

ON SOME NEW INEQUALITIES FOR FUSION FRAMES IN HILBERT SPACES

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(Communicated by I. Perić)

Abstract. Recently fusion frame was considered as a generalization of frame in Hilbert spaces. In this paper, we establish several new inequalities for fusion frames with a scalar in Hilbert spaces. It is shown that the results we obtained can immediately lead to the existing corresponding results when we choose suitable scalars.

1. Introduction

The concept of a frame in Hilbert spaces was first introduced by Duffin and Schaffer [10] in 1952 to study some problems in nonharmonic Fourier series. Frames were reintroduced and developed in 1986 by Daubechies at al [9], and popularized from then on. Nice properties of frames make them very useful in the characterization of function spaces and other fields of applications such as signal processing [16], coding theory [17], sampling theory [3] and more. We refer the reader to [8, 14] for an introduction to the frame theory and its applications.

Later on, fusion frame, which we also call it a frame of subspaces, was first proposed by Casazza and Kutyniok in [5] and reintroduced in [6]. Fusion frame is a natural generalization of frame theory and related to the construction of global frames from local frames in Hilbert spaces. Due to this characterization, fusion frames have been applied for distributed processing [6], optimal transmission by packet encoding [2], compressed sensing [4], filter bank [7], high energy physics [15], etc.

Balan et al. [1] found some Parseval equalities when they studied the optimal decomposition of a Parseval frames in a Hilbert space. Then, Găvrute [11] developed some identities and inequalities about discrete frames and the authors in [20] generalized these identities to alternate dual frames and got some general results.

These equalities and inequalities have been used for reconstructing signal without information about the phase. However, a number of new applications have emerged which cannot be modeled naturally by one single frame system. In order to reconstruct signal without phase in a wireless sensor network, we need to study some equalities and inequalities of fusion frames. Some authors have extended the equalities and inequalities for frames and dual frames in Hilbert spaces to fusion frames and dual fusion frames, respectively (see [13, 19]).

Mathematics subject classification (2010): 42C15, 47B40.

Keywords and phrases: Fusion frame, frame operator, inequality.

In this paper, we give some new inequalities for fusion frames with a scalar $\lambda \in [0, 2]$. We show that inequalities and equalities of [13, 19] can be obtained for special values of $\lambda = 1$ and $\lambda = \frac{1}{2}$, respectively. We use different techniques to prove our results. Moreover, we also give some inequalities for fusion frames with $\lambda \in [1, 2]$.

Throughout the paper, let \mathcal{H} be a Hilbert spaces and let I be a countable index set. $I_{\mathcal{H}}$ denotes the identity operator on \mathcal{H} . If W is a closed subspace of \mathcal{H} , we denote the orthogonal projection of \mathcal{H} onto W by π_W .

DEFINITION 1. Let $\{W_i\}_{i \in I}$ be a sequence of closed subspaces in \mathcal{H} , $\{w_i\}_{i \in I}$ be a family of weights, i.e., $w_i > 0$ for all $i \in I$. $\{(W_i, w_i)\}_{i \in I}$ is called a fusion frame for \mathcal{H} , if there exist two positive constants A, B such that

$$A\|f\|^2 \leq \sum_{i \in I} w_i^2 \|\pi_{W_i}(f)\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

The numbers A, B are called the fusion frame bounds. The family $\{(W_i, w_i)\}_{i \in I}$ is called an A -tight fusion frame if $A = B$, it is a Parseval fusion frame if $A = B = 1$, and v -uniform if $w = w_i = w_j$ for all $i, j \in I$. If $\{(W_i, w_i)\}_{i \in I}$ possesses an upper fusion frame bound, but not necessarily a lower bound, we call it a Bessel fusion sequence with Bessel fusion bound B . Moreover we say that $\{W_i\}_{i \in I}$ is an orthonormal fusion basis for \mathcal{H} if $\mathcal{H} = \oplus_{i \in I} W_i$.

For each Bessel fusion sequence $\{(W_i, w_i)\}_{i \in I}$ of \mathcal{H} , we define the representation space associated with $\{W_i\}_{i \in I}$ by

$$\ell^2(\mathcal{H}, I) = \left\{ \{f_i\}_{i \in I} \mid f_i \in W_i \text{ and } \sum_{i \in I} \|f_i\|^2 < \infty \right\}$$

with inner product given by

$$\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle.$$

The frame operator S for $\{(W_i, w_i)\}_{i \in I}$ is defined by

$$S: \mathcal{H} \longrightarrow \mathcal{H}, \quad S(f) = \sum_{i \in I} w_i^2 \pi_{W_i}(f), \quad \forall f \in \mathcal{H}.$$

Casazza and Kutyniok in [5] proved that S is positive, self-adjoint, invertible operator on \mathcal{H} and the following reconstruction formula holds for all $f \in \mathcal{H}$:

$$f = S^{-1}Sf = \sum_{i \in I} w_i^2 S^{-1} \pi_{W_i} f = SS^{-1}f = \sum_{i \in I} w_i^2 \pi_{W_i}(S^{-1}f).$$

In [12] the author gave a more general alternate dual reconstruction formula, that is, given a fusion frame $\{(W_i, w_i)\}_{i \in I}$ with frame operator S and a Bessel sequence $\{(V_i, v_i)\}_{i \in I}$, there is

$$f = \sum_{i \in I} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f, \quad \forall f \in \mathcal{H}.$$

In this case we also call $\{(V_i, v_i)\}_{i \in I}$ an alternate dual of $\{(W_i, w_i)\}_{i \in I}$.

Let $\{(W_i, w_i)\}_{i \in I}$ be a fusion frame, then for any $J \subset I$, we define a bounded linear operator $S_J : \mathcal{H} \rightarrow \mathcal{H}$ by

$$S_J f = \sum_{i \in J} w_i^2 \pi_{W_i} f,$$

and denote $J^c = I \setminus J$.

The following equalities for a fusion frame in a Hilbert space were given in [13, 19].

THEOREM 1. *Let $\{(W_i, w_i)\}_{i \in I}$ be a fusion frame for \mathcal{H} with the fusion frame operator S , $\{(S^{-1}W_i, w_i)\}_{i \in I}$ is the dual fusion frame of $\{(W_i, w_i)\}_{i \in I}$. Then for any $J \subset I$ and any $f \in \mathcal{H}$, we have*

$$\begin{aligned} & \sum_{i \in J} w_i^2 \|\pi_{W_i} f\|^2 + \sum_{i \in I} \|\pi_{W_i} S^{-1} S_{J^c} f\|^2 \\ &= \sum_{i \in J^c} w_i^2 \|\pi_{W_i} f\|^2 + \sum_{i \in I} \|\pi_{W_i} S^{-1} S_J f\|^2 \geq \frac{3}{4} \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2. \end{aligned} \tag{1}$$

THEOREM 2. *Let $\{(W_i, w_i)\}_{i \in I}$ be a fusion frame for \mathcal{H} with the fusion frame operator S and let $\{(V_i, v_i)\}_{i \in I}$ be the alternate dual fusion frame of $\{(W_i, w_i)\}_{i \in I}$. Then, for any $J \subset I$ and any $f \in \mathcal{H}$, we have*

$$\begin{aligned} & \sum_{i \in J} v_i w_i \langle S^{-1} \pi_{W_i} f, \pi_{V_i} f \rangle + \left\| \sum_{i \in J^c} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f \right\|^2 \\ &= \sum_{i \in J^c} v_i w_i \langle S^{-1} \pi_{W_i} f, \pi_{V_i} f \rangle + \left\| \sum_{i \in J} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f \right\|^2 \geq \frac{3}{4} \|f\|^2. \end{aligned} \tag{2}$$

2. The main results and their proofs

To derive our main results, we need the following lemma.

LEMMA 1. *Let P, Q be two self-adjoint bounded linear operators in \mathcal{H} and $P + Q = I_{\mathcal{H}}$, then for any $\lambda \in [0, 2]$ and all $f \in \mathcal{H}$ we have*

$$\|Pf\|^2 + \lambda \langle Qf, f \rangle = \|Qf\|^2 + (2 - \lambda) \langle Pf, f \rangle + (\lambda - 1) \|f\|^2 \geq \left(\lambda - \frac{\lambda^2}{4} \right) \|f\|^2.$$

Proof. Since $P + Q = I_{\mathcal{H}}$, we have

$$\|Pf\|^2 + \lambda \langle Qf, f \rangle = \langle P^2 f, f \rangle + \lambda \langle (I_{\mathcal{H}} - P)f, f \rangle = \langle (P^2 - \lambda P + \lambda I_{\mathcal{H}})f, f \rangle,$$

and

$$\begin{aligned} & \|Qf\|^2 + (2 - \lambda) \langle Pf, f \rangle + (\lambda - 1) \|f\|^2 \\ &= \langle (I_{\mathcal{H}} - P)^2 f, f \rangle + (2 - \lambda) \langle Pf, f \rangle + (\lambda - 1) \|f\|^2 \\ &= \langle (P^2 - \lambda P + \lambda I_{\mathcal{H}})f, f \rangle. \end{aligned}$$

We also have

$$\begin{aligned} \langle (P^2 - \lambda P + \lambda I_{\mathcal{H}})f, f \rangle &= \left\langle \left(P^2 - \lambda P + \frac{\lambda^2}{4} + \lambda I_{\mathcal{H}} - \frac{\lambda^2}{4} I_{\mathcal{H}} \right) f, f \right\rangle \\ &= \left\langle \left(\left(P - \frac{\lambda}{2} I_{\mathcal{H}} \right)^2 + \lambda I_{\mathcal{H}} - \frac{\lambda^2}{4} \right) f, f \right\rangle \\ &= \left\| \left(P - \frac{\lambda}{2} I_{\mathcal{H}} \right) f \right\|^2 + \left(\lambda - \frac{\lambda^2}{4} \right) \|f\|^2 \\ &\geq \left(\lambda - \frac{\lambda^2}{4} \right) \|f\|^2. \quad \square \end{aligned}$$

THEOREM 3. *Let $\{(W_i, w_i)\}_{i \in I}$ be a fusion frame for \mathcal{H} with the fusion frame operator S , $\{(S^{-1}W_i, w_i)\}_{i \in I}$ is the dual fusion frame of $\{(W_i, w_i)\}_{i \in I}$. Then for any $\lambda \in [0, 2]$, for all $J \subset I$ and any $f \in \mathcal{H}$, we have*

$$\begin{aligned} \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2 &\geq \sum_{i \in J^c} w_i^2 \|\pi_{W_i} f\|^2 + \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1} S_J f\|^2 \\ &= \sum_{i \in J} w_i^2 \|\pi_{W_i} f\|^2 + \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1} S_{J^c} f\|^2 \\ &\geq \left(\lambda - \frac{\lambda^2}{4} \right) \sum_{i \in J} w_i^2 \|\pi_{W_i} f\|^2 + \left(1 - \frac{\lambda^2}{4} \right) \sum_{i \in J^c} w_i^2 \|\pi_{W_i} f\|^2. \end{aligned} \tag{3}$$

Proof. Since $S = S_J + S_{J^c}$, it follows that

$$I_{\mathcal{H}} = S^{-1/2} S_J S^{-1/2} + S^{-1/2} S_{J^c} S^{-1/2}.$$

Let $P = S^{-1/2} S_J S^{-1/2}$, $Q = S^{-1/2} S_{J^c} S^{-1/2}$ and let $S^{1/2} f$ instead of $f \in \mathcal{H}$ in Lemma 1, we have

$$\|PS^{1/2} f\|^2 + \lambda \langle QS^{1/2} f, S^{1/2} f \rangle = \langle S^{-1} S_J f, S_J f \rangle + \lambda \langle S_{J^c} f, f \rangle. \tag{4}$$

And

$$\begin{aligned} &\|QS^{1/2} f\|^2 + (2 - \lambda) \langle PS^{1/2} f, S^{1/2} f \rangle + (\lambda - 1) \|S^{1/2} f\|^2 \\ &= \langle S^{-1} S_{J^c} f, S_{J^c} f \rangle + (2 - \lambda) \langle S_J f, f \rangle + (\lambda - 1) \langle S f, f \rangle. \end{aligned} \tag{5}$$

By (4) and (5), we have

$$\langle S^{-1} S_J f, S_J f \rangle + \lambda \langle S_{J^c} f, f \rangle = \langle S^{-1} S_{J^c} f, S_{J^c} f \rangle + (2 - \lambda) \langle S_J f, f \rangle + (\lambda - 1) \langle S f, f \rangle.$$

After subtracting both sides by $\lambda \langle S_{J^c} f, f \rangle$, we obtain

$$\begin{aligned} \langle S^{-1} S_J f, S_J f \rangle &= \langle S^{-1} S_{J^c} f, S_{J^c} f \rangle + (2 - \lambda) \langle S_J f, f \rangle + (\lambda - 1) \langle S f, f \rangle - \lambda \langle S_{J^c} f, f \rangle \\ &= \langle S^{-1} S_{J^c} f, S_{J^c} f \rangle + 2 \langle S_J f, f \rangle - \lambda \langle (S_{J^c} + S_J) f, f \rangle + (\lambda - 1) \langle S f, f \rangle \\ &= \langle S^{-1} S_{J^c} f, S_{J^c} f \rangle + 2 \langle S_J f, f \rangle - \langle S f, f \rangle \\ &= \langle S^{-1} S_{J^c} f, S_{J^c} f \rangle + 2 \langle S_J f, f \rangle - \langle (S_J + S_{J^c}) f, f \rangle \\ &= \langle S^{-1} S_{J^c} f, S_{J^c} f \rangle + \langle S_J f, f \rangle - \langle S_{J^c} f, f \rangle. \end{aligned}$$

Thus,

$$\langle S^{-1}S_Jf, S_Jf \rangle + \langle S_{J^c}f, f \rangle = \langle S^{-1}S_{J^c}f, S_{J^c}f \rangle + \langle S_Jf, f \rangle. \tag{6}$$

On the other hand, we have

$$\langle S^{-1}S_Jf, S_Jf \rangle = \langle SS^{-1}S_Jf, S^{-1}S_Jf \rangle = \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1}S_Jf\|^2. \tag{7}$$

Similarly, we obtain

$$\langle S_{J^c}f, f \rangle = \sum_{i \in J^c} w_i^2 \|\pi_{W_i} f\|^2. \tag{8}$$

$$\langle S_Jf, f \rangle = \sum_{i \in J} w_i^2 \|\pi_{W_i} f\|^2. \tag{9}$$

$$\langle S^{-1}S_{J^c}f, S_{J^c}f \rangle = \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1}S_{J^c}f\|^2. \tag{10}$$

Using (6)–(10), we prove the equality of (3). Next, we prove the first inequality of (3). Since $P = S^{-1/2}S_J S^{-1/2}$, $Q = S^{-1/2}S_{J^c} S^{-1/2}$ are positive operators, then

$$0 \leq PQ = P(I_{\mathcal{H}} - P) = P - P^2 = S^{-1/2}(S_J - S_J S^{-1}S_J)S^{-1/2},$$

from which we conclude that $S_J - S_J S^{-1}S_J \geq 0$. Therefore, By (7) and (8), we have

$$\begin{aligned} \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1}S_Jf\|^2 + \sum_{i \in J^c} w_i^2 \|\pi_{W_i} f\|^2 &= \langle S^{-1}S_Jf, S_Jf \rangle + \langle S_{J^c}f, f \rangle \\ &= \langle S_J S^{-1}S_Jf, f \rangle + \langle S_{J^c}f, f \rangle \\ &\leq \langle S_Jf, f \rangle + \langle S_{J^c}f, f \rangle \\ &= \langle Sf, f \rangle = \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2. \end{aligned}$$

We now prove the last inequality. By Lemma 1 and (4), we have

$$\langle S^{-1}S_Jf, S_Jf \rangle + \lambda \langle S_{J^c}f, f \rangle \geq (\lambda - \lambda^2/4) \langle Sf, f \rangle.$$

And then,

$$\begin{aligned} \langle S^{-1}S_Jf, S_Jf \rangle &\geq (\lambda - \lambda^2/4) \langle Sf, f \rangle - \lambda \langle S_{J^c}f, f \rangle \\ &= \lambda \langle Sf, f \rangle - \lambda \langle S_{J^c}f, f \rangle - \frac{\lambda^2}{4} \langle Sf, f \rangle \\ &= \lambda \langle S_Jf, f \rangle - \frac{\lambda^2}{4} \langle S_Jf, f \rangle - \frac{\lambda^2}{4} \langle S_{J^c}f, f \rangle \\ &= \left(\lambda - \frac{\lambda^2}{4}\right) \langle S_Jf, f \rangle + \left(1 - \frac{\lambda^2}{4}\right) \langle S_{J^c}f, f \rangle - \langle S_{J^c}f, f \rangle. \end{aligned} \tag{11}$$

Hence,

$$\langle S^{-1}S_Jf, S_Jf \rangle + \langle S_{J^c}f, f \rangle \geq \left(\lambda - \frac{\lambda^2}{4}\right) \langle S_Jf, f \rangle + \left(1 - \frac{\lambda^2}{4}\right) \langle S_{J^c}f, f \rangle.$$

Therefore the proof is completed. \square

Theorem 3 leads to a direct consequence as follow.

COROLLARY 1. Let $\{(W_i, w_i)\}_{i \in I}$ be a Parseval fusion frame for \mathcal{H} . Then for any $\lambda \in [0, 2]$, for all $J \subset I$ and any $f \in \mathcal{H}$, we have

$$\begin{aligned} \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2 &\geq \sum_{i \in J^c} w_i^2 \|\pi_{W_i} f\|^2 + \sum_{i \in I} w_i^2 \|\pi_{W_i} S_J f\|^2 \\ &= \sum_{i \in J} w_i^2 \|\pi_{W_i} f\|^2 + \sum_{i \in I} w_i^2 \|\pi_{W_i} S_{J^c} f\|^2 \\ &\geq \left(\lambda - \frac{\lambda^2}{4}\right) \sum_{i \in J} w_i^2 \|\pi_{W_i} f\|^2 + \left(1 - \frac{\lambda^2}{4}\right) \sum_{i \in J^c} w_i^2 \|\pi_{W_i} f\|^2. \end{aligned}$$

REMARK 1. If we take $\lambda = 1$ in Theorem 3, then we obtain the previous inequality in Theorem 1 and Theorem 8 of [13].

Next, We consider scalar $\lambda \in [0, 1]$ and give a generalization of the Theorem 2. We need the following result.

LEMMA 2. Let P, Q be two self-adjoint bounded linear operators in \mathcal{H} and $P + Q = I_{\mathcal{H}}$, then for any $\lambda \in [0, 1]$ and all $f \in \mathcal{H}$ we have

$$P^*P + \lambda(Q^* + Q) = Q^*Q + (1 - \lambda)(P^* + P) + (2\lambda - 1)I_{\mathcal{H}} \geq \lambda(2 - \lambda)I_{\mathcal{H}}.$$

Proof. Since $P + Q = I_{\mathcal{H}}$, we have

$$P^*P + \lambda(Q^* + Q) = P^*P + \lambda(I_{\mathcal{H}} - P^* + I_{\mathcal{H}} - P) = P^*P - \lambda(P^* + P) + 2\lambda I_{\mathcal{H}},$$

and

$$\begin{aligned} &Q^*Q + (1 - \lambda)(P^* + P) + (2\lambda - 1)I_{\mathcal{H}} \\ &= (I_{\mathcal{H}} - P^*)(I_{\mathcal{H}} - P) + (1 - \lambda)(P^* + P) + (2\lambda - 1)I_{\mathcal{H}} \\ &= P^*P - \lambda(P^* + P) + 2\lambda I_{\mathcal{H}}. \end{aligned}$$

We also have

$$\begin{aligned} P^*P - \lambda(P^* + P) + 2\lambda I_{\mathcal{H}} &= P^*P - \lambda(P^* + P) + 2\lambda I_{\mathcal{H}} + \lambda^2 I_{\mathcal{H}} - \lambda^2 I_{\mathcal{H}} \\ &= (P - \lambda I_{\mathcal{H}})^*(P - \lambda I_{\mathcal{H}}) + \lambda(2 - \lambda)I_{\mathcal{H}} \\ &\geq \lambda(2 - \lambda)I_{\mathcal{H}}. \quad \square \end{aligned}$$

THEOREM 4. Let $\{(W_i, w_i)\}_{i \in I}$ be a fusion frame for \mathcal{H} with the fusion frame operator S and let $\{(V_i, v_i)\}_{i \in I}$ be the alternate dual fusion frame of $\{(W_i, w_i)\}_{i \in I}$. Then for any $\lambda \in [0, 1]$, for all $J \subset I$ and $x \in \mathcal{H}$, we have

$$\begin{aligned} &\operatorname{Re} \sum_{i \in J} v_i w_i \langle S^{-1} \pi_{W_i} f, \pi_{V_i} f \rangle + \left\| \sum_{i \in J^c} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f \right\|^2 \\ &= \operatorname{Re} \sum_{i \in J^c} v_i w_i \langle S^{-1} \pi_{W_i} f, \pi_{V_i} f \rangle + \left\| \sum_{i \in J} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f \right\|^2 \\ &\geq (2\lambda - \lambda^2) \operatorname{Re} \sum_{i \in J} v_i w_i \langle S^{-1} \pi_{W_i} f, \pi_{V_i} f \rangle + (1 - \lambda^2) \operatorname{Re} \sum_{i \in J^c} v_i w_i \langle S^{-1} \pi_{W_i} f, \pi_{V_i} f \rangle. \quad (12) \end{aligned}$$

Proof. For any $J \subset I$, we define a bounded linear operator E_J as

$$E_J f = \sum_{i \in J} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f, \quad \forall f \in \mathcal{H}.$$

Clearly, $E_J + E_{J^c} = I_{\mathcal{H}}$. By Lemma 2, we have

$$\begin{aligned} & \langle E_J^* E_J f, f \rangle + \lambda \langle (E_J^* + E_{J^c}) f, f \rangle \\ &= \langle E_J^* E_J f, f \rangle + \lambda \overline{\langle E_{J^c} f, f \rangle} + \lambda \langle E_{J^c} f, f \rangle \end{aligned} \tag{13}$$

$$\begin{aligned} &= \langle E_{J^c}^* E_{J^c} f, f \rangle + (1 - \lambda) \langle (E_J^* + E_J) f, f \rangle + (2\lambda - 1) \|f\|^2 \\ &= \langle E_{J^c}^* E_{J^c} f, f \rangle + (1 - \lambda) (\overline{\langle E_J f, f \rangle} + \langle E_J f, f \rangle) + (2\lambda - 1) \langle I_{\mathcal{H}} f, f \rangle. \end{aligned} \tag{14}$$

Taking real part of (13) and (14), we have

$$\|E_J f\|^2 + 2\lambda \operatorname{Re} \langle E_{J^c} f, f \rangle = \|E_{J^c} f\|^2 + 2(1 - \lambda) \operatorname{Re} \langle E_J f, f \rangle + (2\lambda - 1) \operatorname{Re} \langle I_{\mathcal{H}} f, f \rangle.$$

Thus,

$$\begin{aligned} \|E_J f\|^2 &= \|E_{J^c} f\|^2 + 2(1 - \lambda) \operatorname{Re} \langle E_J f, f \rangle - 2\lambda \operatorname{Re} \langle E_{J^c} f, f \rangle + (2\lambda - 1) \operatorname{Re} \langle I_{\mathcal{H}} f, f \rangle \\ &= \|E_{J^c} f\|^2 + 2 \operatorname{Re} \langle E_J f, f \rangle - 2\lambda \operatorname{Re} \langle (E_J + E_{J^c}) f, f \rangle + (2\lambda - 1) \operatorname{Re} \langle I_{\mathcal{H}} f, f \rangle \\ &= \|E_{J^c} f\|^2 + 2 \operatorname{Re} \langle E_J f, f \rangle - \operatorname{Re} \langle I_{\mathcal{H}} f, f \rangle \\ &= \|E_{J^c} f\|^2 + 2 \operatorname{Re} \langle E_J f, f \rangle - \operatorname{Re} \langle (E_J + E_{J^c}) f, f \rangle \\ &= \|E_{J^c} f\|^2 + \operatorname{Re} \langle E_J f, f \rangle - \operatorname{Re} \langle E_{J^c} f, f \rangle. \end{aligned}$$

Hence,

$$\|E_J f\|^2 + \operatorname{Re} \langle E_{J^c} f, f \rangle = \|E_{J^c} f\|^2 + \operatorname{Re} \langle E_J f, f \rangle. \tag{15}$$

By (15), we have

$$\begin{aligned} & \left\| \sum_{i \in J} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f \right\|^2 + \operatorname{Re} \sum_{i \in J^c} v_i w_i \langle S^{-1} \pi_{W_i} f, \pi_{V_i} f \rangle \\ &= \left\| \sum_{i \in J} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f \right\|^2 + \operatorname{Re} \sum_{i \in J^c} v_i w_i \langle \pi_{V_i} S^{-1} \pi_{W_i} f, f \rangle \\ &= \|E_J f\|^2 + \operatorname{Re} \langle E_{J^c} f, f \rangle = \|E_{J^c} f\|^2 + \operatorname{Re} \langle E_J f, f \rangle \\ &= \left\| \sum_{i \in J^c} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f \right\|^2 + \operatorname{Re} \sum_{i \in J} v_i w_i \langle \pi_{V_i} S^{-1} \pi_{W_i} f, f \rangle \\ &= \left\| \sum_{i \in J^c} v_i w_i \pi_{V_i} S^{-1} \pi_{W_i} f \right\|^2 + \operatorname{Re} \sum_{i \in J} v_i w_i \langle S^{-1} \pi_{W_i} f, \pi_{V_i} f \rangle \end{aligned}$$

We now prove the inequality of (12). By Lemma 2, we have

$$\langle E_J^* E_J f, f \rangle + \lambda \overline{\langle E_{J^c} f, f \rangle} + \lambda \langle E_{J^c} f, f \rangle \geq (2\lambda - \lambda^2) \langle I_{\mathcal{H}} f, f \rangle. \tag{16}$$

Taking real part of (16), we obtain

$$\|E_J f\|^2 + 2\lambda \operatorname{Re} \langle E_{J^c} f, f \rangle \geq (2\lambda - \lambda^2) \operatorname{Re} \langle I_{\mathcal{H}} f, f \rangle,$$

then

$$\begin{aligned} \|E_J f\|^2 &\geq (2\lambda - \lambda^2)\operatorname{Re}\langle I_{\mathcal{H}} f, f \rangle - 2\lambda\operatorname{Re}\langle E_{J^c} f, f \rangle \\ &= (2\lambda - \lambda^2)\operatorname{Re}\langle (E_J + E_{J^c})f, f \rangle - 2\lambda\operatorname{Re}\langle E_{J^c} f, f \rangle \\ &= (2\lambda - \lambda^2)\operatorname{Re}\langle E_J f, f \rangle - \lambda^2\operatorname{Re}\langle E_{J^c} f, f \rangle \\ &= (2\lambda - \lambda^2)\operatorname{Re}\langle E_J f, f \rangle + (1 - \lambda^2)\operatorname{Re}\langle E_{J^c} f, f \rangle - \operatorname{Re}\langle E_{J^c} f, f \rangle. \end{aligned}$$

Hence

$$\|E_J f\|^2 + \langle E_{J^c} f, f \rangle \geq (2\lambda - \lambda^2)\operatorname{Re}\langle E_J f, f \rangle + (1 - \lambda^2)\operatorname{Re}\langle E_{J^c} f, f \rangle.$$

The proof is completed. \square

In the situation of Parseval fusion frames the inequality is of special form.

COROLLARY 2. *Let $\{(W_i, w_i)\}_{i \in I}$ be a Parseval fusion frame for \mathcal{H} and let $\{(V_i, v_i)\}_{i \in I}$ be the alternate dual fusion frame of $\{(W_i, w_i)\}_{i \in I}$. Then, for any $\lambda \in [0, 1]$, for all $J \subset I$ and any $f \in \mathcal{H}$, we have*

$$\begin{aligned} &\operatorname{Re} \sum_{i \in J} v_i w_i \langle \pi_{W_i} f, \pi_{V_i} f \rangle + \left\| \sum_{i \in J^c} v_i w_i \pi_{V_i} \pi_{W_i} f \right\|^2 \\ &= \operatorname{Re} \sum_{i \in J^c} v_i w_i \langle \pi_{W_i} f, \pi_{V_i} f \rangle + \left\| \sum_{i \in J} v_i w_i \pi_{V_i} \pi_{W_i} f \right\|^2 \\ &\geq (2\lambda - \lambda^2)\operatorname{Re} \sum_{i \in J} v_i w_i \langle \pi_{W_i} f, \pi_{V_i} f \rangle + (1 - \lambda^2)\operatorname{Re} \sum_{i \in J^c} v_i w_i \langle \pi_{W_i} f, \pi_{V_i} f \rangle. \end{aligned}$$

REMARK 2. If we take $\lambda = \frac{1}{2}$ in Theorem 4, we can obtain the inequality in Theorem 2 of [19].

In [18] the author presented some inequalities for g-frame in C^* -modules. Next, we will generalize the version of fusion frame for Theorem 2.4 in [18].

Now, we consider scalar $\lambda \in [1, 2]$ and give some exciting inequalities for fusion frames in Hilbert spaces. We first give a simple lemma.

LEMMA 3. *Let P, Q be two self-adjoint, positive and bounded linear operators in \mathcal{H} and $P + Q = I_{\mathcal{H}}$, then for any $\lambda \in [1, 2]$ and all $f \in \mathcal{H}$ we have*

$$\|Pf\|^2 \leq \lambda \langle Pf, f \rangle, \quad \|Qf\|^2 \leq \lambda \langle Qf, f \rangle.$$

Proof. Since P and Q are positive operators, we have

$$0 \leq PQ = P(I_{\mathcal{H}} - Q) = P - P^2.$$

Then, for any $\lambda \in [1, 2]$ and any $f \in \mathcal{H}$ we obtain

$$\begin{aligned} \|Pf\|^2 + \lambda \langle Qf, f \rangle &= \langle P^2 f, f \rangle + \lambda \langle Qf, f \rangle \\ &\leq \langle Pf, f \rangle + \lambda \langle (I_{\mathcal{H}} - P)f, f \rangle \\ &= (1 - \lambda) \langle Pf, f \rangle + \lambda \|f\|^2 \leq \lambda \|f\|^2, \end{aligned}$$

it follows that

$$\|Pf\|^2 \leq \lambda \|f\|^2 - \lambda \langle Qf, f \rangle = \lambda \langle Pf, f \rangle.$$

Similarly, we can obtain $\|Qf\|^2 \leq \lambda \langle Qf, f \rangle$. \square

THEOREM 5. *Let $\{(W_i, w_i)\}_{i \in I}$ be a fusion frame for \mathcal{H} with the fusion frame operator S , $\{(S^{-1}W_i, w_i)\}_{i \in I}$ is the dual fusion frame of $\{(W_i, w_i)\}_{i \in I}$. Then for any $\lambda \in [1, 2]$, for all $J \subset I$ and any $f \in \mathcal{H}$, we have*

$$\begin{aligned} 0 &\leq \sum_{i \in J} w_i^2 \|\pi_{W_i} f\|^2 - \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1} S_J f\|^2 \\ &\leq (\lambda - 1) \sum_{i \in J^c} w_i^2 \|\pi_{W_i} f\|^2 + \left(1 - \frac{\lambda}{2}\right)^2 \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2. \end{aligned} \tag{17}$$

Proof. As mentioned in the proof of Theorem 3, we have $S_J - S_J S^{-1} S_J \geq 0$, thus, for all $f \in \mathcal{H}$ we have

$$\sum_{i \in J} w_i^2 \|\pi_{W_i} f\|^2 - \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1} S_J f\|^2 = \langle S_J f, f \rangle - \langle S^{-1} S_J f, S_J f \rangle > 0. \tag{18}$$

On the other hand, by (11) we have

$$\begin{aligned} &\sum_{i \in J} w_i^2 \|\pi_{W_i} f\|^2 - \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1} S_J f\|^2 \\ &= \langle S_J f, f \rangle - \langle S^{-1} S_J f, S_J f \rangle \\ &\leq \langle S_J f, f \rangle - \lambda \langle S_J f, f \rangle + \frac{\lambda^2}{4} \langle S f, f \rangle \\ &= (1 - \lambda) \langle S_J f, f \rangle + \frac{\lambda^2}{4} \langle S f, f \rangle \\ &= (1 - \lambda) \langle (S - S_{J^c}) f, f \rangle + \frac{\lambda^2}{4} \langle S f, f \rangle \\ &= (\lambda - 1) \langle S_{J^c} f, f \rangle + \left(1 - \frac{\lambda}{2}\right)^2 \langle S f, f \rangle \\ &= (\lambda - 1) \sum_{i \in J^c} w_i^2 \|\pi_{W_i} f\|^2 + \left(1 - \frac{\lambda}{2}\right)^2 \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2. \end{aligned} \tag{19}$$

From (18) and (19), the conclusion is holds. \square

THEOREM 6. *Let $\{(W_i, w_i)\}_{i \in I}$ be a fusion frame for \mathcal{H} with the fusion frame operator S , $\{(S^{-1}W_i, w_i)\}_{i \in I}$ is the dual fusion frame of $\{(W_i, w_i)\}_{i \in I}$. Then for any $\lambda \in [1, 2]$, for all $J \subset I$ and any $f \in \mathcal{H}$, we have*

$$\begin{aligned} &\left(2\lambda - \frac{\lambda^2}{2} - 1\right) \sum_{i \in J} w_i^2 \|\pi_{W_i} f\|^2 + \left(1 - \frac{\lambda^2}{2}\right) \sum_{i \in J^c} w_i^2 \|\pi_{W_i} f\|^2 \\ &\leq \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1} S_J f\|^2 + \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1} S_{J^c} f\|^2 \leq \lambda \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2. \end{aligned} \tag{20}$$

Proof. As mentioned in the proof of Theorem 3, from (6) and (11), we have

$$\langle S^{-1}S_{J^c}f, S_{J^c}f \rangle + \langle S_{Jf}, f \rangle \geq \left(\lambda - \frac{\lambda^2}{4}\right) \langle S_{Jf}, f \rangle + \left(1 - \frac{\lambda^2}{4}\right) \langle S_{J^c}f, f \rangle,$$

and then

$$\langle S^{-1}S_{J^c}f, S_{J^c}f \rangle \geq \left(\lambda - \frac{\lambda^2}{4} - 1\right) \langle S_{Jf}, f \rangle + \left(1 - \frac{\lambda^2}{4}\right) \langle S_{J^c}f, f \rangle. \tag{21}$$

By using (11) and (21), we obtain

$$\begin{aligned} & \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1} S_{Jf}\|^2 + \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1} S_{J^c}f\|^2 \\ &= \langle S^{-1} S_{Jf}, S_{Jf} \rangle + \langle S^{-1} S_{J^c}f, S_{J^c}f \rangle \\ &\geq \lambda \langle S_{Jf}, f \rangle - \frac{\lambda^2}{4} \langle Sf, f \rangle + \left(\lambda - \frac{\lambda^2}{4} - 1\right) \langle S_{Jf}, f \rangle + \left(1 - \frac{\lambda^2}{4}\right) \langle S_{J^c}f, f \rangle \\ &= \left(2\lambda - \frac{\lambda^2}{2} - 1\right) \langle S_{Jf}, f \rangle + \left(1 - \frac{\lambda^2}{2}\right) \langle S_{J^c}f, f \rangle \\ &= \left(2\lambda - \frac{\lambda^2}{2} - 1\right) \sum_{i \in J} w_i^2 \|\pi_{W_i} f\|^2 + \left(1 - \frac{\lambda^2}{2}\right) \sum_{i \in J^c} w_i^2 \|\pi_{W_i} f\|^2. \end{aligned}$$

Next, we prove the last inequality of (20). Since $P = S^{-1/2} S_J S^{-1/2}$, $Q = S^{-1/2} S_{J^c} S^{-1/2}$ are positive and self-adjoint operators, by Lemma 3, we have

$$\begin{aligned} & \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1} S_{Jf}\|^2 + \sum_{i \in I} w_i^2 \|\pi_{W_i} S^{-1} S_{J^c}f\|^2 \\ &= \langle S^{-1} S_{Jf}, S_{Jf} \rangle + \langle S^{-1} S_{J^c}f, S_{J^c}f \rangle \\ &= \langle S^{-1/2} S_{Jf}, S^{-1/2} S_{Jf} \rangle + \langle S^{-1/2} S_{J^c}f, S^{-1/2} S_{J^c}f \rangle \\ &= \langle S^{-1/2} S_J S^{-1/2} S^{1/2} f, S^{-1/2} S_J S^{-1/2} S^{1/2} f \rangle \\ &\quad + \langle S^{-1/2} S_{J^c} S^{-1/2} S^{1/2} f, S^{-1/2} S_{J^c} S^{-1/2} S^{1/2} f \rangle \\ &\leq \lambda \langle S^{-1/2} S_J S^{-1/2} S^{1/2} f, S^{1/2} f \rangle + \lambda \langle S^{-1/2} S_{J^c} S^{-1/2} S^{1/2} f, S^{1/2} f \rangle \\ &= \lambda \langle S_{Jf}, f \rangle + \lambda \langle S_{J^c}f, f \rangle = \lambda \langle (S_J + S_{J^c})f, f \rangle \\ &= \lambda \langle Sf, f \rangle = \lambda \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2. \end{aligned}$$

The proof is completed. \square

In case of Parseval fusion frames, we immediately get the following result.

COROLLARY 3. *Let $\{(W_i, w_i)\}_{i \in I}$ be a Parseval fusion frame for \mathcal{H} . Then for*

any $\lambda \in [1, 2]$, for all $J \subset I$ and any $f \in \mathcal{H}$, we have

$$\begin{aligned} 0 &\leq \sum_{i \in J} w_i^2 \|\pi_{W_i} f\|^2 - \sum_{i \in I} w_i^2 \|\pi_{W_i} S_J f\|^2 \\ &\leq (\lambda - 1) \sum_{i \in J^c} w_i^2 \|\pi_{W_i} f\|^2 + \left(1 - \frac{\lambda}{2}\right)^2 \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2, \end{aligned}$$

and

$$\begin{aligned} &\left(2\lambda - \frac{\lambda^2}{2} - 1\right) \sum_{i \in J} w_i^2 \|\pi_{W_i} f\|^2 + \left(1 - \frac{\lambda^2}{2}\right) \sum_{i \in J^c} w_i^2 \|\pi_{W_i} f\|^2 \\ &\leq \sum_{i \in I} w_i^2 \|\pi_{W_i} S_J f\|^2 + \sum_{i \in I} w_i^2 \|\pi_{W_i} S_{J^c} f\|^2 \leq \lambda \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2. \end{aligned}$$

REMARK 3. If we take $\lambda = 1$ in Theorem 5 and Theorem 6, we can obtain the version of fusion frame for Theorem 2.4 in [18].

Acknowledgements. The research is supported by the National Natural Science Foundation of China (11271001).

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(Received October 7, 2016)

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