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A MONOTONICITY PROPERTY INVOLVING THE GENERALIZED ELLIPTIC INTEGRAL OF THE FIRST KIND

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Abstract. In this paper, we prove that the function

$$r \to Y(r) = \frac{\mathscr{K}_a(r)}{\sin(\pi a) r'^2 \log(e^{R(a)/2}/r')} - \frac{1}{r'^2}$$

is strictly increasing from $(0,1)$ onto $(\pi/[R(a)\sin(\pi a)]-1, a(1-a))$ for all $a \in (0,1/2]$, where $r' = \sqrt{1-r^2}$, $\mathscr{K}_a(r)$ is the generalized elliptic integral of the first kind, $R(a) = -2\gamma - \psi(a) - \psi(1-a)$, ψ is the classical psi function and $\gamma = 0.57721566\cdots$ is the Euler-Mascheroni constant.

1. Introduction

For real numbers a, b and c with $c \neq 0, -1, -2, \cdots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad (-1 < x < 1), \tag{1.1}$$

where $(a)_n$ is the shifted factorial function defined by $(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ for $n = 1,2,\cdots$ and $(a)_0 = 1$ for $a \neq 0$, $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ $(x > 0)$ is the classical gamma function and the double inequality

$$\frac{1}{(x+a)^{1-a}} < \frac{\Gamma(x+a)}{\Gamma(x+1)} < \frac{1}{x^{1-a}} \tag{1.2}$$

holds for all $a \in (0,1)$ and $x > 0$ (See [1, 2]).

The following asymptotic formula for the Gaussian hypergeometric function can be found in the literature [3, 1.48]:

$$B(a,b)F(a,b;a+b;x) + \log(1-x) = R(a,b) + O((1-x)\log(1-x)) \quad (x \to 1), \tag{1.3}$$

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where

$$R(a,b) = -\psi(a) - \psi(b) - 2\gamma,$$

$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt = \Gamma(p)\Gamma(q)/\Gamma(p+q)\quad (p,q>0)$ is the classical Beta function, $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the psi function and $\gamma = \lim_{n\to\infty}(\sum_{k=1}^{n} 1/k - \log n) = 0.57721566\ldots$ is the Euler-Mascheroni constant.

It is well known that the Gaussian hypergeometric function $F(a,b;c;x)$ has many important properties and applications in mathematics, physics and engineering. In particular, many important functions are the special or limiting cases of the Gaussian hypergeometric function.

Let $a \in (0,1)$ and $r \in (0,1)$. Then the generalized elliptic integral $\mathscr{K}_a(r)$ [4, 5] of the first kind is defined by

$$\mathscr{K}_a(r) = \frac{\pi}{2} F(a, 1-a; 1; r^2). \tag{1.4}$$

Clearly, $\mathscr{K}_a(0^+) = \pi/2$ and $\mathscr{K}_a(1^-) = \infty$. If $a = 1/2$, then $\mathscr{K}_a(r)$ reduces to the complete elliptic integral $\mathscr{K}_{1/2}(r) \equiv \mathscr{K}(r)$ of the first kind. Recently, the generalized elliptic integral $\mathscr{K}_a(r)$ has attracted the attention of many researchers. From the symmetry of (1.4), we assume that $a \in (0, 1/2]$ in what follows.

Carlson and Gustafson [6] proved that the inequality

$$\frac{\mathscr{K}(r)}{\log\left(\frac{4}{r'}\right)} < \frac{4}{3+r^2}$$

holds for all $r \in (0,1)$, where and in what follows $r' = \sqrt{1-r^2}$.

Anderson et. al. [7] conjectured that

$$\frac{\mathscr{K}(r)}{\log\left(\frac{4}{r'}\right)} > \frac{9}{8+r^2}$$

for all $r \in (0,1)$. It was proved by Rühnau in [8].

In [9, 10], the authors proved that the double inequality

$$1 + \lambda r'^2 < \frac{\mathscr{K}(r)}{\log\left(\frac{4}{r'}\right)} < 1 + \mu r'^2$$

holds for all $r \in (0,1)$ if and only if $\lambda \leqslant \pi/(4\log 2) - 1$ and $\mu \geqslant 1/4$.

Wang et. al. [11] proved that the double inequality

$$1 + \alpha r'^2 < \frac{\mathscr{K}_a(r)}{\sin(\pi a)\log\left(\frac{e^{R(a)/2}}{r'}\right)} < 1 + \beta r'^2 \tag{1.5}$$

holds for all $a \in (0, 1/2]$ and $r \in (0,1)$ if and only if $\alpha \leqslant \pi/[R(a)\sin(\pi a)] - 1$ and $\beta \geqslant a(1-a)$ and the two-sided inequality

$$\frac{R^2(x)}{(1+x-x^2)R(x) - 1} < \frac{\pi}{\sin(\pi x)} < \left(1 + x - x^2\right) R(x) \tag{1.6}$$

takes place for all $x \in (0,1/2]$, where

$$R(x) = R(x,1-x) = -2\gamma - \psi(x) - \psi(1-x). \tag{1.7}$$

In [12], the authors proved that the function

$$J(r) = \frac{r'^2 \mathscr{K}_a(r)}{\frac{\mathscr{K}_a(r)}{\sin(\pi a)} - \log\left(\frac{e^{R(a)/2}}{r'}\right)} \tag{1.8}$$

is strictly decreasing from $(0,1)$ onto $(\sin(\pi a)/[a(1-a)], \pi\sin(\pi a)/[\pi - R(a)\sin(\pi a)])$ for all $a \in (0,1/2]$.

The main purpose of this paper is to prove that the function

$$Y(r) = \frac{\mathscr{K}_a(r)}{\sin(\pi a) r'^2 \log(e^{R(a)/2}/r')} - \frac{1}{r'^2} \tag{1.9}$$

is strictly increasing from $(0,1)$ onto $(\pi/[R(a)\sin(\pi a)] - 1, a(1-a))$ for all $a \in (0,1/2]$.

2. Lemmas

In order to prove our main result we need several lemmas, which we present in this section.

LEMMA 2.1. (See [3, Theorem 1.25]) *Let $-\infty < a < b < \infty$, $f,g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

LEMMA 2.2. (See [13]) *Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ be two real power series converging on $(-r,r)$ ($r > 0$) with $b_k > 0$ for all k. If the non-constant sequence $\{a_k/b_k\}$ is increasing (decreasing) for all k, then the function $A(t)/B(t)$ is strictly increasing (decreasing) on $(0,r)$.*

LEMMA 2.3. (See [14, Theorem 2.1]) *Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ be two real power series converging on $(-r,r)$ and $b_k > 0$ for all k, and $H_{A,B}(t) = A'(t)B(t)/B'(t) - A(t)$. Suppose that for certain $m \in \mathbb{N}$, the non-constant sequence $\{a_k/b_k\}$ is increasing (decreasing) for $0 \leqslant k \leqslant m$ and decreasing (increasing) for $k \geqslant m$. Then the function $A(t)/B(t)$ is strictly increasing (decreasing) on $(0,r)$ if and only if $H_{A,B}(r^-) \geqslant (\leqslant) 0$. Moreover, if $H_{A,B}(r^-) < (>) 0$, then there exists $t_0 \in (0,r)$ such that the function $A(t)/B(t)$ is strictly increasing (decreasing) on $(0,t_0)$ and strictly decreasing (increasing) on (t_0,r).*

LEMMA 2.4. (See [15, Theorem 9]) *Let $a<b$, f and g be two differentiable functions on (a,b), and $H_{f,g}(x)=f'(x)g(x)/g'(x)-f(x)$. If $f(b^-)=g(b^-)=0$, $g'(x)<0$ for $x\in(a,b)$, $H_{f,g}(a^+)<0$, and there exists $c\in(a,b)$ such that $f'(x)/g'(x)$ is strictly decreasing on (a,c) and strictly increasing on (c,b), then $f(x)/g(x)$ is strictly increasing on (a,b).*

LEMMA 2.5. *Let $Y(r)$ be defined by (1.9). Then*

$$\lim_{r\to1^-}Y(r)=a(1-a).$$

Proof. From (1.9) we clearly see that $Y(r)$ can be rewritten as

$$Y(r)=\frac{\frac{\mathscr{K}_a(r)}{\sin(\pi a)}-\log\frac{e^{R(a)/2}}{r'}}{r'^2\mathscr{K}_a(r)}\frac{2\mathscr{K}_a(r)}{R(a)-\log(1-r^2)}. \tag{2.1}$$

It follows from (1.3), (1.4) and the monotonicity of the function $J(r)$ given by (1.8) together with the identity $\Gamma(a)\Gamma(1-a)=\pi/\sin(\pi a)$ that

$$\lim_{r\to1^-}\frac{\frac{\mathscr{K}_a(r)}{\sin(\pi a)}-\log\frac{e^{R(a)/2}}{r'}}{r'^2\mathscr{K}_a(r)}=\frac{a(1-a)}{\sin(\pi a)}, \tag{2.2}$$

$$\lim_{r\to1^-}\frac{2\mathscr{K}_a(r)}{R(a)-\log(1-r^2)}=\frac{\pi}{B(a,1-a)}=\frac{\pi\Gamma(1)}{\Gamma(a)\Gamma(1-a)}=\sin(\pi a). \tag{2.3}$$

Therefore, Lemma 2.5 follows easily from (2.1)–(2.3). □

LEMMA 2.6. *Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be defined by*

$$u_n=1-\frac{\Gamma(a+n)\Gamma(1-a+n)}{n!(n-1)!},\quad v_1=R(a)-1,\quad v_n=\frac{1}{n-1}\ (n\geqslant2).$$

Then the the sequence $\{u_n/v_n\}$ is strictly increasing for $n\geqslant2$ and $a\in(0,1/2]$.

Proof. Let $a\in(0,1/2]$, $n\geqslant2$ and

$$h_n=\frac{\left(n+1+a-a^2\right)\Gamma(n+a)\Gamma(n+1-a)}{(n+1)!(n-1)!}.$$

Then simple computations lead to

$$\frac{u_{n+1}}{v_{n+1}}-\frac{u_n}{v_n}=1-h_n, \tag{2.4}$$

$$\frac{h_{n+1}}{h_n}-1=\frac{a(1-a)(2-a)(a+1)}{n(n+2)(n+1+a-a^2)}>0. \tag{2.5}$$

It follows from (1.2) that

$$\frac{n(n+1+a-a^2)}{(n+1)^2} < h_n = \frac{n(n+1+a-a^2)}{n+1}\frac{\Gamma(n+a)}{\Gamma(n+1)}\frac{\Gamma(n+1-a)}{\Gamma(n+1)} < \frac{n(n+1+a-a^2)}{n(n+1)},$$

$$\lim_{n\to\infty} h_n = 1. \tag{2.6}$$

Therefore, Lemma 2.6 follows easily from (2.4)-(2.6). □

LEMMA 2.7. *Let $R(x)$ be defined by (1.7). Then*

$$R(x) \geqslant 4\log 2$$

for all $x \in (0,1/2]$.

Proof. It follows from the function $x \mapsto \xi(x) = 1/[x(1-x)] - R(x)$ is strictly increasing form $(0,1/2]$ onto $(1,4-4\log 2]$ given in [11, Lemma 2.1] and the function $1/[x(1-x)]$ is strictly decreasing from $(0,1/2]$ onto $[4,\infty)$ that $R(x) = 1/[x(1-x)] - \xi(x)$ is strictly decreasing from $(0,1/2]$ onto $[4\log 2,\infty)$. □

3. Main results

THEOREM 3.1. *Let $Y(r)$ be defined by (1.9). Then $Y(r)$ is strictly increasing from $(0,1)$ onto $(\pi/[R(a)\sin(\pi a)]-1, a(1-a))$ for all $a \in (0,1/2]$.*

Proof. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be defined as in Lemma 2.6, $x = r^2 \in (0,1)$, $f(x) = \pi F(a,1-a;1;x)/\sin(\pi a) - [R(a) - \log(1-x)]$, $g(x) = (1-x)[R(a) - \log(1-x)]$, $D_+ = \{a | u_2/v_2 - u_1/v_1 \geqslant 0\}$ and $D_- = \{a | u_2/v_2 - u_1/v_1 < 0\}$. Then from (1.1), (1.4), (1.9) and the identity $\Gamma(a)\Gamma(1-a) = \pi/\sin(\pi a)$ we clearly see that

$$Y(r) = \frac{f(x)}{g(x)}, \tag{3.1}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} g(x) = 0, \tag{3.2}$$

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \frac{\pi}{R(a)\sin(\pi a)} - 1, \tag{3.3}$$

$$\begin{aligned} f'(x) &= \frac{\pi}{\sin(\pi a)}\sum_{n=1}^{\infty}\frac{(a)_n(1-a)_n}{n!(n-1)!}x^{n-1} - \frac{1}{1-x} \\ &= \sum_{n=1}^{\infty}\left[\frac{\Gamma(a+n)\Gamma(1-a+n)}{n!(n-1)!} - 1\right]x^{n-1} = -\sum_{n=1}^{\infty} u_n x^{n-1}, \end{aligned} \tag{3.4}$$

$$g'(x) = 1 - R + \log(1-x) = -(R-1) - \sum_{n=2}^{\infty}\frac{x^{n-1}}{n-1} = -\sum_{n=1}^{\infty} v_n x^{n-1}, \tag{3.5}$$

$$\frac{f'(x)}{g'(x)}=\frac{\sum_{n=1}^{\infty}u_nx^{n-1}}{\sum_{n=1}^{\infty}v_nx^{n-1}}. \tag{3.6}$$

We divide the proof into two cases.

Case 1 $a\in D_+$. Then from Lemma 2.6 we know that the non-constant sequence $\{u_n/v_n\}_{n=1}^{\infty}$ is increasing, and Lemma 2.2 and (3.6) lead to the conclusion that $f'(x)/g'(x)$ is strictly increasing on $(0,1)$. It follows from Lemma 2.1 and (3.2) together with the monotonicity of $f'(x)/g'(x)$ that $f(x)/g(x)$ is strictly increasing on $(0,1)$. Therefore, the desired result follows from Lemma 2.5, (3.1) and (3.3) together with the monotonicity of $f(x)/g(x)$.

Case 2 $a\in D_-$. Then Lemma 2.6 implies that the non-constant sequence $\{u_n/v_n\}$ is decreasing for $1\leqslant n\leqslant 2$ and increasing for $n\geqslant 2$.

We claim that $H_{f',g'}(1^-)=\lim_{x\to 1^-}[f''(x)g'(x)/g''(x)-f'(x)]>0$ for all $a\in(0,1/2]$. Indeed, if there exists $a_0\in(0,1/2]$ such that $H_{f',g'}(1^-)\leqslant 0$, then Lemma 2.3 and (3.6) together with the piecewise monotonicity of the sequence $\{u_n/v_n\}_{n=1}^{\infty}$ lead to the conclusion that $f'(x)/g'(x)$ is strictly decreasing on $(0,1)$. It follows from Lemmas 2.1 and 2.5 together with (3.1)-(3.3) and the monotonicity of $f'(x)/g'(x)$ that $f(x)/g(x)$ is strictly decreasing on $(0,1)$ and

$$a_0(1-a_0)<Y(r)<\frac{\pi}{R(a_0)\sin(\pi a_0)}-1. \tag{3.7}$$

We clearly see that inequality (3.7) contradicts with the second inequality of (1.6).

From Lemma 2.3 and $H_{f',g'}(1^-)>0$ together with (3.6) and the piecewise monotonicity of the sequence $\{u_n/v_n\}_{n=1}^{\infty}$ we know that there exists $x_0\in(0,1)$ such that $f'(x)/g'(x)$ is strictly decreasing on $(0,x_0)$ and strictly increasing on $(x_0,1)$.

It follows from (1.1), (1.6), (3.4), (3.5) and Lemma 2.7 that

$$g'(x)=-[R(a)-1]-\log\frac{1}{1-x}<0 \tag{3.8}$$

for $a\in(0,1/2]$ and

$$\begin{aligned}
\lim_{x\to 0^+}H_{f,g}(x) &= \lim_{x\to 0^+}\left[\frac{f'(x)}{g'(x)}g(x)-f(x)\right] \qquad (3.9)\\
&= \lim_{x\to 0^+}\left[\frac{\frac{\pi}{\sin(\pi a)}\sum_{n=1}^{\infty}\frac{(a)_n(1-a)_n}{n!(n-1)!}x^{n-1}-\frac{1}{1-x}}{1-R(a)+\log(1-x)}(1-x)\left(R(a)-\log(1-x)\right)\right]\\
&\quad -\lim_{x\to 0^+}\left[\frac{\pi}{\sin(\pi a)}\sum_{n=0}^{\infty}\frac{(a)_n(1-a)_n}{(n!)^2}x^n-\left(R(a)-\log(1-x)\right)\right]\\
&= \frac{R(a)\left(\frac{\pi a(1-a)}{\sin(\pi a)}-1\right)}{1-R(a)}-\left(\frac{\pi}{\sin(\pi a)}-R(a)\right)\\
&= -\frac{\left(1+a-a^2\right)R(a)-1}{R(a)-1}\left[\frac{\pi}{\sin(\pi a)}-\frac{R^2(a)}{(1+a-a^2)R(a)-1}\right]<0.
\end{aligned}$$

From Lemma 2.4, (3.2), (3.8), (3.9) and the piecewise monotonicity of $f'(x)/g'(x)$ we know that $f(x)/g(x)$ is strictly increasing on $(0,1)$. Therefore, Theorem 3.1 follows from Lemma 2.5, (3.1) and (3.3) together with the monotonicity of $f'(x)/g'(x)$. □

Theorem 3.1 leads to Corollary 3.2 immediately.

COROLLARY 3.2. *The double inequality (1.5) holds for all* $a \in (0,1/2]$ *and* $r \in (0,1)$ *if and only if* $\alpha \leqslant \pi/[R(a)\sin(\pi a)] - 1$ *and* $\beta \geqslant a(1-a)$.

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