(y)|^m |f(y)| dy \\ & = C \sum_{k=0}^{\infty} \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ l(Q) = 2^{k+1} l(Q_0)}} \frac{1}{l(Q)^{n-\alpha}} \chi_Q(x) \int_{|x-y| < l(Q)} |b(x) - b(y)|^m |f(y)| dy. \end{aligned}$$

If $x \in Q$ and $|x - y| < l(Q)$, then $y \in 3Q$. Therefore we have

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ l(Q) = 2^{k+1} l(Q_0)}} \frac{1}{l(Q)^{n-\alpha}} \chi_Q(x) \int_{|x-y| < l(Q)} |b(x) - b(y)|^m |f(y)| dy \\ & \leq \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \not\supseteq Q_0}} \frac{1}{l(Q)^{n-\alpha}} \chi_Q(x) \int_{3Q} |b(x) - b(y)|^m |f(y)| dy. \end{aligned}$$

By $(a + b)^m \leq C(a^m + b^m)$, we obtain

$$\begin{aligned} & \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \not\supseteq Q_0}} \frac{1}{l(Q)^{n-\alpha}} \chi_Q(x) \int_{3Q} |b(x) - m_{w^{q_1}, Q_0}(b) + m_{w^{q_1}, Q_0}(b) - b(y)|^m |f(y)| dy \\ & = C \left(\sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \not\supseteq Q_0}} \frac{1}{l(Q)^{n-\alpha}} \chi_Q(x) |b(x) - m_{w^{q_1}, Q_0}(b)|^m \int_{3Q} |f(y)| dy \right) \\ & \quad + \left(\sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \not\supseteq Q_0}} \frac{1}{l(Q)^{n-\alpha}} \chi_Q(x) \int_{3Q} |m_{w^{q_1}, Q_0}(b) - b(y)|^m |f(y)| dy \right) \\ & = C(I_{\infty}(x) + II_{\infty}(x)). \end{aligned}$$

We evaluate $I_{\infty}(x)$. Recall that $I_{\infty}(x)$ is given by:

$$I_{\infty}(x) = \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \not\supseteq Q_0}} \frac{1}{l(Q)^{n-\alpha}} \chi_Q(x) |b(x) - m_{w^{q_1}, Q}(b)|^m \left(\int_{3Q} |f(y)| dy \right).$$

By Hölder’s inequality for $p > 1$, we have

$$I_\infty(x) = \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \not\supseteq Q_0}} \frac{1}{l(Q)^{n-\alpha}} \chi_Q(x) |b(x) - m_{w^{q_1}, Q}(b)|^m \left(\int_{3Q} |f(y)|^p w(y)^p dy \right)^{\frac{1}{p}} \\ \times \left(\int_{3Q} w(y)^{-p'} dy \right)^{\frac{1}{p'}}.$$

By taking weighted Morrey norm, we have

$$I_\infty(x) \leq \|f\|_{L^{p,\lambda}(w^p, w^{q_1})} \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \not\supseteq Q_0}} \frac{\chi_Q(x)}{l(Q)^{n-\alpha}} |b(x) - m_{w^{q_1}, Q}(b)|^m w^{q_1}(3Q)^{\frac{\lambda}{p}} \\ \times \left(\int_{3Q} w(y)^{-p'} dy \right)^{\frac{1}{p'}}.$$

This implies that

$$\left(\frac{1}{w^{q_1}(Q_0)^\lambda} \int_{Q_0} I_\infty(x)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \\ \leq \|f\|_{L^{p,\lambda}(w^p, w^{q_1})} \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \not\supseteq Q_0}} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} w^{q_1}(3Q)^{\frac{\lambda}{p}} \left(\int_{3Q} w(y)^{-p'} dy \right)^{\frac{1}{p'}} \\ \times \left(\int_{Q_0} |b(x) - m_{w^{q_1}, Q}(b)|^{mq_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}}.$$

Next, we check that the following inequality holds:

$$\left(\int_{Q_0} |b(x) - m_{w^{q_1}, Q}(b)|^{mq_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \leq C(1+k^m) \|b\|_{\text{BMO}}^m w^{q_1}(Q_0)^{\frac{1}{q_2}}. \tag{14}$$

If $Q \in \mathcal{D}(\mathbb{R}^n)$ and $Q \not\supseteq Q_0$, then there exists $k = 1, 2, \dots$, such that $Q =: Q_k \not\supseteq Q_{k-1} \not\supseteq \dots \not\supseteq Q_1 \not\supseteq Q_0$, where $Q_1, \dots, Q_k \in \mathcal{D}(\mathbb{R}^n)$ and $|Q_j| = 2^n |Q_{j-1}|$. By the triangle inequality, we have

$$|b(x) - b_{w^{q_1}, Q}(b)| \\ = |b(x) - m_{w^{q_1}, Q_0}(b) + m_{w^{q_1}, Q_0}(b) - m_{w^{q_1}, Q_1}(b) + m_{w^{q_1}, Q_1}(b) - \dots \\ + m_{w^{q_1}, Q_{k-1}}(b) - m_{w^{q_1}, Q_k}(b)| \\ \leq |b(x) - m_{w^{q_1}, Q_0}(b)| + |m_{w^{q_1}, Q_0}(b) - m_{w^{q_1}, Q_1}(b)| + \dots + |m_{w^{q_1}, Q_{k-1}}(b) - m_{w^{q_1}, Q_k}(b)|.$$

Therefore we have for $j = 1, 2, \dots, k$

$$\begin{aligned} \left| m_{w^{q_1}, 2^{j-1}Q_{j-1}}(b) - m_{w^{q_1}, Q_j}(b) \right| &= \frac{1}{w^{q_1}(Q_{j-1})} \left| \int_{Q_{j-1}} (b(x) - m_{w^{q_1}, Q_j}(b)) w(x)^{q_1} dx \right| \\ &\leq \frac{1}{w^{q_1}(Q_{j-1})} \int_{Q_{j-1}} |b(x) - m_{w^{q_1}, Q_j}(b)| w(x)^{q_1} dx \\ &\leq \frac{1}{w^{q_1}(Q_{j-1})} \int_{Q_j} |b(x) - m_{w^{q_1}, Q_j}(b)| w(x)^{q_1} dx. \end{aligned}$$

Since $w^{q_1} \in A_{1+\frac{q_1}{p}}(\mathbb{R}^n)$, we have $w^{q_1}(2Q) \leq Cw^{q_1}(Q)$. This implies that

$$\left| m_{w^{q_1}, Q_{j-1}}(b) - m_{w^{q_1}, Q_j}(b) \right| \leq \frac{C}{w^{q_1}(Q_j)} \int_{Q_j} |b(x) - m_{w^{q_1}, Q_j}(b)| w(x)^{q_1} dx.$$

By the inequality (6) in Lemma 10,

$$\frac{1}{w^{q_1}(Q_j)} \int_{Q_j} |b(x) - m_{w^{q_1}, Q_j}(b)| w(x)^{q_1} dx \leq C \|b\|_{\text{BMO}}.$$

Therefore we obtain

$$|b(x) - m_{w^{q_1}, Q}(b)|^m \leq (|b(x) - m_{w^{q_1}, Q_0}(b)| + Ck \|b\|_{\text{BMO}})^m.$$

We obtain the following inequality:

$$|b(x) - m_{w^{q_1}, Q}(b)|^m \leq C (|b(x) - m_{w^{q_1}, Q_0}(b)|^m + k^m \|b\|_{\text{BMO}}^m).$$

Hence we have

$$\begin{aligned} &\left(\int_{Q_0} |b(x) - m_{w^{q_1}, Q}(b)|^{mq_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \\ &\leq C \left(\int_{Q_0} (|b(x) - m_{w^{q_1}, Q_0}(b)|^m + k^m \|b\|_{\text{BMO}}^m)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}}. \end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned} &\left(\int_{Q_0} (|b(x) - m_{w^{q_1}, Q_0}(b)|^m + k^m \|b\|_{\text{BMO}}^m)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \\ &\leq \left(\int_{Q_0} |b(x) - m_{w^{q_1}, Q_0}(b)|^{mq_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} + \left(\int_{Q_0} (k^m \|b\|_{\text{BMO}}^m)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \\ &= \left(\int_{Q_0} |b(x) - m_{w^{q_1}, Q_0}(b)|^{mq_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} + k^m \|b\|_{\text{BMO}} w^{q_1}(Q_0)^{\frac{1}{q_2}}. \end{aligned}$$

By the inequality (6) in Lemma 10, we have

$$\left(\int_{Q_0} |b(x) - m_{w^{q_1}, Q_0}(b)|^{mq_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \leq C \|b\|_{\text{BMO}}^m w^{q_1}(Q_0)^{\frac{1}{q_2}}.$$

Therefore we obtain the inequality (14).

We return to the estimate $I_\infty(x)$. By the inequality (14), we have

$$\begin{aligned} & \left(\frac{1}{w^{q_1}(Q_0)^\lambda} \int_{Q_0} I_\infty(x)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \\ & \leq C \|b\|_{\text{BMO}}^m \|f\|_{L^{p, \lambda}(w^p, w^{q_1})} \\ & \quad \times \sum_{k=1}^\infty (1+k^m) \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q = Q_k \supseteq Q_0}} \frac{w^{q_1}(Q_0)^{\frac{1-\lambda}{q_2}}}{l(Q)^{n-\alpha}} w^{q_1}(3Q)^{\frac{\lambda}{p}} \left(\int_{3Q} w(y)^{-p'} dy \right)^{\frac{1}{p'}} \\ & = C \|b\|_{\text{BMO}}^m \|f\|_{L^{p, \lambda}(w^p, w^{q_1})} w^{q_1}(Q_0)^{\frac{1-\lambda}{q_2}} \\ & \quad \times \sum_{k=1}^\infty (1+k^m) \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q = Q_k \supseteq Q_0}} w^{q_1}(3Q)^{\frac{\lambda-1}{q_2}} \frac{|3Q|^{\frac{1}{q_1} + \frac{1}{p'}}}{|Q|^{1-\frac{\alpha}{n}}} \left(\int_{3Q} w(x)^{q_1} dx \right)^{\frac{1}{q_1}} \left(\int_{3Q} w(y)^{-p'} dy \right)^{\frac{1}{p'}}. \end{aligned}$$

Since $w \in A_{p, q_1}(\mathbb{R}^n)$, we have

$$\begin{aligned} & \left(\frac{1}{w^{q_1}(Q_0)^\lambda} \int_{Q_0} I_\infty(x)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \\ & \leq C [w]_{A_{p, q_1}} \|b\|_{\text{BMO}}^m \|f\|_{L^{p, \lambda}(w^p, w^{q_1})} w^{q_1}(Q_0)^{\frac{1-\lambda}{q_2}} \\ & \quad \times \sum_{k=1}^\infty \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q = Q_k \supseteq Q_0}} \frac{|3Q|^{\frac{1}{q_1} + \frac{1}{p'}}}{|Q|^{1-\frac{\alpha}{n}}} w^{q_1}(3Q)^{\frac{\lambda-1}{q_2}} (1+k^m). \end{aligned}$$

Since $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$, we have

$$\begin{aligned} & \left(\frac{1}{w^{q_1}(Q_0)^\lambda} \int_{Q_0} I_\infty(x)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \\ & \leq C [w]_{A_{p, q_1}} \|b\|_{\text{BMO}}^m \|f\|_{L^{p, \lambda}(w^p, w^{q_1})} w^{q_1}(Q_0)^{\frac{1-\lambda}{q_2}} \sum_{k=1}^\infty \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q = Q_k \supseteq Q_0}} w^{q_1}(3Q)^{\frac{\lambda-1}{q_2}} (1+k^m). \end{aligned}$$

Since $w^{q_1} \in A_{1+\frac{q_1}{p}}(\mathbb{R}^n)$, there exists $D > 1$ such that

$$w^{q_1}(3Q) \geq Dw^{q_1}(Q) = Dw^{q_1}(2^{k+1}Q_0) \geq D^{k+1}w^{q_1}(Q_0).$$

Hence, we have

$$w^{q_1}(3Q)^{\frac{\lambda-1}{q_2}} \leq \left(D^{k+1} w^{q_1}(Q_0) \right)^{\frac{\lambda-1}{q_2}}.$$

Therefore we have

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q=Q_k \not\supseteq Q_0}} w^{q_1}(3Q)^{\frac{\lambda-1}{q_2}} (1+k^m) &\leq \sum_{k=1}^{\infty} \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q=Q_k \not\supseteq Q_0}} \left(D^{k+1} w^{q_1}(Q_0) \right)^{\frac{\lambda-1}{q_2}} (1+k^m) \\ &= w^{q_1}(Q_0)^{\frac{\lambda-1}{q_2}} \sum_{k=1}^{\infty} \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q=Q_k \not\supseteq Q_0}} D^{(k+1)\frac{\lambda-1}{q_2}} (1+k^m) \\ &= w^{q_1}(Q_0)^{\frac{\lambda-1}{q_2}} \sum_{k=1}^{\infty} D^{(k+1)\frac{\lambda-1}{q_2}} (1+k^m). \end{aligned}$$

Since the series $\sum_{k=1}^{\infty} D^{(k+1)\frac{\lambda-1}{q_2}} (1+k^m)$ is convergent, we have

$$w^{q_1}(Q_0)^{\frac{\lambda-1}{q_2}} \sum_{k=1}^{\infty} D^{(k+1)\frac{\lambda-1}{q_2}} (1+k^m) \leq C w^{q_1}(Q_0)^{\frac{\lambda-1}{q_2}}.$$

Therefore we have

$$\left(\frac{1}{w^{q_1}(Q_0)^\lambda} \int_{Q_0} I_\infty(x)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \leq C[w]_{A_{p,q_1}} \cdot \|b\|_{\text{BMO}}^m \cdot \|f\|_{L^{p,\lambda}(w^{p,q_1})}.$$

Next, we evaluate $II_\infty(x)$. By Hölder’s inequality for a small number $\theta_1 > 1$,

$$\begin{aligned} II_\infty(x) &\leq \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \not\supseteq Q_0}} \frac{1}{l(Q)^{n-\alpha}} \chi_Q(x) \left(\int_{3Q} |m_{w^{q_1},Q}(b) - b(y)|^{m\theta'_1} dy \right)^{\frac{1}{\theta'_1}} \\ &\quad \times \left(\int_{3Q} |f(y)|^{\theta_1} dy \right)^{\frac{1}{\theta_1}} |3Q|. \end{aligned}$$

Since $|m_{w^{q_1},Q}(b) - b(y)| \leq |m_{w^{q_1},Q}(b) - m_{w^{q_1},3Q}(b)| + |m_{w^{q_1},3Q}(b) - b(y)|$, we have

$$\begin{aligned} &\left(\int_{3Q} |m_{w^{q_1},Q}(b) - b(y)|^{m\theta'_1} dy \right)^{\frac{1}{\theta'_1}} \\ &\leq \left(\int_{3Q} |m_{w^{q_1},Q}(b) - m_{w^{q_1},3Q}(b)|^{m\theta'_1} dy \right)^{\frac{1}{\theta'_1}} + \left(\int_{3Q} |m_{w^{q_1},3Q}(b) - b(y)|^{m\theta'_1} dy \right)^{\frac{1}{\theta'_1}} \\ &= |m_{w^{q_1},Q}(b) - m_{w^{q_1},3Q}(b)|^m + \left(\int_{3Q} |m_{w^{q_1},3Q}(b) - b(y)|^{m\theta'_1} dy \right)^{\frac{1}{\theta'_1}} \end{aligned}$$

$$= \left| \frac{1}{w^{q_1}(Q)} \int_Q (b(y) - m_{w^{q_1}, 3Q}(b)) w(y)^{q_1} dy \right|^m + \left(\int_{3Q} |m_{w^{q_1}, 3Q}(b) - b(y)|^{m\theta'_1} dy \right)^{\frac{1}{\theta'_1}}.$$

Since $w^{q_1} \in A_{1+\frac{q_1}{p'}}(\mathbb{R}^n)$, we have $w^{q_1}(3Q) \leq Cw^{q_1}(Q)$. Therefore we obtain

$$\begin{aligned} & \left(\frac{1}{w^{q_1}(Q)} \int_Q |b(y) - m_{w^{q_1}, 3Q}(b)| w(y)^{q_1} dy \right)^m \\ & \leq \left(\frac{C}{w^{q_1}(3Q)} \int_{3Q} |b(y) - m_{w^{q_1}, 3Q}(b)| w(y)^{q_1} dy \right)^m. \end{aligned}$$

By the inequalities (6) and (7) in Lemma 10,

$$\left(\int_{3Q} |m_{w^{q_1}, Q}(b) - b(y)|^{m\theta'_1} dy \right)^{\frac{1}{\theta'_1}} \leq C \|b\|_{\text{BMO}}^m.$$

This implies the following estimate:

$$II_\infty(x) \leq C \|b\|_{\text{BMO}}^m \cdot \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \supset Q_0}} |Q|^{\frac{\alpha}{n}} \chi_Q(x) \left(\int_{3Q} |f(y)|^{\theta_1} dy \right)^{\frac{1}{\theta_1}}.$$

By Hölder’s inequality for $\frac{p}{\theta_1} > 1$, we have

$$\begin{aligned} II_\infty(x) & \leq C \|b\|_{\text{BMO}}^m \cdot \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \supset Q_0}} |Q|^{\frac{\alpha}{n}} \chi_Q(x) \left(\int_{3Q} |f(y)|^p w(y)^p dy \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_{3Q} w(y)^{-\theta_1 \cdot \left(\frac{p}{\theta_1}\right)'} dy \right)^{\frac{1}{\theta_1} \cdot \frac{1}{\left(\frac{p}{\theta_1}\right)'}}. \end{aligned}$$

By Lemma 1 for $\theta_1 \left(\frac{p}{\theta_1}\right)' > p'$, we have

$$\begin{aligned} II_\infty(x) & \leq C \|b\|_{\text{BMO}}^m \cdot \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \supset Q_0}} |Q|^{\frac{\alpha}{n}} \chi_Q(x) \left(\int_{3Q} |f(y)|^p w(y)^p dy \right)^{\frac{1}{p}} \cdot \left(\int_{3Q} w(y)^{-p'} dy \right)^{\frac{1}{p'}} \\ & \leq C \|b\|_{\text{BMO}}^m \cdot \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \supset Q_0}} |Q|^{\frac{\alpha}{n}} \chi_Q(x) \frac{1}{|3Q|^{\frac{1}{p}}} \left(\frac{1}{w^{q_1}(3Q)^\lambda} \int_{3Q} |f(y)|^p w(y)^p dy \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_{3Q} w(y)^{-p'} dy \right)^{\frac{1}{p'}} w^{q_1}(3Q)^{\frac{\lambda}{p}}. \end{aligned}$$

Taking the weighted Morrey norm, we have

$$\begin{aligned}
 I_{\infty}(x) &\leq C \|b\|_{\text{BMO}}^m \cdot \|f\|_{L^{p,\lambda}(w^p, w^{q_1})} \\
 &\quad \times \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \supset Q_0}} |Q|^{\frac{\alpha}{n}} \chi_Q(x) \frac{1}{|3Q|^{\frac{1}{p}}} \left(\int_{3Q} w(y)^{-p'} dy \right)^{\frac{1}{p'}} w^{q_1}(3Q)^{\frac{\lambda}{p}}.
 \end{aligned}$$

Since $w \in A_{p,q_1}(\mathbb{R}^n)$, we have

$$\begin{aligned}
 I_{\infty}(x) &\leq C [w]_{A_{p,q_1}} \|b\|_{\text{BMO}}^m \cdot \|f\|_{L^{p,\lambda}(w^p, w^{q_1})} \\
 &\quad \times \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \supset Q_0}} |Q|^{\frac{\alpha}{n}} \chi_Q(x) \frac{1}{|Q|^{\frac{1}{p}}} \left(\int_{3Q} w(y)^{q_1} dy \right)^{-\frac{1}{q_1}} w^{q_1}(3Q)^{\frac{\lambda}{p}}.
 \end{aligned}$$

This implies that

$$I_{\infty}(x) \leq C [w]_{A_{p,q_1}} \|b\|_{\text{BMO}}^m \cdot \|f\|_{L^{p,\lambda}(w^p, w^{q_1})} \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n), \\ Q \supset Q_0}} \left(\int_{3Q} w(y)^{q_1} dy \right)^{\frac{\lambda-1}{q_2}}.$$

Next, we examine that the series is convergent. Since $w^{q_1} \in A_{1+\frac{q_1}{p}}(\mathbb{R}^n)$, there exists $D > 1$ such that

$$\begin{aligned}
 I_{\infty}(x) &\leq C [w]_{A_{p,q_1}} \|b\|_{\text{BMO}}^m \cdot \|f\|_{L^{p,\lambda}(w^p, w^{q_1})} \sum_{k=0}^{\infty} D^{(k+1)\frac{\lambda-1}{q_2}} \left(\int_{Q_0} w(y)^{q_1} dy \right)^{\frac{\lambda-1}{q_2}} \\
 &\leq C [w]_{A_{p,q_1}} \|b\|_{\text{BMO}}^m \cdot \|f\|_{L^{p,\lambda}(w^p, w^{q_1})} \left(\int_{Q_0} w(y)^{q_1} dy \right)^{\frac{\lambda-1}{q_2}} \sum_{k=0}^{\infty} D^{(k+1)\frac{\lambda-1}{q_2}} \\
 &\leq C [w]_{A_{p,q_1}} \|b\|_{\text{BMO}}^m \cdot \|f\|_{L^{p,\lambda}(w^p, w^{q_1})} w^{q_1}(Q_0)^{\frac{\lambda-1}{q_2}}.
 \end{aligned}$$

This implies that

$$\left(\frac{1}{w^{q_1}(Q_0)^{\lambda}} \int_{Q_0} I_{\infty}(x)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \leq C [w]_{A_{p,q_1}} \|b\|_{\text{BMO}}^m \cdot \|f\|_{L^{p,\lambda}(w^p, w^{q_1})}.$$

Next, we evaluate $\int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f_0(y)| dy$. By telescoping $m_{w^{q_1}, Q}(b)$, we have

$$\begin{aligned}
 &\int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f_0(y)| dy \\
 &= \int_{3Q_0} \frac{|b(x) - b(y)|^m}{|x - y|^{n-\alpha}} |f(y)| dy \\
 &\leq C \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^{\alpha} \int_{3Q} |b(x) - b(y)|^m |f(y)| dy \cdot \chi_Q(x)
 \end{aligned}$$

$$\begin{aligned} &\leq C \left(\sum_{Q \in \mathcal{D}(Q_0)} l(Q)^\alpha \int_{3Q} |b(x) - m_{w^{q_1}, Q}(b)|^m |f(y)| dy \cdot \chi_Q(x) \right) \\ &\quad + C \left(\sum_{Q \in \mathcal{D}(Q_0)} l(Q)^\alpha \int_{3Q} |m_{w^{q_1}, Q}(b) - b(y)|^m |f(y)| dy \cdot \chi_Q(x) \right) \\ &= C(I_0(x) + II_0(x)). \end{aligned}$$

We consider $I_0(x)$. Firstly, it is obvious that the following holds:

$$I_0(x) = \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^\alpha |b(x) - m_{w^{q_1}, Q}(b)|^m \left(\int_{3Q} |f(y)| dy \right) \chi_Q(x).$$

We apply Lemma 8.

Let

$$\mathcal{D}_0(Q_0) := \left\{ Q \in \mathcal{D}(Q_0); \int_{3Q} |f(y)| dy \leq \gamma A \right\}$$

and

$$\mathcal{D}_{k,j}(Q_0) := \left\{ Q \in \mathcal{D}(Q_0); Q \subset Q_{k,j}, \gamma A^k < \int_{3Q} |f(y)| dy \leq \gamma A^{k+1} \right\}.$$

Then we have

$$\mathcal{D}(Q_0) = \mathcal{D}_0(Q_0) \cup \left(\bigcup_{k,j} \mathcal{D}_{k,j}(Q_0) \right).$$

By the duality, we have

$$\left(\int_{Q_0} I_0(x)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} = \sup_{\|g w^{-\frac{q_1}{q_2}}\|_{L^{q'_2}(Q_0)} = 1} \int_{Q_0} I_0(x) |g(x)| dx.$$

Let $g \in L^{q'_2} \left(w^{-\frac{q'_2}{q_2} q_1} \right)$ satisfies $g \geq 0$, $\text{supp}(g) \subset Q_0$ and $\|g w^{-\frac{q_1}{q_2}}\|_{L^{q'_2}(Q_0)} = 1$.

$$\begin{aligned} \int_{Q_0} I_0(x) g(x) dx &= \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^\alpha \left(\int_{3Q} |f(y)| dy \right) \left(\int_Q |b(x) - m_{w^{q_1}, Q}(b)|^m g(x) dx \right) \\ &= \left(\sum_{Q \in \mathcal{D}_0(Q_0)} + \sum_{k,j} \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} \right) l(Q)^\alpha \left(\int_{3Q} |f(y)| dy \right) \\ &\quad \times \left(\int_Q |b(x) - m_{w^{q_1}, Q}(b)|^m g(x) dx \right) \\ &= A_0 + \sum_{k,j} A_{k,j}. \end{aligned}$$

We evaluate $A_{k,j}$. By the definition of $\mathcal{D}_{k,j}(Q_0)$,

$$\begin{aligned} A_{k,j} &\leq \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} \gamma A^{k+1} \left(\int_Q |b(x) - m_{w^{q_1}, Q}(b)|^m g(x) dx \right) \\ &= A l(Q_{k,j})^\alpha \left(\int_{3Q_{k,j}} |f(y)| dy \right) \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} \left(\int_Q |b(x) - m_{w^{q_1}, Q}(b)|^m g(x) dx \right). \end{aligned}$$

By Hölder’s inequality for a small number $\theta_2 > 1$, we obtain

$$\begin{aligned} A_{k,j} &\leq A l(Q_{k,j})^\alpha \left(\int_{3Q_{k,j}} |f(y)| dy \right) \\ &\quad \times \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} \left(\int_Q |b(x) - m_{w^{q_1}, Q}(b)|^{m\theta_2'} w(x)^{q_1} dx \right)^{\frac{1}{\theta_2}} \left(\int_Q w(x)^{-\frac{\theta_2}{\theta_2'} q_1} g(x)^{\theta_2} dx \right)^{\frac{1}{\theta_2}}. \end{aligned}$$

Since $w(x)^{-\frac{\theta_2}{\theta_2'} q_1} g(x)^{\theta_2} = \left(\frac{g(x)}{w(x)^{q_1}} \right)^{\theta_2} w(x)^{q_1}$, we have

$$\begin{aligned} A_{k,j} &\leq A l(Q_{k,j})^\alpha \left(\int_{3Q_{k,j}} |f(y)| dy \right) \\ &\quad \times \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} w^{q_1}(Q) \left(\frac{1}{w^{q_1}(Q)} \int_Q |b(x) - m_{w^{q_1}, Q}(b)|^{m\theta_2'} w(x)^{q_1} dx \right)^{\frac{1}{\theta_2}} \\ &\quad \times \left(\frac{1}{w^{q_1}(Q)} \int_Q \left(\frac{g(x)}{w(x)^{q_1}} \right)^{\theta_2} w(x)^{q_1} dx \right)^{\frac{1}{\theta_2}}. \end{aligned}$$

By the inequality (6) in Lemma 10, we have

$$\begin{aligned} A_{k,j} &\leq A \|b\|_{\text{BMO}}^m l(Q_{k,j})^\alpha \left(\int_{3Q_{k,j}} |f(y)| dy \right) \\ &\quad \times \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} w^{q_1}(Q) \left(\frac{1}{w^{q_1}(Q)} \int_Q \left(\frac{g(x)}{w(x)^{q_1}} \right)^{\theta_2} w(x)^{q_1} dx \right)^{\frac{1}{\theta_2}}. \end{aligned}$$

If $y \in Q$, then we have

$$\frac{1}{w^{q_1}(Q)} \int_Q \left(\frac{g(x)}{w(x)^{q_1}} \right)^{\theta_2} w(x)^{q_1} dx \leq M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (y).$$

This gives us the following estimate:

$$\begin{aligned} A_{k,j} &\leq A \|b\|_{\text{BMO}}^m l(Q_{k,j})^\alpha \left(\int_{3Q_{k,j}} |f(y)| dy \right) \\ &\quad \times \sum_{Q \in \mathcal{D}_{k,j}(Q_0)} \left(\int_Q M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (y)^{\frac{1}{\theta_2}} w(y)^{q_1} dy \right). \end{aligned}$$

Therefore we obtain the following by the definition of $\mathcal{D}_{k,j}(Q_0)$:

$$A_{k,j} \leq A \|b\|_{\text{BMO}}^m l(Q_{k,j})^\alpha \left(\int_{3Q_{k,j}} |f(y)| dy \right) \left(\int_{Q_{k,j}} M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (y)^{\frac{1}{\theta_2}} w(y)^{q_1} dy \right).$$

Since $w^{q_1}(Q_{k,j})w^{q_1}(Q_{k,j})^{-1} = 1$, we have

$$\begin{aligned} A_{k,j} &\leq A \|b\|_{\text{BMO}}^m l(Q_{k,j})^\alpha \left(\int_{3Q_{k,j}} |f(y)| dy \right) w^{q_1}(Q_{k,j}) \\ &\quad \times \left(\frac{1}{w^{q_1}(Q_{k,j})} \int_{Q_{k,j}} M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (y)^{\frac{1}{\theta_2}} w(y)^{q_1} dy \right). \end{aligned}$$

By $w^{q_1}(Q_{k,j}) \leq Cw^{q_1}(E_{k,j})$, we obtain

$$\begin{aligned} A_{k,j} &\leq CA \|b\|_{\text{BMO}}^m w^{q_1}(E_{k,j}) l(Q_{k,j})^\alpha \left(\int_{3Q_{k,j}} |f(y)| dy \right) \\ &\quad \times \left(\frac{1}{w^{q_1}(Q_{k,j})} \int_{Q_{k,j}} M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (y)^{\frac{1}{\theta_2}} w(y)^{q_1} dy \right). \end{aligned}$$

Therefore we have

$$\begin{aligned} A_{k,j} &\leq CA \|b\|_{\text{BMO}}^m \int_{E_{k,j}} l(Q_{k,j})^\alpha \left(\int_{3Q_{k,j}} |f(y)| dy \right) \\ &\quad \times \left(\frac{1}{w^{q_1}(Q_{k,j})} \int_{Q_{k,j}} M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (y)^{\frac{1}{\theta_2}} w(y)^{q_1} dy \right) w(x)^{q_1} dx. \end{aligned}$$

If $x \in Q_{k,j}$, then we have

$$l(Q_{k,j})^\alpha \left(\int_{3Q_{k,j}} |f(y)| dy \right) \leq M_\alpha f(x)$$

and

$$\frac{1}{w^{q_1}(Q_{k,j})} \int_{Q_{k,j}} M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (y)^{\frac{1}{\theta_2}} w(y)^{q_1} dy \leq M_{w^{q_1}} \left(M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (\cdot)^{\frac{1}{\theta_2}} \right) (x).$$

This implies that

$$A_{k,j} \leq CA \|b\|_{\text{BMO}}^m \int_{E_{k,j}} M_\alpha f(x) \cdot M_{w^{q_1}} \left(M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (\cdot)^{\frac{1}{\theta_2}} \right) (x) w(x)^{q_1} dx.$$

By a similar argument, we obtain

$$A_0 \leq CA \|b\|_{\text{BMO}}^m \int_{E_0} M_\alpha f(x) \cdot M_{w^{q_1}} \left(M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (\cdot)^{\frac{1}{\theta_2}} \right) (x) w(x)^{q_1} dx.$$

We sum up A_0 and $A_{k,j}$:

$$A_0 + \sum_{k,j} A_{k,j} \leq CA \|b\|_{\text{BMO}}^m \int_{Q_0} M_{\alpha} f(x) \cdot M_{w^{q_1}} \left(M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (\cdot)^{\frac{1}{\theta_2}} \right) (x) w(x)^{q_1} dx.$$

By Hölder’s inequality for $q_2 > 1$, we have

$$\begin{aligned} A_0 + \sum_{k,j} A_{k,j} &\leq CA \|b\|_{\text{BMO}}^m \left(\int_{Q_0} M_{\alpha} f(x)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \\ &\quad \times \left(\int_{Q_0} M_{w^{q_1}} \left(M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (\cdot)^{\frac{1}{\theta_2}} \right) (x)^{q_2'} w(x)^{q_1} dx \right)^{\frac{1}{q_2'}. \end{aligned}$$

By Lemma 7, we obtain

$$\begin{aligned} &\left(\int_{Q_0} M_{w^{q_1}} \left(M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (\cdot)^{\frac{1}{\theta_2}} \right) (x)^{q_2'} w(x)^{q_1} dx \right)^{\frac{1}{q_2'}} \\ &\leq C \left(\int_{\mathbb{R}^n} M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (x)^{\frac{q_2'}{\theta_2}} w(x)^{q_1} dx \right)^{\frac{1}{q_2'}}. \end{aligned}$$

By Lemma 7 again, we obtain

$$\left(\int_{\mathbb{R}^n} M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (x)^{\frac{q_2'}{\theta_2}} w(x)^{q_1} dx \right)^{\frac{1}{q_2'}} \leq C \left(\int_{\mathbb{R}^n} \left(\frac{g(x)}{w(x)^{q_1}} \right) (x)^{\theta_2 \cdot \frac{q_2'}{\theta_2}} w(x)^{q_1} dx \right)^{\frac{1}{q_2'}}.$$

That is,

$$\left(\int_{Q_0} M_{w^{q_1}} \left(M_{w^{q_1}} \left(\left(\frac{g}{w^{q_1}} \right)^{\theta_2} \right) (\cdot)^{\frac{1}{\theta_2}} \right) (x)^{q_2'} w(x)^{q_1} dx \right)^{\frac{1}{q_2'}} \leq C \left\| gw^{-\frac{q_1}{q_2}} \right\|_{L^{q_2'}(Q_0)} = C.$$

Therefore we have

$$\left(\frac{1}{w^{q_1}(Q_0)^\lambda} \int_{Q_0} I_0(x)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \leq CA \|b\|_{\text{BMO}}^m \|M_{\alpha} f\|_{L^{q_2, \lambda}(w^{q_1}, w^{q_1})}.$$

By Theorem 2, we obtain

$$\left(\frac{1}{w^{q_1}(Q_0)^\lambda} \int_{Q_0} I_0(x)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \leq C \|b\|_{\text{BMO}}^m \|f\|_{L^{p, \lambda}(w^p, w^{q_1})}.$$

Lastly, we evaluate $II_0(x)$. By the duality, we have

$$\begin{aligned} &\left(\int_{Q_0} II_0(x)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \\ &= \sup \left\{ \int_{Q_0} II_0(x) \cdot h(x) dx : h \text{ satisfies that } \left\| hw^{-\frac{q_1}{q_2}} \right\|_{L^{q_2'}(Q_0)} = 1 \right\}. \end{aligned}$$

Let $h \geq 0$, $\text{supp}(h) \subset Q_0$ and $\|hw^{-\frac{q_1}{q_2}}\|_{L^{q_2'}(Q_0)} = 1$.

$$\begin{aligned} \int_{Q_0} I_0(x)h(x)dx &= C \sum_{Q \in \mathcal{D}(Q_0)} \int_{Q_0} l(Q)^\alpha \left(\int_{3Q} |m_{w^{q_1}, Q}(b) - b(y)|^m |f(y)|dy \right) \chi_Q(x) \cdot h(x)dx \\ &= C \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^\alpha \left(\int_{3Q} |m_{w^{q_1}, Q}(b) - b(y)|^m |f(y)|dy \right) \int_Q h(x)dx. \end{aligned}$$

By Hölder’s inequality for $\theta_3 > 1$, we obtain

$$\begin{aligned} &\sum_{Q \in \mathcal{D}(Q_0)} l(Q)^\alpha \left(\int_{3Q} |m_{w^{q_1}, Q}(b) - b(y)|^m |f(y)|dy \right) \int_Q h(x)dx \\ &\leq \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^\alpha \left(\int_{3Q} |m_{w^{q_1}, Q}(b) - b(y)|^{m\theta_3'} dy \right)^{\frac{1}{\theta_3}} \left(\int_{3Q} |f(y)|^{\theta_3} dy \right)^{\frac{1}{\theta_3}} \int_Q h(x)dx. \end{aligned}$$

By (7) in Lemma 10, we have

$$\int_{Q_0} I_0(x)h(x)dx \leq C \|b\|_{\text{BMO}}^m \sum_{Q \in \mathcal{D}(Q_0)} l(Q)^\alpha \left(\int_{3Q} |f(y)|^{\theta_3} dy \right)^{\frac{1}{\theta_3}} \int_Q h(x)dx.$$

We apply Lemma 8 to $\beta = \theta_3$ and $W = w$. Let

$$\mathcal{D}_0(Q_0; \theta_3) = \left\{ Q \in \mathcal{D}(Q_0); \left(\int_{3Q} |f(y)|^{\theta_3} dy \right)^{\frac{1}{\theta_3}} \leq \gamma(\theta_3)A(\theta_3) \right\}$$

and

$$\begin{aligned} \mathcal{D}_{k,j}(Q_0; \theta_3) &= \left\{ Q \in \mathcal{D}(Q_0; \theta_3); Q \subset Q_{k,j}(\theta_3), \gamma(\theta_3)A(\theta_3)^k \right. \\ &\quad \left. < \left(\int_{3Q} |f(y)|^{\theta_3} dy \right)^{\frac{1}{\theta_3}} \leq \gamma(\theta_3)A(\theta_3)^{k+1} \right\}. \end{aligned}$$

Then, we have

$$\mathcal{D}_0(Q_0; \theta_3) = \mathcal{D}_0(Q_0; \theta_3) \cup \left(\bigcup_{k,j} \mathcal{D}_{k,j}(Q_0, \theta_3) \right).$$

$$\begin{aligned} \int_{Q_0} I_0(x)h(x)dx &= C \|b\|_{\text{BMO}}^m \left(\sum_{Q \in \mathcal{D}_0(Q_0, \theta_3)} + \sum_{k,j} \sum_{Q \in \mathcal{D}_{k,j}(Q_0, \theta_3)} \right) \\ &\quad \times l(Q)^\alpha \left(\int_{3Q} |f(y)|^{\theta_3} dy \right)^{\frac{1}{\theta_3}} \int_Q h(x)dx \\ &= C \|b\|_{\text{BMO}}^m \left(B_0(\theta_3) + \sum_{k,j} B_{k,j}(\theta_3) \right). \end{aligned}$$

We evaluate $B_{k,j}(\theta_3)$. By the definition of $\mathcal{D}_{k,j}(Q_0, \theta_3)$, we have

$$\begin{aligned} B_{k,j}(\theta_3) &< \gamma(\theta_3)A(\theta_3)^{k+1} \sum_{Q \in \mathcal{D}_{k,j}(Q_0; \theta_3)} \int_Q h(x)dx \\ &\leq A(\theta_3)l(Q_{k,j}(\theta_3))^\alpha \left(\int_{3Q_{k,j}(\theta_3)} |f(y)|^{\theta_3} dy \right)^{\frac{1}{\theta_3}} \sum_{Q \in \mathcal{D}_{k,j}(Q_0; \theta_3)} \int_Q h(x)dx \\ &= A(\theta_3)l(Q_{k,j}(\theta_3))^\alpha \left(\int_{3Q_{k,j}(\theta_3)} |f(y)|^{\theta_3} dy \right)^{\frac{1}{\theta_3}} \\ &\quad \times \left(\frac{1}{w^{q_1}(Q_{k,j}(\theta_3))} \int_{Q_{k,j}(\theta_3)} h(x)dx \right) w^{q_1}(Q_{k,j}(\theta_3)). \end{aligned}$$

By $w^{q_1}(Q_{k,j}(\theta_3)) \leq Cw^{q_1}(E_{k,j}(\theta_3))$, we have

$$\begin{aligned} B_{k,j}(\theta_3) &\leq CA(\theta_3)l(Q_{k,j}(\theta_3))^\alpha \left(\int_{3Q_{k,j}(\theta_3)} |f(y)|^{\theta_3} dy \right)^{\frac{1}{\theta_3}} \\ &\quad \times \left(\frac{1}{w^{q_1}(Q_{k,j}(\theta_3))} \int_{Q_{k,j}(\theta_3)} h(x)dx \right) w^{q_1}(E_{k,j}(\theta_3)) \\ &= CA(\theta_3) \left(\int_{E_{k,j}(\theta_3)} l(Q_{k,j}(\theta_3))^\alpha \left(\int_{3Q_{k,j}(\theta_3)} |f(y)|^{\theta_3} dy \right)^{\frac{1}{\theta_3}} \right. \\ &\quad \left. \times \left(\frac{1}{w^{q_1}(Q_{k,j}(\theta_3))} \int_{Q_{k,j}(\theta_3)} h(x)dx \right) w(y)^{q_1} dy \right) \\ &\leq CA(\theta_3) \int_{E_{k,j}(\theta_3)} M_{\alpha\theta_3} \left(|f|^{\theta_3} \right) (y)^{\frac{1}{\theta_3}} \cdot M_{w^{q_1}} \left(\frac{h}{w^{q_1}} \right) (y) w(y)^{q_1} dy. \end{aligned}$$

A similar argument gives us the following inequality:

$$B_0(\theta_3) \leq CA(\theta_3) \int_{E_0(\theta_3)} M_{\alpha\theta_3} \left(|f|^{\theta_3} \right) (y)^{\frac{1}{\theta_3}} \cdot M_{w^{q_1}} \left(\frac{h}{w^{q_1}} \right) (y) w(y)^{q_1} dy.$$

This implies that

$$B_0(\theta_3) + \sum_{k,j} B_{k,j}(\theta_3) \leq CA(\theta_3) \int_{Q_0} M_{\alpha\theta_3} \left(|f|^{\theta_3} \right) (y)^{\frac{1}{\theta_3}} \cdot M_{w^{q_1}} \left(\frac{h}{w^{q_1}} \right) (y) w(y)^{q_1} dy.$$

By Hölder’s inequality for $q_2 > 1$, we obtain

$$\begin{aligned} B_0(\theta_3) + \sum_{k,j} B_{k,j}(\theta_3) &\leq CA(\theta_3) \left(\int_{Q_0} M_{\alpha\theta_3} \left(|f|^{\theta_3} \right) (y)^{\frac{q_2}{\theta_3}} w(y)^{q_1} dy \right)^{\frac{1}{q_2}} \\ &\quad \times \left(\int_{Q_0} M_{w^{q_1}} \left(\frac{h}{w^{q_1}} \right) (y)^{q_2'} w(y)^{q_1} dy \right)^{\frac{1}{q_2}}. \end{aligned}$$

By Lemma 7, we have

$$B_0(\theta_3) + \sum_{k,j} B_{k,j}(\theta_3) \leq CA(\theta_3) \left(\int_{Q_0} M_{\alpha\theta_3} (|f|^{\theta_3}) (y)^{\frac{q_2}{\theta_3}} w(y)^{q_1} dy \right)^{\frac{1}{q_2}} \\ \times \left(\int_{\mathbb{R}^n} \left(\frac{h(y)}{w(y)^{q_1}} \right)^{q'_2} w(y)^{q_1} dy \right)^{\frac{1}{q'_2}}.$$

By $\text{supp}(h) \subset Q_0$, we obtain

$$\left(\int_{\mathbb{R}^n} \left(\frac{h(y)}{w(y)^{q_1}} \right)^{q'_2} w(y)^{q_1} dy \right)^{\frac{1}{q'_2}} = \|hw^{-\frac{q_1}{q_2}}\|_{L^{q'_2}(Q_0)} = 1.$$

Hence we have

$$B_0(\theta_3) + \sum_{k,j} B_{k,j}(\theta_3) \leq CA(\theta_3) \left(\int_{Q_0} M_{\alpha\theta_3} (|f|^{\theta_3}) (y)^{\frac{q_2}{\theta_3}} w(y)^{q_1} dy \right)^{\frac{\theta_3}{q_2} \cdot \frac{1}{\theta_3}}.$$

This implies that the following estimate holds:

$$\left(\frac{1}{w^{q_1}(Q_0)^\lambda} \int_{Q_0} H_0(x)^{q_2} w(x)^{q_1} dx \right)^{\frac{1}{q_2}} \leq CA(\theta_3) \|M_{\alpha\theta_3} (|f|^{\theta_3})\|_{L^{\frac{q_2}{\theta_3}, \lambda}(w^{q_1}, w^{q_1})}^{\frac{1}{\theta_3}} \\ = CA(\theta_3) \|M_{\alpha\theta_3} (|f|^{\theta_3})\|_{L^{\frac{q_2}{\theta_3}, \lambda}(w^{\theta_3 \frac{q_1}{\theta_3}}, w^{\theta_3 \frac{q_1}{\theta_3}})}^{\frac{1}{\theta_3}}.$$

We choose a small number $\theta_3 > 1$ that the following holds: $0 < \alpha\theta_3 < n$ and $w^{\theta_3} \in A_{\frac{p}{\theta_3}, \frac{q_1}{\theta_3}}(\mathbb{R}^n)$. Since

$$\frac{\theta_3}{q_2} = \frac{\theta_3}{p} - \frac{\alpha\theta_3}{n(1-\lambda)} \quad \text{and} \quad \frac{\theta_3}{q_1} = \frac{\theta_3}{p} - \frac{\alpha\theta_3}{n},$$

by Theorem 2, we obtain

$$\|M_{\alpha\theta_3} (|f|^{\theta_3})\|_{L^{\frac{q_2}{\theta_3}, \lambda}(w^{\theta_3 \frac{q_1}{\theta_3}}, w^{\theta_3 \frac{q_1}{\theta_3}})}^{\frac{1}{\theta_3}} \leq C \| |f|^{\theta_3} \|_{L^{\frac{p}{\theta_3}, \lambda}(w^{\theta_3 \frac{p}{\theta_3}}, w^{\theta_3 \frac{q_1}{\theta_3}})}^{\frac{1}{\theta_3}} \\ = C \|f\|_{L^{p, \lambda}(w^p, w^{q_1})} \cdot \square$$

Next, we prove Theorem 2.

Proof of Theorem 2. By Lemma 2, we have

$$\|I_{\alpha} f\|_{L^{q_2, \lambda}(w^{q_1}, w^{q_1})} \leq C \|M_{\alpha} f\|_{L^{q_2, \lambda}(w^{q_1}, w^{q_1})}.$$

By Lemma 3, we have

$$\|M_{\alpha}f\|_{L^{q_2,\lambda}(w^{q_1},w^{q_1})} \leq C[w]_{A_{p,q_1}}^{\theta} \left\| M_{w^{q_1},\frac{\alpha p}{a}} \left(\frac{|fw^{\theta}|^{\frac{p}{a}}}{w^{q_1}} \right) \right\|_{L^{\frac{aq_2}{p},\lambda}(w^{q_1},w^{q_1})}.$$

Taking $w^{q_1} = V$ and $F = \frac{(fw^{\theta})^{\frac{p}{a}}}{w^{q_1}}$, we have

$$\left\| M_{w^{q_1},\frac{\alpha p}{a}} \left(\frac{|fw^{\theta}|^{\frac{p}{a}}}{w^{q_1}} \right) \right\|_{L^{\frac{aq_2}{p},\lambda}(w^{q_1},w^{q_1})}^{\frac{a}{p}} = \left\| M_{V,\frac{\alpha p}{a}}(F) \right\|_{L^{\frac{aq_2}{p},\lambda}(V,V)}^{\frac{a}{p}}.$$

Since

$$\frac{p}{aq_2} = \frac{1}{a} - \frac{\frac{\alpha p}{a}}{n} \frac{1}{1-\lambda},$$

by Lemma 4, we obtain

$$\begin{aligned} \left\| M_{V,\frac{\alpha p}{a}}(F) \right\|_{L^{\frac{aq_2}{p},\lambda}(V,V)}^{\frac{a}{p}} &\leq C \|F\|_{L^{a,\lambda}(V,V)}^{\frac{a}{p}} \\ &= C \sup_{\substack{Q \subset \mathbb{R}^n, \\ Q:\text{cube}}} \left(\frac{1}{w^{q_1}(Q)^{\lambda}} \int_Q f(x)^p w(x)^{q_1 - q_1 a + \theta p} dx \right)^{\frac{1}{p}}. \end{aligned}$$

Since $q_1 - q_1 a + p\theta = p$, we obtain the desired inequality:

$$\sup_{\substack{Q \subset \mathbb{R}^n, \\ Q:\text{cube}}} \left(\frac{1}{w^{q_1}(Q)^{\lambda}} \int_Q f(x)^p w(x)^{q_1 - q_1 a + \theta p} dx \right)^{\frac{1}{p}} = \|f\|_{L^{p,\lambda}(w^p,w^{q_1})}.$$

Therefore we obtain

$$\|I_{\alpha}f\|_{L^{q_2,\lambda}(w^{q_1},w^{q_1})} \leq C[w]_{A_{p,q_1}(\mathbb{R}^n)}^{\theta} \cdot \|f\|_{L^{p,\lambda}(w^p,w^{q_1})}.$$

Hence we get the desired result. \square

Next, we prove Theorem 3.

Proof of Theorem 3. By Lemma 5, we have

$$\|I_{\Omega,\alpha}(f)\|_{L^{q_2,\lambda}(v^{q_1},v^{q_1})} \leq C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \left\| M_{\alpha s'}(f^{s'})^{1/s'} \right\|_{L^{q_2,\lambda}(v^{q_1},v^{q_1})}.$$

The scaling law yields

$$\left\| M_{\alpha s'}(f^{s'})^{1/s'} \right\|_{L^{q_2,\lambda}(v^{q_1},v^{q_1})} = \left\| M_{\alpha s'}(f^{s'}) \right\|_{L^{\frac{q_2}{s'},\lambda}(v^{q_1},v^{q_1})}^{\frac{1}{s'}}.$$

Since $\frac{s'}{q_2} = \frac{s'}{p} - \frac{\alpha s'}{n(1-\lambda)}$ and $\frac{s'}{q_1} = \frac{s'}{p} - \frac{\alpha s'}{n}$, by Theorem 2, we have

$$\left\| M_{\alpha s'}(f^{s'}) \right\|_{L^{\frac{q_2}{s'}, \lambda}(v^{q_1}, v^{q_1})} \leq C \left\| f^{s'} \right\|_{L^{\frac{p}{s'}, \lambda}(v^p, v^{q_1})}.$$

The scaling law yields

$$\left\| f^{s'} \right\|_{L^{\frac{p}{s'}, \lambda}(v^p, v^{q_1})} = \|f\|_{L^{p, \lambda}(v^p, v^{q_1})}^{s'}.$$

This gives us the inequality (3):

$$\left\| I_{\Omega, \alpha}(f) \right\|_{L^{q_2, \lambda}(v^{q_1}, v^{q_1})} \leq C \|f\|_{L^{p, \lambda}(v^p, v^{q_1})}. \quad \square$$

Lastly, we prove Theorem 4.

Proof of Theorem 4. By Lemma 11, we have

$$\left\| I_{\alpha, m}(\vec{f}) \right\|_{L^{q_2, \lambda}(v^{q_1}, v^{q_1})} \leq \left\| \mathcal{M}_{\alpha, m}(\vec{f}) \right\|_{L^{q_2, \lambda}(v^{q_1}, v^{q_1})}.$$

By Lemma 13, we obtain

$$\left\| \mathcal{M}_{\alpha, m}(\vec{f}) \right\|_{L^{q_2, \lambda}(v^{q_1}, v^{q_1})} \leq C \left[\left(v^{\frac{p}{p_1}}, \dots, v^{\frac{p}{p_m}} \right) \right]_{A_{\vec{P}, q_1}}^\theta \left\| \mathcal{M}_{V, \frac{\vec{P}}{a}, q_1}(\vec{F}) \right\|_{L^{q_2, \lambda}(V, V)},$$

where $F_j(x) = \frac{|f_j(x)|v(x)^{\theta \cdot \frac{p}{p_j}}}{v(x)^{q_1 \cdot \frac{a}{p_j}}}$ and $V(x) = v(x)^{q_1}$. By Lemma 14, we have

$$\begin{aligned} \left\| \mathcal{M}_{V, \frac{\vec{P}}{a}, q_1}(\vec{F}) \right\|_{L^{q_2, \lambda}(V, V)} &\leq \left(\sup_{\substack{Q \subset \mathbb{R}^n \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{1}{p_j}} \right)^{1 - \frac{p}{q_2}} \\ &\times \left\| \left(\mathcal{M}_V \left(\vec{F}^{\frac{\vec{P}}{a}} \right) (\cdot)^{\frac{a}{\vec{P}}} \right)^{\frac{p}{q_2}} \right\|_{L^{q_2, \lambda}(V, V)}. \end{aligned}$$

By Lemma 15, we have

$$\left\| \left(\mathcal{M}_V \left(\vec{F}^{\frac{\vec{P}}{a}} \right) (\cdot)^{\frac{a}{\vec{P}}} \right)^{\frac{p}{q_2}} \right\|_{L^{q_2, \lambda}(V, V)} \leq C \left(\sup_{\substack{Q \subset \mathbb{R}^n \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{1}{p_j}} \right)^{\frac{p}{q_2}}.$$

These imply that

$$\left\| I_{\alpha, m}(\vec{f}) \right\|_{L^{q_2, \lambda}(v^{q_1}, v^{q_1})} \leq C \sup_{\substack{Q \subset \mathbb{R}^n \\ Q: \text{cube}}} \prod_{j=1}^m \left(\frac{1}{V(Q)^\lambda} \int_Q |F_j(y_j)|^{p_j} V(y_j) dy_j \right)^{\frac{1}{p_j}}.$$

Lemma 16 gives us the inequality (4). \square

5. Appendix

According to [31], for $f \in C_0^\infty(\mathbb{R}^n)$, we have

$$|f(x)| \leq C \sum_{j=1}^n I_1 \left| \frac{\partial f}{\partial x_j} \right| (x).$$

Therefore by Theorem 2, we obtain a type of the Fefferman-Phong inequality:

COROLLARY 2. *Let $n \geq 2$, $0 < \lambda < 1 - \frac{1}{n}$ and $1 < p < n(1 - \lambda)$. Define q_1 and q_2 by*

$$\frac{1}{q_2} = \frac{1}{p} - \frac{1}{n} \frac{1}{1 - \lambda} \quad \text{and} \quad \frac{1}{q_1} = \frac{1}{p} - \frac{1}{n}.$$

Suppose that $w \in A_{p,q_1}(\mathbb{R}^n)$, then for $f \in C_0^\infty(\mathbb{R}^n)$, we obtain

$$\|f\|_{L^{q_2,\lambda}(w^{q_1},w^{q_1})} \leq C \|\nabla f\|_{L^{p,\lambda}(w^p,w^{q_1})}.$$

According to [26], if $n \geq 3$, $f \in C_0^\infty(\mathbb{R}^n)$, then we have $f = CI_2(\Delta f)$. Therefore, we obtain the following inequality.

COROLLARY 3. *Let $n \geq 3$, $0 < \lambda < 1 - \frac{2}{n}$ and $1 < p < \frac{n}{2}(1 - \lambda)$. Define q_1 and q_2 by*

$$\frac{1}{q_1} = \frac{1}{p} - \frac{2}{n} \quad \text{and} \quad \frac{1}{q_2} = \frac{1}{p} - \frac{2}{n} \frac{1}{1 - \lambda}.$$

Suppose that $w \in A_{p,q_1}(\mathbb{R}^n)$, then for $f \in C_0^\infty(\mathbb{R}^n)$, we have

$$\|f\|_{L^{q_2,\lambda}(w^{q_1},w^{q_1})} \leq C \|\Delta f\|_{L^{p,\lambda}(w^p,w^{q_1})}.$$

We give the counterexample of Remark 10.

EXAMPLE 1. *Let $\alpha = -\frac{1}{2} \left(\frac{1}{q} + \frac{1}{p} \right)$ and $w(x) = |x|^\alpha$. If $q < p$, then $w \in A_{p,q}(\mathbb{R})$.*

However, we have

$$\begin{cases} w^q \in A_q(\mathbb{R}) \\ w^{-p'} \notin A_{p'}(\mathbb{R}). \end{cases}$$

Proof of Example 1. Firstly we check $w \in A_{p,q}(\mathbb{R})$. By Remark 4, $|x|^\alpha \in A_{p,q}(\mathbb{R})$ if and only if $|x|^{\alpha q} \in A_{1+\frac{q}{p}}(\mathbb{R})$. On the other hand, By [10, pp. 286], $|x|^\alpha \in A_p(\mathbb{R})$ if and only if $-1 < \alpha < p - 1$. Combined with the property, $|x|^\alpha \in A_{p,q}(\mathbb{R})$ if and only if $-\frac{1}{q} < \alpha < \frac{1}{p'}$. Therefore the matters are reduced to check the following: $-\frac{1}{q} < \alpha < \frac{1}{p'}$. Since $\alpha < 0$, we have $\alpha < \frac{1}{p'}$. Since $\frac{1}{q} > \frac{1}{p}$, we have $-\frac{1}{q} = \frac{1}{2} \left(-\frac{1}{q} - \frac{1}{q} \right) < \alpha$. This implies that $w(x) \in A_{p,q}(\mathbb{R})$.

Next we prove that $|x|^{\alpha q} \in A_q(\mathbb{R})$. By a similar argument, we have only to check the following:

$$-\frac{1}{q} < \alpha < \frac{1}{q'}.$$

Since $\alpha < 0$, we have $\alpha < \frac{1}{q}$. Hence we have $w^q \in A_q(\mathbb{R})$. Lastly, we disprove $|x|^{-\alpha p'} \in A_{p'}(\mathbb{R})$. Since $\frac{1}{q} > \frac{1}{p}$, we have $\alpha < -\frac{1}{p}$. On the other hand, $|x|^{-\alpha p'} \in A_{p'}(\mathbb{R})$ if and only if $-\frac{1}{p} < \alpha < \frac{1}{p'}$. This implies that $|x|^{-\alpha p'} \notin A_{p'}(\mathbb{R})$. \square

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