Mathematical
Inequalities
& Applications
Volume 20, Number 3 (2017), 801–813 doi:10.7153/mia-20-50

FRACTIONAL INTEGRAL OPERATORS ON CENTRAL MORREY SPACES

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(*Communicated by B. Opic*)

Abstract. We consider the boudedness of fractional integral operators on localized (central) Morrey spaces and investigate the relation between the Adams inequality and the Spanne inequality.

1. Introduction

The classical Morrey spaces were introduced by Morrey [10] to investigate the local behavior of solutions to second order elliptic differential equations. Spanne (see Peetre [12, Theorem 5.4]) and Adams [1] studied the boundedness of fractional integral operators on Morrey spaces. García-Cuerva and Herrero [6] and Alvarez, Lakey and Guzman-Partida [2] considered localized (central) Morrey spaces.

We show that a localized Adams inequality does not hold (Proposition 1), but it holds for radial functions when $n \geqslant 2$ (Theorem 1). We also prove a theorem which interpolates between Adams inequality and Spanne inequality on central Morrey spaces (Theorem 2). In sections 5 and 6 we show that our results are optimal by giving counterexamples.

We define fractional integral operators.

DEFINITION 1.

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

We define Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ and localized (central) Morrey spaces $L^{p,\lambda}(0)$.

DEFINITION 2. Let $1 < p < \infty$ and $0 < \lambda < 1$.

$$L^{p,\lambda}(R^n) = \left\{ f : \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \left(\sup_{x\in\mathbb{R}^n, R>0} \frac{1}{|B(x,R)|^\lambda} \int_{B(x,R)} |f(y)|^p\, dy \right)^{1/p} < \infty \right\},$$

$$L^{p,\lambda}(0) = \left\{ f : \|f\|_{L^{p,\lambda}(0)} = \left(\sup_{R>0} \frac{1}{|B(0,R)|^\lambda} \int_{B(0,R)} |f(y)|^p\, dy \right)^{1/p} < \infty \right\}$$

where $B(x,R)$ is the ball centered at x and the radius R. Compare with Definition 4 below.

Mathematics subject classification (2010): 42B20, 42B25.
Keywords and phrases: The Adams inequality, Morrey space, central Morrey space, fractional integral operator, radial function.

In this paper the following three indices q_1, q_2 and μ_1 are very important.

DEFINITION 3.

$$\frac{1}{q_1}=\frac{1}{p}-\frac{\alpha}{n},\quad \frac{1}{q_2}=\frac{1}{p}-\frac{\alpha}{n(1-\lambda)}\quad \text{and}\quad \frac{\mu_1}{q_1}=\frac{\lambda}{p}.$$

Spanne and Adams proved the following inequalities.

THEOREM A. (Spanne)

$$\|I_\alpha f\|_{L^{q_1,\mu_1}(\mathbb{R}^n)}\leqslant C\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}\quad \textit{where}\quad 1<p<\frac{n}{\alpha}.$$

THEOREM B. (Adams)

$$\|I_\alpha f\|_{L^{q_2,\lambda}(\mathbb{R}^n)}\leqslant C\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}\quad \textit{where}\quad 1<p<\frac{n(1-\lambda)}{\alpha}.$$

Throughout this paper we will let C denote a positive constant whose value may change from line to line, but which is independent of essential parameters.

REMARK. Since $\|f\|_{L^{q_1,\mu_1}(\mathbb{R}^n)}\leqslant\|f\|_{L^{q_2,\lambda}(\mathbb{R}^n)}$, Theorem B improves Theorem A when $1<p<n(1-\lambda)/\alpha$.

We also know the following result, see Fu, Lin and Lu [5]. This is a special case of the theorem by Burenkov, Gogatishvili, Guliyev and Mustafayev [3]. They investigate more general local Morrey-type spaces.

THEOREM C.

$$\|I_\alpha f\|_{L^{q_1,\mu_1}(0)}\leqslant C\|f\|_{L^{p,\lambda}(0)}\quad \textit{where}\quad 1<p<\frac{n}{\alpha}.$$

However I_α is not bounded from $L^{p,\lambda}(0)$ to $L^{q_2,\lambda}(0)$, see Proposition 1 below. We know only following trivial corollary of Theorem B.

THEOREM B'.

$$\|I_\alpha f\|_{L^{q_2,\lambda}(0)}\leqslant C\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}\quad \textit{where}\quad 1<p<\frac{n(1-\lambda)}{\alpha}.$$

The indices q_1, q_2 and μ_1 satisfy the following relation:

$$\frac{1-\mu_1}{q_1}=\frac{1-\lambda}{q_2}=\frac{1-\lambda}{p}-\frac{\alpha}{n}.$$

By Hölder's inequality we have

$$\|f\|_{L^{q_1,\mu_1}(0)}\leqslant\|f\|_{L^{q,\mu}(0)}\leqslant\|f\|_{L^{q_2,\lambda}(0)}\leqslant\|f\|_{L^{q_2,\lambda}(\mathbb{R}^n)}$$

where

$$\frac{1-\mu}{q} = \frac{1-\lambda}{p} - \frac{\alpha}{n}, \quad q_1 < q < q_2 \quad \text{and} \quad \lambda < \mu < \mu_1.$$

Therefore we estimate $\|I_\alpha f\|_{L^{q,\mu}(0)}$ and we want to obtain a theorem which interpolates between Theorems B' and C. For this purpose we introduce new function spaces. The localized (central) Morrey space is localized at the origin. We consider the space localized at x.

DEFINITION 4.

$$L^{p,\lambda}(x) = \left\{ f : \|f\|_{L^{p,\lambda}(x)} = \left(\sup_{R>0} \frac{1}{|B(x,R)|^\lambda} \int_{B(x,R)} |f(y)|^p \, dy \right)^{1/p} < \infty \right\}.$$

REMARKS. $\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \|f\|_{L^{p,\lambda}(x)}$. The space $L^{p,\lambda}(x)$ is a paticular case of the space $LM^{\{x\}}_{p\theta,\omega}$ introduced by Gogatishvili and Mustafayev [7].

2. Theorems

Our results are the following.

PROPOSITION 1. *I_α is not bounded from $L^{p,\lambda}(0)$ to $L^{q,\mu}(0)$ where $\frac{1-\mu}{q} = \frac{1-\lambda}{p} - \frac{\alpha}{n}$, $q_1 < q \leqslant q_2$ and $\lambda \leqslant \mu < \mu_1$.*

Proof. Let $R > 10$, $x_R = (R,0,\dots,0) \in \mathbb{R}^n$ and $f_R(x) = \chi_{B(x_R,1)}(x)$. Then

$$\|f_R\|_{L^{p,\lambda}(0)} \leqslant CR^{-n\lambda/p}.$$

On the contrary $I_\alpha f(x) \geqslant C$ if $x \in B(x_R,1)$. Therefore

$$\|I_\alpha f_R\|_{L^{q,\mu}(0)} \geqslant CR^{-n\mu/q}.$$

Since $\mu/q < \lambda/p$, we have

$$\lim_{R\to\infty} \frac{\|I_\alpha f_R\|_{L^{q,\mu}(0)}}{\|f_R\|_{L^{p,\lambda}(0)}} = \infty. \qquad \square$$

As Proposition 1 shows, a localized Adams inequality does not hold, but we obtain the following theorem.

THEOREM 1. *Let $n \geqslant 2$. If* (i) $p\alpha \geqslant 1$ *or* (ii) $p\alpha < 1$ *and* $0 < \lambda \leqslant 1 - 1/n$, *then for any radial functions f,*

$$\|I_\alpha f\|_{L^{q_2,\lambda}(0)} \leqslant C\|f\|_{L^{p,\lambda}(0)}. \tag{1}$$

REMARKS. This is an example of a well known general phenomenon: under suitable assumptions of symmetry, notably radial symmetry, classical estimates admit substantial improvements, see for example [11], and [8] in the context of Morrey space.

By the definition of q_2, we always assume that $0<\lambda<1-p\alpha/n$. We shall show that the condition $\lambda \leqslant 1-1/n$ in (ii) is optimal by giving a counterexample in section 5. When $n=1$, the inequality (1) does not hold, see the following counterexample.

COUNTEREXAMPLE 1. *When* $n=1$, *let*

$$f_k(x)=\chi_{[-k-1,-k]}(x)+\chi_{[k,k+1]}(x).$$

Same as the proof of Proposition 1 we can show

$$\lim_{k\to\infty}\frac{\|I_\alpha f_k\|_{L^{q,\mu}(0)}}{\|f_k\|_{L^{p,\lambda}(0)}}=\infty.$$

When we remove the condition that f is radial, we obtain the following theorem which interpolates between Theorems B' and C.

THEOREM 2. *Let* $1<p<\frac{n(1-\lambda)}{\alpha}$, $q_1<q<q_2$, $\lambda<\mu<\mu_1$ *and*

$$\frac{1-\mu}{q}=\frac{1-\lambda}{p}-\frac{\alpha}{n}.$$

Then

$$\|I_\alpha f\|_{L^{q,\mu}(0)} \leqslant C\|f\|_{L^{p,\lambda}(0)}^{1-\frac{p\alpha}{n(1-\lambda)}}\left(\sup_{R>0}\frac{1}{R^n}\int_{|x|<R}\|f\|_{L^{p,\lambda}(x)}^{\frac{p\alpha}{n(1-\lambda)}\frac{qq_2}{q_2-q}}\,dx\right)^{\frac{q_2-q}{qq_2}}. \tag{2}$$

REMARKS. Note that (2) is same as (3).

$$\|I_\alpha f\|_{L^{q,\mu}(0)} \leqslant C\|f\|_{L^{p,\lambda}(0)}^{\frac{p}{q_2}}\left(\sup_{R>0}\frac{1}{R^n}\int_{|x|<R}\|f\|_{L^{p,\lambda}(x)}^{\frac{pq\alpha}{n(\mu-\lambda)}}\,dx\right)^{\frac{1}{q}-\frac{1}{q_2}}. \tag{3}$$

We also obtain

$$\|I_\alpha f\|_{L^{q,\mu}(x_0)} \leqslant C\|f\|_{L^{p,\lambda}(x_0)}^{1-\frac{p\alpha}{n(1-\lambda)}}\left(\sup_{R>0}\frac{1}{R^n}\int_{|x-x_0|<R}\|f\|_{L^{p,\lambda}(x)}^{\frac{p\alpha}{n(1-\lambda)}\frac{qq_2}{q_2-q}}\,dx\right)^{\frac{q_2-q}{qq_2}} \tag{4}$$

for all $x_0\in\mathbb{R}^n$. Using $\|f\|_{L^{p,\lambda}(x)}\leqslant\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}$ in (4), we have

$$\|I_\alpha f\|_{L^{q,\mu}(x_0)} \leqslant C\|f\|_{L^{p,\lambda}(\mathbb{R}^n)},$$

and we obtain

$$\|I_\alpha f\|_{L^{q,\mu}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

This is reduced from Theorem B and Hölder's inequality.

In (2), if we take $q = q_2$ and $\mu = \lambda$, formally, then we have

$$\|I_\alpha f\|_{L^{q_2,\lambda}(0)} \leqslant C\|f\|_{L^{p,\lambda}(0)}^{1-\frac{p\alpha}{n(1-\lambda)}} \sup_{x\in\mathbb{R}^n} \|f\|_{L^{p,\lambda}(x)}^{\frac{p\alpha}{n(1-\lambda)}} \leqslant C\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

This is Theorem B'.

If we take $q = q_1$ in (2), then

$$\|I_\alpha f\|_{L^{q_1,\mu_1}(0)} \leqslant C\|f\|_{L^{p,\lambda}(0)}^{\frac{p}{q_2}} \left(\sup_{R>0} \frac{1}{R^n} \int_{|x|<R} \|f\|_{L^{p,\lambda}(x)}^{\frac{p}{\lambda}} dx\right)^{\frac{1}{q_1}-\frac{1}{q_2}}.$$

This inequality is worse than Theorem C. However we shall show in section 6 that the indices $\frac{p\alpha}{n(1-\lambda)}$ and $\frac{qq_2}{q_2-q}$ in Theorem 2 are optimal when $q > q_1$. Therefore the central Morrey space $L^{q_1,\mu_1}(0)$ for Spanne indices q_1 and μ_1 is a special function space.

3. Proof of Theorem 1

Let M be the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{R>0} \frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y)|\,dy.$$

The following Lemma is implicit in [12] and proved in [5, Proposition 1.1].

LEMMA 1.

$$\|Mf\|_{L^{p,\lambda}(0)} \leqslant C\|f\|_{L^{p,\lambda}(0)}.$$

We define some variants of the maximal operator.

DEFINITION 5. For $0 < \beta < n$,

$$M_\beta f(x) = \sup_{R>0} \frac{1}{|B(x,R)|^{1-\beta/n}} \int_{B(x,R)} |f(y)|\,dy.$$

DEFINITION 6.

$$m_\beta f(x) = \sup_{R>|x|} \frac{1}{R^{n-\beta}} \int_{|y|<R} |f(y)|\,dy.$$

For a radial function $f(x) = f_0(|x|)$,

$$\tilde{m}_\beta f(x) = \sup_{0<R<\frac{|x|}{2}} \frac{1}{R^{1-\beta}} \int_{|x|-R}^{|x|+R} |f_0(t)|\,dt.$$

We know the following two lemmas.

LEMMA 2. ([1], p. 768)

$$|I_\alpha f(x)| \leqslant C(M_\beta f(x))^{\alpha/\beta}(Mf(x))^{1-\alpha/\beta} \quad where \quad \alpha < \beta.$$

LEMMA 3. ([4], Lemma 3.1) *For any radial functions f,*

$$M_\beta f(x) \leqslant Cm_\beta f(x) \quad if \quad \beta \geqslant 1,$$
$$M_\beta f(x) \leqslant C\big(m_\beta f(x) + \tilde{m}_\beta f(x)\big) \quad if \quad \beta < 1.$$

By Lemma 3 we obtain the following lemma.

LEMMA 4. *Let $\beta = \frac{n(1-\lambda)}{p}$. For any radial functions f,*

$$M_\beta f(x) \leqslant C\|f\|_{L^{p,\lambda}(0)} \quad if \quad \beta \geqslant 1,$$
$$M_\beta f(x) \leqslant C\big(\|f\|_{L^{p,\lambda}(0)} + \tilde{m}_\beta f(x)\big) \quad if \quad \beta < 1.$$

Proof. By Hölder's inequality,

$$\frac{1}{R^{n-\beta}}\int_{|y|<R}|f(y)|\,dy \leqslant CR^{-n+\beta+n/p'+n\lambda/p}\left(\frac{1}{R^{n\lambda}}\int_{|y|<R}|f(y)|^p\,dy\right)^{1/p}$$
$$\leqslant C\|f\|_{L^{p,\lambda}(0)},$$

and we have $m_\beta f(x) \leqslant C\|f\|_{L^{p,\lambda}(0)}$. □

LEMMA 5. *Let $\beta = \frac{n(1-\lambda)}{p}$. If $\lambda \leqslant 1-1/n$, then for any radial functions f,*

$$\tilde{m}_\beta f(x) \leqslant C\|f\|_{L^{p,\lambda}(0)}.$$

Proof. Assume that $f(x) = f_0(|x|)$ and $R < \frac{|x|}{2}$.

$$\frac{1}{R^{1-\beta}}\int_{|x|-R}^{|x|+R}|f_0(t)|\,dt \leqslant CR^{\beta-1}\left(\int_{|x|-R}^{|x|+R}|f_0(t)|^p\,dt\right)^{1/p}R^{1/p'}$$
$$\leqslant CR^{\beta-1+1/p'}|x|^{(1-n)/p}\left(\int_{|x|-R}^{|x|+R}|f_0(t)|^p t^{n-1}\,dt\right)^{1/p}$$
$$\leqslant CR^{\beta-1/p}|x|^{(1-n+n\lambda)/p}\left(\frac{1}{(2|x|)^{n\lambda}}\int_{|y|\leqslant 2|x|}|f(y)|^p\,dy\right)^{1/p}$$
$$\leqslant C\|f\|_{L^{p,\lambda}(0)},$$

since $\lambda \leqslant 1-1/n$. □

LEMMA 6. *Let $\beta = \frac{n(1-\lambda)}{p}$. If $\lambda \leqslant 1-1/n$, then for any radial functions f,*

$$M_\beta f(x) \leqslant C\|f\|_{L^{p,\lambda}(0)}.$$

Proof. By Lemmas 4 and 5 we have the desired result. □

Proof of Theorem 1. Let $\beta = \frac{n(1-\lambda)}{p}$. Note that $\alpha < \beta$. We have by Lemmas 2 and 6

$$|I_\alpha f(x)| \leqslant C\|f\|_{L^{p,\lambda}(0)}^{\alpha/\beta}(Mf(x))^{1-\alpha/\beta}.$$

Since $(1-\alpha/\beta)q_2 = p$, we have by Lemma 1

$$\begin{aligned}\left(\frac{1}{R^{n\lambda}}\int_{|x|<R}|I_\alpha f(x)|^{q_2}\,dx\right)^{1/q_2} &\leqslant C\|f\|_{L^{p,\lambda}(0)}^{\alpha/\beta}\left(\frac{1}{R^{n\lambda}}\int_{|x|<R}Mf(x)^p\,dx\right)^{1/q_2}\\ &\leqslant C\|f\|_{L^{p,\lambda}(0)}^{\alpha/\beta}\|Mf\|_{L^{p,\lambda}(0)}^{p/q_2}\\ &\leqslant C\|f\|_{L^{p,\lambda}(0)}^{\alpha/\beta}\|f\|_{L^{p,\lambda}(0)}^{p/q_2} = C\|f\|_{L^{p,\lambda}(0)}. \quad \square\end{aligned}$$

4. Proof of Theorem 2

The following Lemmas 7 and 8 are trivial from the definitions.

LEMMA 7.

$$|B(x,R)|^{\alpha/n-1}\int_{B(x,R)}|f(y)|\,dy \leqslant |B(x,R)|^{\alpha/n-(1-\lambda)/p}\|f\|_{L^{p,\lambda}(x)}.$$

LEMMA 8.

$$|B(x,R)|^{\alpha/n-1}\int_{B(x,R)}|f(y)|\,dy \leqslant |B(x,R)|^{\alpha/n}Mf(x).$$

Proof of Theorem 2. We follow the argument in Hedberg [9, p. 506]. By Lemmas 7 and 8, we have

$$\begin{aligned}|I_\alpha f(x)| &\leqslant \sum_{j=-\infty}^{\infty}\int_{B(x,2^j)}\frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy\\ &\leqslant C\sum_{j=-\infty}^{\infty}|B(x,2^j)|^{\alpha/n-1}\int_{B(x,2^j)}|f(y)|\,dy\\ &\leqslant C\sum_{j=-\infty}^{N}|B(x,2^j)|^{\alpha/n}Mf(x) + C\sum_{j=N+1}^{\infty}|B(x,2^j)|^{\alpha/n-(1-\lambda)/p}\|f\|_{L^{p,\lambda}(x)}.\end{aligned}$$

Minimizing this inequality we obtain

$$|I_\alpha f(x)| \leqslant C\|f\|_{L^{p,\lambda}(x)}^{p\alpha/n(1-\lambda)}Mf(x)^{1-p\alpha/n(1-\lambda)}.$$

We take s such that

$$\left(1-\frac{p\alpha}{n(1-\lambda)}\right)qs = p.$$

Note that $s>1$, and let $t=s/(s-1)$. By Hölder's inequality and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|B(0,R)|^{\mu}}\int_{B(0,R)}|I_\alpha f(x)|^q\,dx\\
&\leqslant C\frac{|B(0,R)|^{\lambda/s}}{|B(0,R)|^{\mu}}\left(\frac{1}{|B(0,R)|^{\lambda}}\int_{B(0,R)}(Mf(x))^p\,dx\right)^{1/s}\left(\int_{B(0,R)}\|f\|_{L^{p,\lambda}(x)}^{\frac{pq\alpha t}{n(1-\lambda)}}\,dx\right)^{1/t}\\
&\leqslant C\|Mf\|_{L^{p,\lambda}(0)}^{p/s}\left(\sup_{R>0}\frac{1}{R^n}\int_{|x|<R}\|f\|_{L^{p,\lambda}(x)}^{\frac{pq\alpha t}{n(1-\lambda)}}\,dx\right)^{1/t}\\
&\leqslant C\|f\|_{L^{p,\lambda}(0)}^{(1-\frac{p\alpha}{n(1-\lambda)})q}\left(\sup_{R>0}\frac{1}{R^n}\int_{|x|<R}\|f\|_{L^{p,\lambda}(x)}^{\frac{p\alpha}{n(1-\lambda)}\frac{qq_2}{q_2-q}}\,dx\right)^{\frac{q_2-q}{q_2}}. \quad\square
\end{aligned}$$

5. A Counterexample for Theorem 1

In this section we shall show the condition $\lambda\leqslant 1-1/n$ in (ii) of Theorem 1 is optimal by giving a counterexample. For this purpose we use the next lemma.

LEMMA 9. ([11], Lemma 4.1) *Let $n\geqslant 2$, $x'\in S^{n-1}$ and*

$$J(x')=\int_{S^{n-1}}f((x',y'))\,dy',$$

where $f:[-1,1]\to\mathbb{R}^n$ and (x',y') denotes the inner product of x' and y'. Then

$$J(x')=w_{n-2}\int_{-1}^{1}f(t)(1-t^2)^{\frac{n-3}{2}}\,dt,$$

where w_{n-2} denotes the area of S^{n-2}.

COUNTEREXAMPLE 2. *Let $n\geqslant 2$ and $\lambda>1-1/n$. We take $0<\alpha<1$ sufficiently small and $p>1$ sufficiently near 1 such that $\frac{n(1-\lambda)}{\alpha}>1$ and $\frac{1}{p}-\frac{\alpha}{n(1-\lambda)}>0$.*

We define $f(x)=f_0(|x|)$ where

$$f_0(r)=\chi_{[1/2,3/2]}(r)|r-1|^{-1/p}\left(\log\frac{1}{|r-1|}\right)^{-1}.$$

Then $f\in L^{p,\lambda}(0)$ but $I_\alpha f\notin L^{q_2,\lambda}(0)$.

Proof. It is easy to show $f\in L^{p,\lambda}(0)$. We show that $I_\alpha f\notin L^{q_2}$ $(1<|x|<4/3)$. We write $x=|x|x'$.

$$\begin{aligned}
I_\alpha f(x)&=C\int_0^\infty f_0(r)r^{n-1}r^{-n+\alpha}\,dr\int_{S^{n-1}}\frac{dy'}{\left(\left(\frac{|x|}{r}\right)^2-2\frac{|x|}{r}(x',y')+1\right)^{\frac{n-\alpha}{2}}}\\
&=C\int_0^\infty f_0(r)r^{\alpha-1}J_{|x|/r}(x')dr
\end{aligned}$$

where

$$J_s(x') = \int_{S^{n-1}} \frac{dy'}{(s^2 - 2s(x',y') + 1)^{\frac{n-\alpha}{2}}}.$$

By Lemma 9

$$J_s(x') = w_{n-2} \int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{(s^2 - 2st + 1)^{\frac{n-\alpha}{2}}}\, dt.$$

CLAIM.

$$J_s(x') \geqslant C(1-s)^{\alpha-1} \quad \textit{where} \quad 0 < s < 1. \tag{5}$$

If $1 < |x| < \frac{4}{3}$, then we have by Claim

$$\begin{aligned} I_\alpha f(x) &\geqslant C \int_{|x|}^{3/2} r^{\alpha-1} |r-1|^{-1/p} \left(\log \frac{1}{|r-1|}\right)^{-1} \left(1 - \frac{|x|}{r}\right)^{\alpha-1} dr \\ &= C \int_{|x|}^{3/2} |r-1|^{-1/p} \left(\log \frac{1}{|r-1|}\right)^{-1} (r-|x|)^{\alpha-1}\, dr \\ &\geqslant C \int_{|x|}^{3/2} |r-1|^{-1/p} |r-1|^{\alpha-1}\, dr \left(\log \frac{1}{|x|-1}\right)^{-1} \\ &\geqslant C(|x|-1)^{-1/p+\alpha} \left(\log \frac{1}{|x|-1}\right)^{-1}. \end{aligned}$$

Since $q_2(-1/p+\alpha) < -1$, we have $I_\alpha f \notin L^{q_2}$ $(1 < |x| < 4/3)$. □

Proof of Claim. When $0 < s \leqslant 1/2$, $J_s(x') \geqslant C$. Therefore it suffices to show (5) for $1/2 < s < 1$.

$$\begin{aligned} &\int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{(s^2-2st+1)^{\frac{n-\alpha}{2}}}\, dt = \int_{-1}^{1} \frac{(1-t^2)^{\frac{n-3}{2}}}{((1-s)^2 + 2s(1-t))^{\frac{n-\alpha}{2}}}\, dt \\ &\geqslant C \int_0^1 \frac{(1-t)^{\frac{n-3}{2}}}{((1-s)^2 + 1 - t)^{\frac{n-\alpha}{2}}}\, dt = C \int_0^1 \frac{t^{\frac{n-3}{2}}}{((1-s)^2 + t)^{\frac{n-\alpha}{2}}}\, dt \\ &\geqslant C \int_{(1-s)^2}^{2(1-s)^2} \frac{t^{\frac{n-3}{2}}}{((1-s)^2+t)^{\frac{n-\alpha}{2}}}\, dt \geqslant C \frac{(1-s)^{n-3}(1-s)^2}{(1-s)^{n-\alpha}} = C(1-s)^{\alpha-1}. \quad \square \end{aligned}$$

6. Examples for Theorem 2

In this section we shall prove that Theorem 2 improves known results and is optimal by giving examples. For simplicity we consider the one dimensional case, that is, $\frac{1}{q_1} = \frac{1}{p} - \alpha$, $\frac{1}{q_2} = \frac{1}{p} - \frac{\alpha}{1-\alpha} > 0$,

$$\frac{1-\mu_1}{q_1} = \frac{1-\mu}{q} = \frac{1-\lambda}{q_2} = \frac{1-\lambda}{p} - \alpha,$$

and $q_1 < q < q_2$.

DEFINITION 7. We define

$$Z(f)=\|f\|_{L^{p,\lambda}(0)}^{1-\frac{p\alpha}{1-\lambda}}\left(\sup_{R>0}\frac{1}{R}\int_{|x|<R}\|f\|_{L^{p,\lambda}(x)}^{\frac{p\alpha}{1-\lambda}\frac{qq_2}{q_2-q}}\,dx\right)^{\frac{q_2-q}{qq_2}}.$$

The following propositions show that $Z(f)$ controls $\|I_\alpha f\|_{L^{q,\mu}(0)}$.

PROPOSITION 2. *Let* $f_n(x)=\chi_{[n,n+1]}(x)$. *Then*

$$\lim_{n\to\infty}\frac{\|I_\alpha f_n\|_{L^{q,\mu}(0)}}{\|f_n\|_{L^{p,\lambda}(0)}}=\infty, \tag{6}$$

$$\lim_{n\to\infty}\frac{\|I_\alpha f_n\|_{L^{q,\mu}(0)}}{\|f_n\|_{L^{p,\lambda}(\mathbb{R}^1)}}=0, \tag{7}$$

$$0<\liminf_{n\to\infty}\frac{\|I_\alpha f_n\|_{L^{q,\mu}(0)}}{Z(f_n)}\leqslant\limsup_{n\to\infty}\frac{\|I_\alpha f_n\|_{L^{q,\mu}(0)}}{Z(f_n)}<\infty. \tag{8}$$

Proof. (6) is proved in Proposition 1 and the proof of (7) is easy. The third inequality of (8) is obtained from Theorem 2. We shall prove the first inequality of (8). Let $A=\frac{p\alpha}{1-\lambda}\frac{qq_2}{q_2-q}$. Note that

$$\|f_n\|_{L^{p,\lambda}(0)}\approx n^{-\lambda/p}, \tag{9}$$

and

$$\|f_n\|_{L^{p,\lambda}(x)}\approx(|x-n|+1)^{-\lambda/p}.$$

Therefore

$$\sup_{R>0}\frac{1}{R}\int_{|x|<R}\|f_n\|_{L^{p,\lambda}(x)}^{A}\,dx\approx\frac{1}{n}\int_0^n\frac{1}{(|x-n|+1)^{A\lambda/p}}\,dx\approx\frac{1}{n}, \tag{10}$$

since $A\lambda/p>1$. By (9) and (10), we have

$$Z(f_n)\approx n^{\frac{1-\lambda}{q_2}-\frac{1}{q}}. \tag{11}$$

Since $I_\alpha f_n(x)\approx(|x-n|+1)^{\alpha-1}$ and $(1-\alpha)q>1$, we have

$$\|I_\alpha f_n\|_{L^{q,\mu}(0)}\approx n^{-\mu/q}. \tag{12}$$

By (11) and (12), we obtain the desired result. $\square$

We show that the index $\frac{p\alpha}{1-\lambda}$ in Definition 7 is optimal when $q>q_1$.

DEFINITION 8. For $0<B\leqslant 1$, we define

$$Z_B(f)=\|f\|_{L^{p,\lambda}(0)}^{1-B}\left(\sup_{R>0}\frac{1}{R}\int_{|x|<R}\|f\|_{L^{p,\lambda}(x)}^{\frac{Bqq_2}{q_2-q}}\,dx\right)^{\frac{q_2-q}{qq_2}}.$$

PROPOSITION 3. *Let* $f_n(x) = \chi_{[n,n+1]}(x)$.

$$\text{If} \quad \frac{p\alpha}{1-\lambda} < B \leqslant 1 \quad \text{then} \quad \lim_{n\to\infty} \frac{\|I_\alpha f_n\|_{L^{q,\mu}(0)}}{Z_B(f_n)} = 0. \tag{13}$$

$$\text{If} \quad \frac{p}{\lambda}\left(\frac{1}{q} - \frac{1}{q_2}\right) < B < \frac{p\alpha}{1-\lambda} \quad \text{then} \quad \lim_{n\to\infty} \frac{\|I_\alpha f_n\|_{L^{q,\mu}(0)}}{Z_B(f_n)} = \infty. \tag{14}$$

Proof. Let $\widetilde{B} = \frac{Bqq_2}{q_2-q}$. Then $\widetilde{B}\lambda/p > 1$ in either cases. Same as (10) we have

$$\sup_{R>0} \frac{1}{R} \int_{|x|<R} \|f_n\|_{L^{p,\lambda}(x)}^{\widetilde{B}} \, dx \approx \frac{1}{n} \int_0^n \frac{1}{(|x-n|+1)^{\widetilde{B}\lambda/p}} \, dx \approx \frac{1}{n},$$

and

$$Z_B(f_n) \approx n^{-\frac{\lambda}{p}(1-B) - \frac{1}{q} + \frac{1}{q_2}}. \tag{15}$$

By (12) and (15) we can prove (13) and (14) simultaneously. □

PROPOSITION 4. *Assume that* $0 < B \leqslant \frac{p}{\lambda}\left(\frac{1}{q} - \frac{1}{q_2}\right)$. *Then there exists a function* f *such that*

(i) $f \in L^{p,\lambda}(0)$ *and* (ii) $Z_B(f) < \infty$ *but* (iii) $I_\alpha f \notin L^{q,\mu}(0)$.

Proof. We write $\frac{1}{q} = \frac{1}{q_1} - \varepsilon$ where $\varepsilon > 0$ and define

$$f(x) = \frac{1}{|x-1|^{1/p-\varepsilon}} \chi_{\{0<x<2\}}(x).$$

To prove (i) is easy. We prove (iii). Since $I_\alpha f(x) \geqslant C|x-1|^{-(1/p-\varepsilon-\alpha)}$ where $0 < x < 1$ and $(1/p - \varepsilon - \alpha)q = 1$, we have $I_\alpha f \notin L^q_{loc}$. Hence $I_\alpha f \notin L^{q,\mu}(0)$.

For the proof of (ii) we use the following elementary lemma.

LEMMA 10. *Let* $g(x) = |x|^{-A}\chi_{\{|x|<1\}}(x)$ *where* $(1-\lambda)/p < A < 1/p$. *Then*

$$\|g\|_{L^{p,\lambda}(x)} \leqslant C \times \begin{cases} |x|^{\frac{1-\lambda}{p} - A} & \text{if} \quad |x| \leqslant 1, \\ |x|^{-\frac{\lambda}{p}} & \text{if} \quad |x| > 1. \end{cases}$$

Note that $g \notin L^{p,\lambda}(0)$.

Proof. If $|x| \leqslant 1$,

$$\|g\|_{L^{p,\lambda}(x)} \leqslant \left(\frac{C}{|x|^\lambda} \int_{|t|\leqslant|x|} |t|^{-Ap} \, dt\right)^{1/p} = C|x|^{\frac{1-\lambda}{p} - A}.$$

If $|x| \geqslant 1$,

$$\|g\|_{L^{p,\lambda}(x)} \leqslant \left(\frac{C}{|x|^\lambda} \int_{|t|\leqslant 1} |t|^{-Ap} \, dt\right)^{1/p} = C|x|^{\frac{-\lambda}{p}}. \quad \square$$

Using Lemma 10 we have

$$\|f\|_{L^{p,\lambda}(x)} \leqslant C \times \begin{cases} |x-1|^{\frac{-\lambda}{p}+\varepsilon} & \text{if} \quad |x-1| \leqslant 1, \\ |x-1|^{-\frac{\lambda}{p}} & \text{if} \quad |x-1| > 1. \end{cases}$$

Since $\left(\frac{-\lambda}{p}+\varepsilon\right)\frac{Bqq_2}{q_2-q} > -1$ we can prove (ii). □

We shall also show that the index $\frac{qq_2}{q_2-q}$ is optimal by giving a counterexample.

DEFINITION 9. For $1 \leqslant A < \infty$, we define

$$Z^A(f) = \|f\|_{L^{p,\lambda}(0)}^{1-\frac{p\alpha}{1-\lambda}} \left(\sup_{R>0} \frac{1}{R} \int_{|x|<R} \|f\|_{L^{p,\lambda}(x)}^{\frac{Ap\alpha}{1-\lambda}} dx\right)^{1/A}.$$

PROPOSITION 5. *Let* $f_n(x) = \chi_{[n,n+1]}(x)$. *If* $A < \frac{qq_2}{q_2-q}$ *then*

$$\lim_{n\to\infty} \frac{\|I_\alpha f_n\|_{L^{q,\mu}(0)}}{Z^A(f_n)} = \infty.$$

Proof. Same as (10), we have

$$\sup_{R>0} \frac{1}{R} \int_{|x|<R} \|f_n\|_{L^{p,\lambda}(x)}^{\frac{Ap\alpha}{1-\lambda}} dx \approx \frac{1}{n} \int_0^n \frac{1}{(|x-n|+1)^{\frac{Ap\alpha}{1-\lambda}}} dx \approx \frac{1}{n},$$

and by (9), we have

$$Z^A(f_n) \approx n^{-\frac{\lambda}{p}(1-\frac{p\alpha}{1-\lambda})} n^{-\frac{1}{A}}$$

By using (12), we obtain the desired result. □

Acknowledgement. The authors would like to express our gratitude to the referee for his/her careful reading of our manuscript and helpful comments.

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(Received May 19, 2016)

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