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# ITERATED HARDY–TYPE INEQUALITIES INVOLVING SUPREMA

AMIRAN GOGATISHVILI AND RZA CH. MUSTAFAYEV

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*Abstract.* In this paper, the boundedness of the composition of the supremal operators defined, for a non-negative measurable functions $f$ on $(0,\infty)$, by

$$S_u g(t) := \operatorname*{ess\,sup}_{0<\tau\leqslant t} u(\tau)g(\tau),\ t\in(0,\infty),$$

and

$$S_u^* g(t) := \operatorname*{ess\,sup}_{t\leqslant\tau<\infty} u(\tau)g(\tau),\ t\in(0,\infty),$$

where $u$ is a fixed continous weight on $(0,\infty)$, with the Hardy and Copson operators between weighted Lebesgue spaces $L^p(v)$ and $L^q(w)$ are characterized.

The complete solution of the restricted inequalities, that is, inequalities

$$\|S_u(f)\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,v,(0,\infty)},$$

and

$$\|S_u(f)\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,v,(0.\infty)},$$

being satisfied on the cones of monotone functions $f$ on $(0,\infty)$, are given.

Moreover, the complete characterization of the inequality

$$\|T_{u,b} f\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,v,(0,\infty)},$$

being satisfied for every non-negative and non-increasing functions $f$ on $(0,\infty)$, is given for $0<p,q<\infty$, as well. Here the operator $T_{u,b}$ is defined for a measurable non-negative function $f$ on $(0,\infty)$ by

$$(T_{u,b}g)(t) := \sup_{t\leqslant\tau<\infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(s)b(s)\,ds, \qquad t\in(0,\infty),$$

where $u,b$ are two weight functions on $(0,\infty)$ such that $u$ is continuous on $(0,\infty)$ and the function $B(t) := \int_0^t b(s)\,ds$ satisfies $0<B(t)<\infty$ for every $t\in(0,\infty)$.

## 1. Introduction

Throughout the paper we assume that $I := (a,b) \subseteq (0,\infty)$. By $\mathfrak{M}(I)$ we denote the set of all measurable functions on $I$. The symbol $\mathfrak{M}^+(I)$ stands for the collection of all $f\in\mathfrak{M}(I)$ which are non-negative on $I$, while $\mathfrak{M}^+(I;\downarrow)$ and $\mathfrak{M}^+(I;\uparrow)$ are used to denote the subset of those functions which are non-increasing and non-decreasing on $I$, respectively. The family of all weight functions (also called just weights) on $I$, that is, locally integrable non-negative functions on $(0,\infty)$, is given by $\mathscr{W}(I)$.

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For $p \in (0,\infty]$ and $w \in \mathfrak{M}^+(I)$, we define the functional $\|\cdot\|_{p,w,I}$ on $\mathfrak{M}(I)$ by

$$\|f\|_{p,w,I} := \begin{cases} \left(\int_I |f(x)|^p w(x)\,dx\right)^{1/p} & \text{if} \quad p < \infty \\ \operatorname{ess\,sup}_I |f(x)| w(x) & \text{if} \quad p = \infty. \end{cases}$$

If, in addition, $w \in \mathscr{W}(I)$, then the weighted Lebesgue space $L^p(w,I)$ is given by

$$L^p(w,I) = \{f \in \mathfrak{M}(I) : \ \|f\|_{p,w,I} < \infty\}$$

and it is equipped with the quasi-norm $\|\cdot\|_{p,w,I}$.

When $w \equiv 1$ on $I$, we write simply $L^p(I)$ and $\|\cdot\|_{p,I}$ instead of $L^p(w,I)$ and $\|\cdot\|_{p,w,I}$, respectively.

Given an operator $T : \mathfrak{M}^+(0,\infty) \to \mathfrak{M}^+(0,\infty)$, for $0 < p < \infty$ and $u \in \mathfrak{M}^+(0,\infty)$, denote by

$$T_u(g) := T(gu), \qquad g \in \mathfrak{M}^+(0,\infty).$$

Suppose that $f$ is a measurable a.e. finite function on $\mathbb{R}^n$. Then its non-increasing rearrangement $f^*$ is given by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \leqslant t\}, \quad t \in (0,\infty),$$

and let $f^{**}$ denotes the Hardy-Littlewood maximal function of $f^*$, i.e.

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(\tau)\,d\tau, \quad t > 0.$$

Quite many familiar function spaces can be defined by using the non-increasing rearrangement of a function. One of the most important classes of such spaces are the so-called classical Lorentz spaces.

Let $p \in (0,\infty)$ and $w \in \mathscr{W}(0,\infty)$. The classical Lorentz spaces $\Lambda^p(w)$ and $\Gamma^p(w)$ consist of all measurable functions $f$ on $\mathbb{R}^n$ for which $\|f\|_{\Lambda^p(w)} := \|f^*\|_{p,w,(0,\infty)} < \infty$ and $\|f\|_{\Gamma^p(w)} := \|f^{**}\|_{p,w,(0,\infty)} < \infty$, respectively. For more information about the Lorentz $\Lambda$ and $\Gamma$ spaces see e.g. [4] and the references therein.

The Hardy and Copson operators are defined by

$$Hg(t) := \int_0^t g(s)\,ds, \qquad g \in \mathfrak{M}^+(0,\infty),$$

and

$$H^*g(t) := \int_t^\infty g(s)\,ds, \qquad g \in \mathfrak{M}^+(0,\infty),$$

respectively. The operators $H$ and $H^*$ play a prominent role in Real and Harmonic Analysis. There are other operators that are also of interest. For example, certain specific problems such as the description of the behaviour of the fractional maximal operator on classical Lorentz spaces [6] or the optimal pairing problem for Sobolev embeddings [18] or various questions arising in the interpolation theory can be handled in an elegant way with the help of the supremal operators

$$Sg(t) := \operatorname*{ess\,sup}_{0<\tau\leqslant t} g(\tau), \qquad g \in \mathfrak{M}^+(0,\infty),$$

and

$$S^* g(t) := \operatorname*{ess\,sup}_{t \leqslant \tau < \infty} g(\tau), \qquad g \in \mathfrak{M}^+(0,\infty).$$

In this paper, we give the complete characterization of restricted inequalities

$$\|S_u(f)\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,v,(0,\infty)},\ f \in \mathfrak{M}^+((0,\infty);\downarrow), \tag{1.1}$$

$$\|S_u(f)\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,v,(0.\infty)},\ f \in \mathfrak{M}^+((0,\infty);\uparrow), \tag{1.2}$$

and

$$\|S_u^*(f)\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,v,(0,\infty)},\ f \in \mathfrak{M}^+((0,\infty);\uparrow), \tag{1.3}$$

$$\|S_u^*(f)\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,v,(0,\infty)},\ f \in \mathfrak{M}^+((0,\infty);\downarrow). \tag{1.4}$$

Note that inequality (1.1) was characterized in [15]. It should be mentioned here that it was done under some additional condition on weight function $u$, when $q < p$ (cf. [15, Theorem 3.4]).

In this paper, in particular, we give criteria for the validity of the iterated Hardy-type inequalities involving suprema

$$\left\| S_u \left( \int_0^x h \right) \right\|_{q,w,(0,\infty)} \leqslant c\,\|h\|_{p,v,(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty), \tag{1.5}$$

and

$$\left\| S_u \left( \int_x^\infty h \right) \right\|_{q,w,(0,\infty)} \leqslant c\,\|h\|_{p,v,(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty), \tag{1.6}$$

where $0 < q < \infty$, $1 \leqslant p < \infty$, $u$, $w$ and $v$ are weight functions on $(0,\infty)$.

It is worth to mention that the characterizations of the ”dual” inequalities

$$\left\| S_u^* \left( \int_x^\infty h \right) \right\|_{q,w,(0,\infty)} \leqslant c\,\|h\|_{p,v,(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty), \tag{1.7}$$

and

$$\left\| S_u^* \left( \int_0^x h \right) \right\|_{q,w,(0,\infty)} \leqslant c\,\|h\|_{p,v,(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty), \tag{1.8}$$

can be easily obtained from the solutions of inequalities (1.5) - (1.6), respectively, by change of variables. Note that inequality (1.8) has been characterized in [15] in the case $0 < q < \infty$, $1 \leqslant p < \infty$.

We pronounce that the characterizations of inequalities (1.5) - (1.8) are important because many inequalities for classical operators can be reduced to them (for illustrations of this important fact, see, for instance, [13]). These inequalities play an important role in the theory of Morrey spaces and other topics (see [1, 2] and [3]).

Investigation of the weighted iterated Hardy-type inequalities started in [12] and [13]. The papers [12] and [13] do not contain the case of supremal operators. In [23, 24, 25, 26] a unified method was created for all possible values of parameters, including the supremal case. In particular, the inequalities (1.5) - (1.8) were characterized with arbitrary measurable weight $u$ in integral forms in [23]. But these characterizations involve auxiliary functions, which make conditions more complicated. It seems to us that our approach is more natural, we are reducing the unknown problems (1.5) - (1.7) to the known one (1.8) and it gives better conditions from a practical point of view. Recently, another characterization of (1.7) was found in [19].

The fractional maximal operator, $M_\gamma$, $\gamma \in (0,n)$, is defined at $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

$$(M_\gamma f)(x) = \sup_{Q \ni x} |Q|^{\gamma/n-1} \int_Q |f(y)|\,dy, \quad x \in \mathbb{R}^n,$$

where the supremum is extended over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes. It was shown in [6, Theorem 1.1] that

$$(M_\gamma f)^*(t) \lesssim \sup_{t \leqslant \tau < \infty} \tau^{\gamma/n-1} \int_0^\tau f^*(y)\,dy \lesssim (M_\gamma \tilde{f})^*(t), \tag{1.9}$$

for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $t \in (0,\infty)$, where $\tilde{f}(x) = f^*(\omega_n |x|^n)$ and $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. Thus, in order to characterize the boundedness of the fractional maximal operator $M_\gamma$ between classical Lorentz spaces $\Lambda^p(v)$ and $\Lambda^q(w)$, it is necessary and sufficient to characterize the validity of the weighted inequality

$$\left( \int_0^\infty \left[ \sup_{t \leqslant \tau < \infty} \tau^{\gamma/n-1} \int_0^\tau \varphi(y)\,dy \right]^q w(t)\,dt \right)^{1/q} \lesssim \left( \int_0^\infty [\varphi(t)]^p v(t)\,dt \right)^{1/p}$$

for all $\varphi \in \mathfrak{M}^+((0,\infty);\downarrow)$. This last estimate can be interpreted as a restricted weighted inequality for the operator $T_\gamma$, defined by

$$(T_\gamma g)(t) = \sup_{t \leqslant \tau < \infty} \tau^{\gamma/n-1} \int_0^\tau g(y)\,dy, \quad g \in \mathfrak{M}^+(0,\infty), \quad t \in (0,\infty). \tag{1.10}$$

Such a characterization was obtained in [6] for the particular case when $1 < p \leqslant q < \infty$ and in [20, Theorem 2.10] for more general operators and for extended range of $p$ and $q$. Full proofs and some further extensions and applications can be found in [9, 10].

The operator $T_\gamma$ is a typical example of what is called a Hardy-operator involving suprema

$$(T_u g)(t) := \sup_{t \leqslant s < \infty} \frac{u(s)}{s} \int_0^s g(y)\,dy,$$

which combines both the operations (integration and taking the supremum).

In the above-mentioned applications, it is required to characterize a restricted weighted inequality for $T_u$. This amounts to find a necessary and sufficient condition on a triple of weights $(u, v, w)$ such that the inequality

$$\left( \int_0^\infty \left( \sup_{t \leqslant s < \infty} u(s) f^{**}(s) \right)^q w(t)\,dt \right)^{1/q} \lesssim \left( \int_0^\infty f^*(t)^p v(t)\,dt \right)^{1/p} \tag{1.11}$$

holds. Particular examples of such inequalities were studied in [6] and, in a more systematic way, in [15]. Inequality (1.11) was investigated in [16] in the case when

$0 < p \leqslant 1$. The approach used in this paper was based on a new type reduction theorem which showed connection between three types of restricted weighted inequalities.

Such operators have been recently encountered in various research projects. They have been found indispensable in the search for optimal pairs of rearrangement-invariant norms for which a Sobolev-type inequality holds (cf. [18]). They constitute a very useful tool for characterization of the associate norm of an operator-induced norm, which naturally appears as an optimal domain norm in a Sobolev embedding (cf. [21, 22]). Supremum operators are also very useful in limiting interpolation theory as can be seen from their appearance for example in [11, 8, 7, 27].

DEFINITION 1.1. Let $u \in \mathscr{W}(0,\infty) \cap C(0,\infty)$, $b \in \mathscr{W}(0,\infty)$ and $B(t) := \int_0^t b(s)\,ds$. Assume that $b$ is such that $0 < B(t) < \infty$ for every $t \in (0,\infty)$. The operator $T_{u,b}$ is defined at $g \in \mathfrak{M}^+(0,\infty)$ by

$$(T_{u,b}g)(t) := \sup_{t \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(s)b(s)\,ds, \qquad t \in (0,\infty).$$

The operator $T_\gamma$, defined in (1.10), is a particular example of operators $T_{u,b}$. These operators are investigated in [15] and [16].

In this paper we give the complete characterization for the inequality

$$\|T_{u,b}f\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,v,(0,\infty)}, \qquad f \in \mathfrak{M}^+((0,\infty);\downarrow) \tag{1.12}$$

for $0 < q < \infty$, $0 < p < \infty$ (see Theorems 5.1 and 5.5).

Inequality (1.12) was characterized in [15, Theorem 3.5] under the additional condition

$$\sup_{0<t<\infty} \frac{u(t)}{B(t)} \int_0^t \frac{b(\tau)}{u(\tau)}\,d\tau < \infty.$$

Note that the case when $0 < p \leqslant 1 < q < \infty$ was not considered in [15]. It is also worth to mention that in the case when $1 < p < \infty$, $0 < q < p < \infty$, $q \neq 1$ [15, Theorem 3.5] contains only discrete condition. In [16] the new reduction theorem was obtained when $0 < p \leqslant 1$, and this technique allows to characterize inequality (1.12) when $b \equiv 1$, and in the case when $0 < q < p \leqslant 1$ this paper contains only discrete condition. Using the results in [23, 24, 25, 26], another characterization of (1.12) was obtained in [30] and [29]. In particular, (1.12) is precisely characterized in [29, Theorem 1 and Proposition 1], but this characterization involves auxiliary functions, which makes conditions more complicated.

The paper is organized as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. Full characterization of inequalities (1.1) - (1.4) and (1.5) - (1.7) are given in Sections 3 and 4. Finally, the solutions of inequality (1.12) are obtained in Section 5.

## 2. Notations and Preliminaries

Throughout the paper, we always denote by $c$ or $C$ a positive constant, which is independent of main parameters but it may vary from line to line. However a constant with subscript such as $c_1$ does not change in different occurrences. By $a \lesssim b$, ($b \gtrsim a$) we mean that $a \leqslant \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that $a$ and $b$ are equivalent. We will denote by $\mathbf{1}$ the function $\mathbf{1}(x) = 1$, $x \in (0,\infty)$. Unless a special remark is made, the differential element $dx$ is omitted when the integrals under consideration are the Lebesgue integrals. Everywhere in the paper, $u$, $v$ and $w$ are weights.

We need the following notations:

$$V(t) := \int_0^t v, \quad V_*(t) := \int_t^\infty v,$$

$$W(t) := \int_0^t w, \; W_*(t) := \int_t^\infty w.$$

CONVENTION 2.1. We adopt the following conventions:

- Throughout the paper we put $0 \cdot \infty = 0$, $\infty/\infty = 0$ and $0/0 = 0$.
- If $p \in [1,+\infty]$, we define $p'$ by $1/p + 1/p' = 1$.
- If $0 < q < p < \infty$, we define $r$ by $1/r = 1/q - 1/p$.
- If $I = (a,b) \subseteq \mathbb{R}$ and $g$ is monotone function on $I$, then by $g(a)$ and $g(b)$ we mean the limits $\lim_{x\to a+} g(x)$ and $\lim_{x\to b-} g(x)$, respectively.

In this paper we consider operators $T : \mathfrak{M}^+(0,\infty) \to \mathfrak{M}^+(0,\infty)$ satisfying the following conditions:

(i) $T(\lambda f) = \lambda T f$ for all $\lambda \geqslant 0$ and $f \in \mathfrak{M}^+(0,\infty)$;

(ii) $Tf(x) \leqslant cTg(x)$ for almost all $x \in \mathbb{R}_+$ if $f(x) \leqslant g(x)$ for almost all $x \in \mathbb{R}_+$, with constant $c > 0$ independent of $f$ and $g$;

(iii) $T(f+g) \leqslant c(Tf + Tg)$ for all $f,\, g \in \mathfrak{M}^+(0,\infty)$, with a constant $c > 0$ independent of $f$ and $g$.

We recall some known results from [17]. Our formulations of the following theorems, which are more convenient for our future applications, are not exactly the same as in the mentioned paper. But by following the proofs of these theorems in [17], it is not difficult to see that such formulations are also true.

THEOREM 2.2. *Let* $0 < \beta \leqslant \infty$ *and* $1 \leqslant s < \infty$, *and let* $T : \mathfrak{M}^+(0,\infty) \to \mathfrak{M}^+(0,\infty)$ *satisfies conditions* (i) - (iii). *Then the inequality*

$$\|Tf\|_{\beta,w,(0,\infty)} \leqslant c\|f\|_{s,v,(0,\infty)}, \quad f \in \mathfrak{M}^+((0,\infty);\downarrow) \tag{2.1}$$

*holds iff both the inequality*

$$\left\| T\left( \int_x^\infty h \right) \right\|_{\beta,w,(0,\infty)} \leqslant c\|h\|_{s,V^s v^{1-s},(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty) \tag{2.2}$$

*and*

$$\|T\mathbf{1}\|_{\beta,w,(0,\infty)} \leqslant c\|\mathbf{1}\|_{s,v,(0,\infty)} \tag{2.3}$$

*hold.*

THEOREM 2.3. *Let* $0 < \beta \leqslant \infty$ *and* $1 \leqslant s < \infty$, *and let* $T : \mathfrak{M}^+(0,\infty) \to \mathfrak{M}^+(0,\infty)$ *satisfies conditions* (i) - (iii). *Then inequality* (2.1) *holds iff the inequality*

$$\left\| T\left(\frac{1}{V^2(x)}\int_0^x hV\right)\right\|_{\beta,w,(0,\infty)} \leqslant c\|h\|_{s,v^{1-s},(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty) \tag{2.4}$$

*holds.*

THEOREM 2.4. *Let* $0 < \beta \leqslant \infty$ *and* $1 \leqslant s < \infty$, *and let* $T : \mathfrak{M}^+(0,\infty) \to \mathfrak{M}^+(0,\infty)$ *satisfies conditions* (i) - (iii). *Then the inequality*

$$\|Tf\|_{\beta,w,(0,\infty)} \leqslant c\|f\|_{s,v,(0,\infty)}, \quad f \in \mathfrak{M}^+((0,\infty);\uparrow) \tag{2.5}$$

*holds iff both the inequality*

$$\left\| T\left(\int_0^x h\right)\right\|_{\beta,w,(0,\infty)} \leqslant c\|h\|_{s,V_*^s v^{1-s},(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty) \tag{2.6}$$

*and* (2.3) *hold.*

THEOREM 2.5. *Let* $0 < \beta \leqslant \infty$ *and* $1 \leqslant s < \infty$, *and let* $T : \mathfrak{M}^+(0,\infty) \to \mathfrak{M}^+(0,\infty)$ *satisfies conditions* (i) - (iii). *Then inequality* (2.5) *holds iff the inequality*

$$\left\| T\left(\frac{1}{V_*^2(x)}\int_x^\infty hV_*\right)\right\|_{\beta,w,(0,\infty)} \leqslant c\|h\|_{s,v^{1-s},(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty) \tag{2.7}$$

*holds.*

REMARK 2.6. If $\int_0^\infty v = \infty$, then condition (2.3) automatically holds, and in this case this condition disappears in the statements of Theorems 2.2 and 2.4.

Note that Theorems 2.2, 2.3, 2.4 and 2.5 were proved in [17] under weaker assumption than (iii): $T(f + \lambda \mathbf{1}) \leqslant c(Tf + \lambda T\mathbf{1})$ for all $f \in \mathfrak{M}^+(0,\infty)$ and $\lambda \geqslant 0$, with a constant $c > 0$ independent of $f$ and $\lambda$.

Now we recall some reduction theorems for positive monotone operators [14].

THEOREM 2.7. [14, Theorem 3.1] *Let* $0 < q \leqslant \infty$, $1 < p < \infty$, *and let* $T : \mathfrak{M}^+(0,\infty) \to \mathfrak{M}^+(0,\infty)$ *satisfies conditions* (i) - (iii). *Assume that* $u, w \in \mathscr{W}(0,\infty)$ *and* $v \in \mathscr{W}(0,\infty)$ *is such that*

$$\int_0^x v^{1-p'}(t)\,dt < \infty \qquad \textit{for all} \qquad x > 0. \tag{2.8}$$

*Then inequality*

$$\left\| T\left(\int_0^x h\right)\right\|_{q,w,(0,\infty)} \leqslant c\,\|h\|_{p,v,(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty), \tag{2.9}$$

*holds iff*

$$\|T_{\Phi^2} f\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,\phi,(0,\infty)}, \quad f \in \mathfrak{M}^+((0,\infty);\downarrow), \tag{2.10}$$

*holds, where*

$$\phi(x) \equiv \phi\left[v;p\right](x) := \left(\int_0^x v^{1-p'}(t)\,dt\right)^{-p'/(p'+1)} v^{1-p'}(x)$$

*and*

$$\Phi(x) \equiv \Phi\left[v;p\right](x) = \int_0^x \phi(t)\,dt = \left(\int_0^x v^{1-p'}(t)\,dt\right)^{1/(p'+1)}.$$

THEOREM 2.8. [14, Theorem 3.11] *Let* $0 < q \leqslant \infty$, *and let* $T : \mathfrak{M}^+(0,\infty) \to \mathfrak{M}^+(0,\infty)$ *satisfies conditions* (i) - (iii). *Assume that* $u, w \in \mathscr{W}(0,\infty)$ *and* $v \in \mathscr{W}(0,\infty)$ *are such that* $V(x) < \infty$ *for all* $x > 0$. *Then inequality*

$$\left\|T\left(\int_0^x h\right)\right\|_{q,w,(0,\infty)} \leqslant c\,\|h\|_{1,V^{-1},(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty), \tag{2.11}$$

*holds iff*

$$\|T_{V^2} f\|_{q,w,(0,\infty)} \leqslant c\|f\|_{1,v,(0,\infty)}, \quad f \in \mathfrak{M}^+((0,\infty);\downarrow). \tag{2.12}$$

## 3. Supremal operators on the cones of monotone functions

In this section, we give complete characterization of inequalities (1.1) - (1.4). To state the next statements we need the following notations:

$$\overline{u}(t) := \sup_{0<\tau\leqslant t} u(\tau), \qquad \underline{u}(t) := \sup_{t\leqslant\tau<\infty} u(\tau), \qquad (t > 0).$$

For a given weight $v$, $0 \leqslant a < b \leqslant \infty$ and $1 \leqslant p < \infty$, we denote

$$\sigma_p(a,b) = \begin{cases} \left(\int\limits_a^b [v(t)]^{1-p'}\,dt\right)^{1/p'}, & \text{when } 1 < p < \infty, \\ \operatorname*{ess\,sup}\limits_{a<t<b}\,[v(t)]^{-1}, & \text{when } p = 1. \end{cases}$$

Recall the following theorem.

THEOREM 3.1. [15, Theorems 4.1 and 4.4] *Let* $1 \leqslant p < \infty$, $0 < q < \infty$ *and let* $u \in \mathscr{W}(0,\infty) \cap C(0,\infty)$. *Assume that* $v, w \in \mathscr{W}(0,\infty)$ *are such that*

$$0 < V(x) < \infty \qquad \text{and} \qquad 0 < W(x) < \infty \qquad \text{for all} \qquad x > 0.$$

*Then inequality* (1.8) *with the best constant* $c$ *holds if and only if the following holds:*

(i) $p \leqslant q$ *and* $\mathscr{A}_1 < \infty$, *where*

$$\mathscr{A}_1 := \sup_{x>0}\left([\underline{u}]^q(x)W(x) + \int_x^\infty [\underline{u}]^q(t)w(t)\,dt\right)^{1/q} \sigma_p(0,x),$$

*and in this case* $c \approx \mathscr{A}_1$;

(ii) $q < p$ *and* $\mathscr{B}_{11} + \mathscr{B}_{12} < \infty$, *where*

$$\mathscr{B}_{11} := \left( \int_0^\infty \left( \int_x^\infty [\underline{u}]^q(t) w(t)\,dt \right)^{r/p} [\underline{u}]^q(x) \left[ \sigma_p(0,x) \right]^r w(x)\,dx \right)^{1/r},$$

$$\mathscr{B}_{12} := \left( \int_0^\infty W^{r/p}(x) \left[ \sup_{x \leqslant \tau < \infty} \underline{u}(\tau) \sigma_p(0,\tau) \right]^r w(x)\,dx \right)^{1/r},$$

*and in this case* $c \approx \mathscr{B}_{11} + \mathscr{B}_{12}$.

Using change of variables $x = 1/t$, we can easily obtain the following statement.

THEOREM 3.2. *Let* $1 \leqslant p < \infty$, $0 < q < \infty$ *and let* $u \in \mathscr{W}(0,\infty) \cap C(0,\infty)$. *Assume that* $v, w \in \mathscr{W}(0,\infty)$ *are such that*

$$0 < V_*(x) < \infty \qquad \text{and} \qquad 0 < W_*(x) < \infty \qquad \text{for all} \qquad x > 0.$$

*Then inequality* (1.6) *with the best constant* $c$ *holds if and only if the following holds:*

(i) $p \leqslant q$ *and* $\mathscr{A}_2 < \infty$, *where*

$$\mathscr{A}_2 := \sup_{x>0} \left( [\overline{u}]^q(x) W_*(x) + \int_0^x [\overline{u}]^q(t) w(t)\,dt \right)^{1/q} \sigma_p(x,\infty),$$

*and in this case* $c \approx \mathscr{A}_2$;

(ii) $q < p$ *and* $\mathscr{B}_{21} + \mathscr{B}_{22} < \infty$, *where*

$$\mathscr{B}_{21} := \left( \int_0^\infty \left( \int_0^x [\overline{u}]^q(t) w(t)\,dt \right)^{r/p} [\overline{u}]^q(x) \left[ \sigma_p(x,\infty) \right]^r w(x)\,dx \right)^{1/r},$$

$$\mathscr{B}_{22} := \left( \int_0^\infty W_*^{r/p}(x) \left[ \sup_{0 < \tau \leqslant x} \overline{u}(\tau) \sigma_p(\tau,\infty) \right]^r w(x)\,dx \right)^{1/r},$$

*and in this case* $c \approx \mathscr{B}_{21} + \mathscr{B}_{22}$.

*Proof.* Obviously, inequality (1.6) is satisfied with the best constant $c$ iff

$$\left\| S^*_{\tilde{u}} \left( \int_0^x h \right) \right\|_{q,\tilde{w},(0,\infty)} \leqslant c\,\|h\|_{p,\tilde{v},(0,\infty)},\ h \in \mathfrak{M}^+ \tag{3.1}$$

holds, where

$$\tilde{u}(t) = u\left(\frac{1}{t}\right),\ \tilde{w}(t) = w\left(\frac{1}{t}\right)\frac{1}{t^2},\ \tilde{v}(t) = v\left(\frac{1}{t}\right)\left(\frac{1}{t^2}\right)^{1-p},\ t > 0.$$

Using Theorem 3.1, and then applying substitution of variables mentioned above three times, we get the statement.

THEOREM 3.3. *Let* $0 < p, q < \infty$ *and let* $u \in \mathscr{W}(0,\infty) \cap C(0,\infty)$. *Assume that* $v, w \in \mathscr{W}(0,\infty)$ *are such that*

$$0 < V_*(x) < \infty \qquad \text{and} \qquad 0 < W_*(x) < \infty \qquad \text{for all} \qquad x > 0.$$

*Then inequality* (1.1) *with the best constant* $c$ *holds if and only if the following holds:*

(i) $p \leqslant q$ *and* $\mathfrak{A}_1 + \|S_u(\mathbf{1})\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{s,v,(0,\infty)} < \infty$, *where*

$$\mathfrak{A}_1 := \sup_{x>0} \left( [\overline{u}]^q(x) W_*(x) + \int_0^x [\overline{u}]^q(t) w(t)\, dt \right)^{1/q} V^{-1/p}(x),$$

*and in this case* $c \approx \mathfrak{A}_1 + \|S_u(\mathbf{1})\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{s,v,(0,\infty)}$;

(ii) $q < p$ *and* $\mathfrak{B}_{11} + \mathfrak{B}_{12} + \|S_u(\mathbf{1})\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{s,v,(0,\infty)} < \infty$, *where*

$$\mathfrak{B}_{11} := \left( \int_0^\infty \left( \int_0^x [\overline{u}]^q(t) w(t)\, dt \right)^{r/p} [\overline{u}]^q(x) V^{-r/p}(x) w(x)\, dx \right)^{1/r},$$

$$\mathfrak{B}_{12} := \left( \int_0^\infty W_*^{r/p}(x) \left[ \sup_{0<\tau\leqslant x} \overline{u}(\tau) V^{-1/p}(\tau) \right]^r w(x)\, dx \right)^{1/r},$$

*and in this case* $c \approx \mathfrak{B}_{11} + \mathfrak{B}_{12} + \|S_u(\mathbf{1})\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{s,v,(0,\infty)}$.

*Proof.* It is easy to see that inequality (1.1) holds iff

$$\|S_{u^p}(f)\|_{q/p,w,(0,\infty)} \leqslant c^p \|f\|_{1,v,(0,\infty)},\ f \in \mathfrak{M}^+((0,\infty);\downarrow) \tag{3.2}$$

holds. By Theorem 2.2, (3.2) holds iff both

$$\left\| S_{u^p}\left( \int_x^\infty h \right) \right\|_{q/p,w,(0,\infty)} \leqslant c^p \|h\|_{1,V,(0,\infty)},\ h \in \mathfrak{M}^+, \tag{3.3}$$

and

$$\|S_u(\mathbf{1})\|_{q,w,(0,\infty)} \leqslant c \|\mathbf{1}\|_{s,v,(0,\infty)} \tag{3.4}$$

hold. In order to complete the proof, it remains to apply Theorem 3.2.

Using change of variables $x = 1/t$, we can easily obtain the following "dual" statement.

THEOREM 3.4. *Let* $0 < p, q < \infty$ *and let* $u \in \mathscr{W}(0,\infty) \cap C(0,\infty)$. *Assume that* $v, w \in \mathscr{W}(0,\infty)$ *are such that*

$$0 < V(x) < \infty \qquad \text{and} \qquad 0 < W(x) < \infty \qquad \text{for all} \qquad x > 0.$$

*Then inequality* (1.3) *with the best constant* $c$ *holds if and only if the following holds:*

(i) $p \leqslant q$ *and* $\mathfrak{A}_2 + \|S_u^*(\mathbf{1})\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{s,v,(0,\infty)} < \infty$, *where*

$$\mathfrak{A}_2 := \sup_{x>0} \left( [\underline{u}]^q(x) W(x) + \int_x^\infty [\underline{u}]^q(t) w(t)\, dt \right)^{1/q} V_*^{-1/p}(x),$$

*and in this case* $c \approx \mathfrak{A}_2 + \|S_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,v,(0,\infty)}$;

(ii) $q < p$ *and* $\mathfrak{B}_{21} + \mathfrak{B}_{22} + \|S_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,v,(0,\infty)} < \infty$, *where*

$$\mathfrak{B}_{21} := \left( \int_0^\infty \left( \int_x^\infty [\underline{u}]^q(t) w(t)\,dt \right)^{r/p} [\underline{u}]^q(x) V_*^{-r/p}(x) w(x)\,dx \right)^{1/r},$$

$$\mathfrak{B}_{22} := \left( \int_0^\infty W^{r/p}(x) \left[ \sup_{x \leqslant \tau < \infty} \underline{u}(\tau) V_*^{-1/p}(\tau) \right]^r w(x)\,dx \right)^{1/r},$$

*and in this case* $c \approx \mathfrak{B}_{21} + \mathfrak{B}_{22} + \|S_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,v,(0,\infty)}$.

*Proof.* It is easy to see that inequality (1.3) is satisfied with the best constant $c$ iff

$$\|S_{\tilde{u}} f\|_{q,\tilde{w},(0,\infty)} \leqslant c\,\|f\|_{p,\tilde{v},(0,\infty)},\ f \in \mathfrak{M}^+((0,\infty);\downarrow) \tag{3.5}$$

holds, where

$$\tilde{u}(t) = u\left(\frac{1}{t}\right),\ \tilde{w}(t) = w\left(\frac{1}{t}\right)\frac{1}{t^2},\ \tilde{v}(t) = v\left(\frac{1}{t}\right)\frac{1}{t^2},\ t > 0.$$

Using Theorem 3.3, and then applying substitution of variables mentioned above three times, we get the statement.

THEOREM 3.5. *Let* $0 < p, q < \infty$ *and let* $u \in \mathscr{W}(0,\infty) \cap C(0,\infty)$. *Assume that* $v, w \in \mathscr{W}(0,\infty)$ *are such that*

$$0 < V_*(x) < \infty \qquad \text{and} \qquad 0 < W_*(x) < \infty \qquad \text{for all} \qquad x > 0.$$

*Then inequality* (1.2) *with the best constant* $c$ *holds if and only if the following holds:*

(i) $p \leqslant q$ *and* $\mathfrak{A}_3 + \|S_u(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0.\infty)} < \infty$, *where*

$$\mathfrak{A}_3 := \sup_{x>0} \left( \left[ \sup_{0<\tau\leqslant x} \frac{u(\tau)^p}{V_*(\tau)^2} \right]^{q/p} W_*(x) + \int_0^x \left[ \sup_{0<\tau\leqslant t} \frac{u(\tau)^p}{V_*(\tau)^2} \right]^{q/p} w(t)\,dt \right)^{1/q} [V_*]^{1/p}(x),$$

*and in this case* $c \approx \mathfrak{A}_3 + \|S_u(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0.\infty)}$;

(ii) $0 < q < p < \infty$ *and* $\mathfrak{B}_{31} + \mathfrak{B}_{32} + \|S_u(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0.\infty)} < \infty$, *where*

$$\mathfrak{B}_{31} := \left( \int_0^\infty \left( \int_0^x \left[ \sup_{0<\tau\leqslant t} \frac{u(\tau)^p}{V_*(\tau)^2} \right]^{q/p} w(t)dt \right)^{r/p} [V_*]^{-r/p}(x) \left[ \sup_{0<\tau\leqslant x} \frac{u(\tau)^p}{V_*(\tau)^2} \right]^{q/p} w(x)dx \right)^{1/r},$$

$$\mathfrak{B}_{32} := \left( \int_0^\infty W_*^{r/p}(x) \left[ \sup_{0<\tau\leqslant x} \left[ \sup_{0<\tau\leqslant t} \frac{u(\tau)^p}{V_*(\tau)^2} \right] V_*(\tau) \right]^{r/p} w(x)\,dx \right)^{1/r}.$$

*and in this case* $c \approx \mathfrak{B}_{31} + \mathfrak{B}_{32} + \|S_u(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0.\infty)}$.

*Proof.* It is easy to see that inequality (1.2) holds iff

$$\|S_{u^p}(f)\|_{q/p,w,(0,\infty)} \leqslant c^p \|f\|_{1,v,(0,\infty)}, \quad f \in \mathfrak{M}^+((0,\infty);\uparrow) \tag{3.6}$$

holds. By Theorem 2.5 with $T = S_{u^p}$ we see that inequality (3.6) is satisfied with the best constant $c$ iff both

$$\left\| S_{u^p/V_*^2}\left(\int_x^\infty h\right)\right\|_{q/p,w,(0,\infty)} \leqslant c\|h\|_{1,1/V_*,(0,\infty)},\ h \in \mathfrak{M}^+(0,\infty),$$

and

$$\|S_u(\mathbf{1})\|_{q,w,(0,\infty)} \leqslant c\|\mathbf{1}\|_{p,v,(0.\infty)}$$

hold. It remains to apply Theorem 3.2.

The following "dual" statement holds true.

THEOREM 3.6. *Let $0 < p,q < \infty$ and let $u \in \mathscr{W}(0,\infty)\cap C(0,\infty)$. Assume that $v, w \in \mathscr{W}(0,\infty)$ are such that*

$$0 < V(x) < \infty \qquad \textit{and} \qquad 0 < W(x) < \infty \qquad \textit{for all} \qquad x > 0.$$

*Then inequality (1.4) with the best constant $c$ holds if and only if the following holds:*

(i) *$p \leqslant q$ and $\mathfrak{A}_4 + \|S_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0.\infty)} < \infty$, where*

$$\mathfrak{A}_4 := \sup_{x>0}\left(\left[\sup_{x\leqslant\tau<\infty}\frac{u(\tau)^p}{V(\tau)^2}\right]^{q/p} W(x) + \int_x^\infty \left[\sup_{t\leqslant\tau<\infty}\frac{u(\tau)^p}{V(\tau)^2}\right]^{q/p} w(t)\,dt\right)^{1/q} V^{1/p}(x),$$

*and in this case $c \approx \mathfrak{A}_4 + \|S_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0.\infty)}$;*

(ii) *$q < p$ and $\mathfrak{B}_{41} + \mathfrak{B}_{42} + \|S_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0.\infty)} < \infty$, where*

$$\mathfrak{B}_{41} := \left(\int_0^\infty\left(\int_x^\infty\left[\sup_{t\leqslant\tau<\infty}\frac{u(\tau)^p}{V(\tau)^2}\right]^{q/p} w(t)dt\right)^{r/p} V^{-r/p}(x)\left[\sup_{x\leqslant\tau<\infty}\frac{u(\tau)^p}{V(\tau)^2}\right]^{q/p} w(x)dx\right)^{1/r},$$

$$\mathfrak{B}_{42} := \left(\int_0^\infty W^{r/p}(x)\left[\sup_{x\leqslant\tau<\infty}\left[\sup_{t\leqslant\tau<\infty}\frac{u(\tau)^p}{V(\tau)^2}\right]V(\tau)\right]^{r/p} w(x)\,dx\right)^{1/r},$$

*and in this case $c \approx \mathfrak{B}_{41} + \mathfrak{B}_{42} + \|S_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0.\infty)}$.*

*Proof.* Obviously, (1.4) is satisfied with the best constant $c$ iff

$$\|S_{\tilde{u}}f\|_{q,\tilde{w},(0,\infty)} \leqslant c\,\|f\|_{p,\tilde{v},(0,\infty)},\ f \in \mathfrak{M}^+((0,\infty);\uparrow)$$

holds, where

$$\tilde{u}(t) = u\left(\frac{1}{t}\right),\ \tilde{w}(t) = w\left(\frac{1}{t}\right)\frac{1}{t^2},\ \tilde{v}(t) = v\left(\frac{1}{t}\right)\frac{1}{t^2},\ t > 0.$$

Using Theorem 3.5, and then applying substitution of variables mentioned above three times, we get the statement.

## 4. Iterated inequalities with supremal operators

In this section we characterize inequalities (1.5) and (1.7).

The following theorem is true.

THEOREM 4.1. *Let* $1 < p < \infty$, $0 < q < \infty$ *and let* $u \in \mathscr{W}(0,\infty) \cap C(0,\infty)$. *Assume that* $v, w \in \mathscr{W}(0,\infty)$ *are such that*

$$0 < \int_0^x v^{1-p'}(t)\,dt < \infty \qquad \text{and} \qquad 0 < W_*(x) < \infty \qquad \text{for all} \qquad x > 0.$$

*Recall that*

$$\phi\big[v;p\big](x) = \left(\int_0^x v^{1-p'}(t)\,dt\right)^{-p'/(p'+1)} v^{1-p'}(x),\ x > 0,$$

$$\Phi\big[v;p\big](x) = \left(\int_0^x v^{1-p'}(t)\,dt\right)^{1/(p'+1)},\ x > 0.$$

*Denote by*

$$\Phi_1(x) := \sup_{0<\tau\leqslant x} u(\tau)\Phi^2[v;p](\tau),\ x > 0.$$

*Then inequality* (1.5) *with the best constant* $c$ *holds if and only if the following holds:*

(i) $p \leqslant q$ *and* $\mathscr{A}_3 + \|S_{u\Phi^2[v;p]}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,\phi[v;p],(0,\infty)} < \infty$, *where*

$$\mathscr{A}_3 := \sup_{x>0}\left([\Phi_1]^q(x)W_*(x) + \int_0^x [\Phi_1]^q(t)w(t)\,dt\right)^{1/q} \Phi[v;p]^{-1/p}(x),$$

*and in this case* $c \approx \mathscr{A}_3 + \|S_{u\Phi^2[v;p]}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,\phi[v;p],(0,\infty)}$;

(ii) $q < p$ *and* $c \approx \mathscr{B}_{31} + \mathscr{B}_{32} + \|S_{u\Phi^2[v;p]}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,\phi[v;p],(0,\infty)} < \infty$, *where*

$$\mathscr{B}_{31} := \left(\int_0^\infty \left(\int_0^x [\Phi_1]^q(t)w(t)\,dt\right)^{r/p} [\Phi_1]^q(x)\Phi[v;p]^{-r/p}(x)w(x)\,dx\right)^{1/r},$$

$$\mathscr{B}_{32} := \left(\int_0^\infty W_*^{r/p}(x)\left[\sup_{0<\tau\leqslant x} \Phi_1(\tau)\Phi[v;p]^{-1/p}(\tau)\right]^r w(x)\,dx\right)^{1/r},$$

*and in this case* $c \approx \mathscr{B}_{31} + \mathscr{B}_{32} + \|S_{u\Phi^2[v;p]}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,\phi[v;p],(0,\infty)}$.

*Proof.* By Theorem 2.7 with $T = S_u$ it is clear that inequality (1.5) with the best constant $c$ holds if and only if the inequality

$$\big\|S_{u\Phi^2[v;p]}(f)\big\|_{q,w,(0,\infty)} \leqslant C\,\|f\|_{p,\phi[v;p],(0,\infty)},\ f \in \mathfrak{M}^+((0,\infty);\downarrow) \tag{4.1}$$

holds. Moreover, $c \approx C$. Now the statement follows by Theorem 3.3.

THEOREM 4.2. *Let $0<q<\infty$ and let $u \in \mathscr{W}(0,\infty)\cap C(0,\infty)$. Assume that $v, w \in \mathscr{W}(0,\infty)$ are such that $0<V(x)<\infty$ and $0<W_*(x)<\infty$ for all $x>0$. Denote by*

$$V_1(x) := \sup_{0<\tau\leqslant x} u(\tau)V^2(\tau),\ x>0.$$

*Then inequality*

$$\left\| S_u\left(\int_0^x h\right)\right\|_{q,w,(0,\infty)} \leqslant c\,\|h\|_{1,V^{-1},(0,\infty)}, \tag{4.2}$$

*with the best constant $c$ holds if and only if the following holds:*

(i) *$p\leqslant q$ and $\mathscr{A}_3^1 + \|S_{uV^2}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)}<\infty$, where*

$$\mathscr{A}_3^1 := \sup_{x>0}\left([V_1]^q(x)\int_x^\infty w(t)\,dt + \int_0^x [V_1]^q(t)w(t)\,dt\right)^{1/q} V^{-1/p}(x),$$

*and in this case $c\approx \mathscr{A}_3^1 + \|S_{uV^2}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)}$;*

(ii) *$q<p$ and $\mathscr{B}_{31}^1 + \mathscr{B}_{31}^1 + \|S_{uV^2}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)}<\infty$, where*

$$\mathscr{B}_{31}^1 := \left(\int_0^\infty \left(\int_0^x [V_1]^q(t)w(t)\,dt\right)^{r/p} [V_1]^q(x)V^{-r/p}(x)w(x)\,dx\right)^{1/r},$$

$$\mathscr{B}_{32}^1 := \left(\int_0^\infty \left(\int_x^\infty w(t)\,dt\right)^{r/p} \left[\sup_{0<\tau\leqslant x}[V_1](\tau)V^{-1/p}(\tau)\right]^r w(x)\,dx\right)^{1/r},$$

*and in this case $c\approx \mathscr{B}_{31}^1 + \mathscr{B}_{31}^1 + \|S_{uV^2}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)}$.*

*Proof.* Applying Theorem 2.8 to the operator $S_u$ we see that inequality (4.2) with the best constant $c$ holds iff the inequality

$$\left\|S_{uV^2}(f)\right\|_{q,w,(0,\infty)} \leqslant C\,\|f\|_{1,v,(0,\infty)},\ f\in\mathfrak{M}^+((0,\infty);\downarrow) \tag{4.3}$$

holds. Moreover, $c\approx C$. Now the statement follows by Theorem 3.3.

The following ”dual” statements also hold true.

THEOREM 4.3. *Let $1<p<\infty$, $0<q<\infty$ and let $u\in\mathscr{W}(0,\infty)\cap C(0,\infty)$. Assume that $v, w\in\mathscr{W}(0,\infty)$ are such that*

$$0<\int_x^\infty v^{1-p'}(t)\,dt<\infty \qquad \text{and} \qquad 0<W(x)<\infty \qquad \text{for all} \qquad x>0.$$

*Denote by*

$$\psi[v;s](x) := \left(\int_x^\infty v^{1-s'}(t)\,dt\right)^{-\frac{s'}{s'+1}} v^{1-s'}(x),\ x>0,$$

$$\Psi[v;s](x) := \left(\int_x^\infty v^{1-s'}(t)\,dt\right)^{\frac{1}{s'+1}},\ x>0,$$

$$\Psi_1(x) := \sup_{x \leqslant \tau < \infty} u(\tau)\Psi^2[v;p](\tau),\ x > 0.$$

*Then inequality* (1.7) *with the best constant* $c$ *holds if and only if the following holds:*

(i) $p \leqslant q$ *and* $\mathscr{A}_4 + \|S_{u\Psi^2[v;p]}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,\psi[v;p],(0,\infty)} < \infty$, *where*

$$\mathscr{A}_4 := \sup_{x>0}\left([\Psi_1]^q(x)W(x) + \int_x^\infty [\Psi_1]^q(t)w(t)\,dt\right)^{1/q}\Psi[v;p]^{-1/p}(x),$$

*and in this case* $c \approx \mathscr{A}_4 + \|S_{u\Psi^2[v;p]}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,\psi[v;p],(0,\infty)}$;

(ii) $q < p$ *and* $c \approx \mathscr{B}_{41} + \mathscr{B}_{42} + \|S_{u\Psi^2[v;p]}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,\psi[v;p],(0,\infty)} < \infty$, *where*

$$\mathscr{B}_{41} := \left(\int_0^\infty \left(\int_x^\infty [\Psi_1]^q(t)w(t)\,dt\right)^{r/p}[\Psi_1]^q(x)\Psi[v;p]^{-r/p}(x)w(x)\,dx\right)^{1/r},$$

$$\mathscr{B}_{42} := \left(\int_0^\infty W^{r/p}(x)\left[\sup_{x\leqslant\tau<\infty}\Psi_1(\tau)\Psi[v;p]^{-1/p}(\tau)\right]^r w(x)\,dx\right)^{1/r},$$

*and in this case* $c \approx \mathscr{B}_{41} + \mathscr{B}_{42} + \|S_{u\Psi^2[v;p]}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,\psi[v;p],(0,\infty)}$.

*Proof.* Obviously, (1.7) is satisfied with the best constant $c$ if and only if

$$\left\|S_{\tilde{u}}\left(\int_0^x h\right)\right\|_{q,\tilde{w},(0,\infty)} \leqslant c\,\|h\|_{p,\tilde{v},(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty)$$

holds, where

$$\tilde{u}(t) = u\left(\frac{1}{t}\right),\ \tilde{w}(t) = w\left(\frac{1}{t}\right)\frac{1}{t^2},\ \tilde{v}(t) = v\left(\frac{1}{t}\right)\frac{1}{t^2},\ t > 0.$$

Using Theorem 4.1, and then applying substitution of variables mentioned above three times, we get the statement.

THEOREM 4.4. *Let* $0 < q < \infty$ *and let* $u \in \mathscr{W}(0,\infty)\cap C(0,\infty)$. *Assume that* $v, w \in \mathscr{W}(0,\infty)$ *are such that* $0 < V_*(x) < \infty$ *and* $0 < W_*(x) < \infty$ *for all* $x > 0$. *Denote by*

$$V_1^*(x) := \sup_{x\leqslant\tau<\infty} u(\tau)V_*^2(\tau),\ x > 0.$$

*Then inequality*

$$\left\|S_u^*\left(\int_x^\infty h\right)\right\|_{q,w,(0,\infty)} \leqslant c\,\|h\|_{1,V_*^{-1},(0,\infty)}, \tag{4.4}$$

*with the best constant* $c$ *holds if and only if the following holds:*

(i) $p \leqslant q$ *and* $\mathscr{A}_4^1 + \|S_{uV_*^2}^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)} < \infty$, *where*

$$\mathscr{A}_4^1 := \sup_{x>0}\left([V_1^*]^q(x)W_*(x) + \int_0^x [V_1^*]^q(t)w(t)\,dt\right)^{1/q} V^{-1/p}(x),$$

*and in this case* $c \approx \mathscr{A}_4^1 + \big\|S^*_{uV_*^2}(\mathbf{1})\big\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{p,v,(0,\infty)}$;

(ii) $q < p$ *and* $\mathscr{B}_{41}^1 + \mathscr{B}_{42}^1 + \big\|S^*_{uV_*^2}(\mathbf{1})\big\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{p,v,(0,\infty)} < \infty$, *where*

$$\mathscr{B}_{41}^1 := \left(\int_0^\infty \left(\int_x^\infty [V_1^*]^q(t)w(t)\,dt\right)^{r/p} [V_1^*]^q(x)V_*^{-r/p}(x)w(x)\,dx\right)^{1/r},$$

$$\mathscr{B}_{42}^1 := \left(\int_0^\infty W_*^{r/p}(x)\left[\sup_{x\leqslant\tau<\infty}[V_1^*](\tau)V^{-1/p}(\tau)\right]^r w(x)\,dx\right)^{1/r},$$

*and in this case* $c \approx \mathscr{B}_{41}^1 + \mathscr{B}_{42}^1 + \big\|S^*_{uV_*^2}(\mathbf{1})\big\|_{q,w,(0,\infty)} / \|\mathbf{1}\|_{p,v,(0,\infty)}$.

*Proof.* By change of variables $x = 1/t$, it is easy to see that inequality (4.4) holds iff

$$\left\|S_{\tilde{u}}\left(\int_0^x h\right)\right\|_{q,\tilde{w},(0,\infty)} \leqslant c\,\|h\|_{1,\tilde{V}^{-1},(0,\infty)},\ h \in \mathfrak{M}^+ \tag{4.5}$$

holds, where

$$\tilde{u}(t) = u\left(\frac{1}{t}\right),\ \tilde{w}(t) = w\left(\frac{1}{t}\right)\frac{1}{t^2},\ \tilde{V}(t) = \int_0^t v\left(\frac{1}{y}\right)\frac{1}{y^2}\,dy,\ t > 0.$$

Applying Theorem 4.2, and then using substitution of variables mentioned above three times, we get the statement.

## 5. Hardy-operator involving suprema - $T_{u,b}$

In this section we give complete characterization of inequality (1.12).

### 5.1. The case $1 \leqslant p < \infty$

The following theorem is true.

THEOREM 5.1. *Let* $0 < q < \infty$, $1 \leqslant p < \infty$ *and let* $u \in \mathscr{W}(0,\infty) \cap C(0,\infty)$. *Assume that* $b, v, w \in \mathscr{W}(0,\infty)$ *are such that*

$$0 < B(t) < \infty,\ 0 < V(x) < \infty \text{ and } 0 < W(x) < \infty \text{ for all } x > 0.$$

*Then inequality* (1.12) *with the best constant* $c$ *holds if and only if the following holds:*

(i) $1 < p \leqslant q$ *and* $A_1 + A_2 < \infty$, *where*

$$A_1 := \sup_{x>0}\left(\left[\sup_{x\leqslant\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q W(x) + \int_x^\infty \left[\sup_{t\leqslant\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q w(t)\,dt\right)^{1/q} \times \left(\int_0^x \left(\frac{B(y)}{V(y)}\right)^{p'} v(y)\,dy\right)^{1/p'},$$

$$A_2 := \sup_{x>0}\left(\left[\sup_{x\leqslant\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q W(x) + \int_x^\infty \left[\sup_{t\leqslant\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q w(t)\,dt\right)^{1/q}$$

$$\times \left( \int_0^x V^{p'}(y)v(y)\,dy \right)^{1/p'},$$

*and in this case* $c \approx A_1 + A_2$;

(ii) $1 = p \leqslant q$ *and* $B_1 + B_2 < \infty$, *where*

$$B_1 := \sup_{x>0} \left( \left[ \sup_{x \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q W(x) + \int_x^\infty \left[ \sup_{t \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t)\,dt \right)^{1/q} \left( \sup_{0<y\leqslant x} \frac{B(y)}{V(y)} \right),$$

$$B_2 := \sup_{x>0} \left( \left[ \sup_{x \leqslant \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q W(x) + \int_x^\infty \left[ \sup_{t \leqslant \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q w(t)\,dt \right)^{1/q} V(x),$$

*and in this case* $c \approx B_1 + B_2$;

(iii) $\max\{q,1\} < p$ *and* $C_1 + C_2 + C_3 + C_4 < \infty$, *where*

$$C_1 := \left( \int_0^\infty \left( \int_x^\infty \left[ \sup_{t \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t)\,dt \right)^{r/p} \left[ \sup_{x \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q \right.$$
$$\left. \times \left( \int_0^x \left( \frac{B(y)}{V(y)} \right)^{p'} v(y)\,dy \right)^{r/p'} w(x)\,dx \right)^{1/r},$$

$$C_2 := \left( \int_0^\infty W^{r/p}(x) \left[ \sup_{x \leqslant \tau < \infty} \left[ \sup_{\tau \leqslant y < \infty} \frac{u(y)}{B(y)} \right] \left( \int_0^\tau \left( \frac{B(y)}{V(y)} \right)^{p'} v(y)\,dy \right)^{1/p'} \right]^r w(x)\,dx \right)^{1/r},$$

$$C_3 := \left( \int_0^\infty \left( \int_x^\infty \left[ \sup_{t \leqslant \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q w(t)\,dt \right)^{r/p} \left[ \sup_{x \leqslant \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q \right.$$
$$\left. \times \left( \int_0^x V^{p'}(y)v(y)\,dy \right)^{r/p'} w(x)\,dx \right)^{1/r},$$

$$C_4 := \left( \int_0^\infty W^{r/p}(x) \left[ \sup_{x \leqslant \tau < \infty} \left[ \sup_{\tau \leqslant y < \infty} \frac{u(y)}{V^2(y)} \right] \left( \int_0^\tau V^{p'}(y)v(y)\,dy \right)^{1/p'} \right]^r w(x)\,dx \right)^{1/r},$$

*and in this case* $c \approx C_1 + C_2 + C_3 + C_4$;

(iv) $q < 1 = p$ *and* $D_1 + D_2 + D_3 + D_4 < \infty$, *where*

$$D_1 := \left( \int_0^\infty \left( \int_x^\infty \left[ \sup_{t \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t)\,dt \right)^{r/p} \left[ \sup_{x \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q \right.$$
$$\left. \times \left( \sup_{0<y\leqslant x} \frac{B(y)}{V(y)} \right)^r w(x)\,dx \right)^{1/r},$$

$$D_2 := \left( \int_0^\infty W^{r/p}(x) \left[ \sup_{x \leqslant \tau < \infty} \left[ \sup_{\tau \leqslant y < \infty} \frac{u(y)}{B(y)} \right] \left( \sup_{0<y\leqslant \tau} \frac{B(y)}{V(y)} \right) \right]^r w(x)\,dx \right)^{1/r},$$

$$D_3 := \left( \int_0^\infty \left( \int_x^\infty \left[ \sup_{t \leqslant \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q w(t)\,dt \right)^{r/p} \left[ \sup_{x \leqslant \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q V^r(x) w(x)\,dx \right)^{1/r},$$

$$D_4 := \left( \int_0^\infty W^{r/p}(x) \left[ \sup_{x \leqslant \tau < \infty} \left[ \sup_{\tau \leqslant y < \infty} \frac{u(y)}{V^2(y)} \right] V(\tau) \right]^r w(x)\,dx \right)^{1/r},$$

*and in this case* $c \approx D_1 + D_2 + D_3 + D_4$.

*Proof.* By Theorem 2.2, (1.12) holds iff both

$$\left\| T_{u,b}\left(\int_x^\infty h\right)\right\|_{q,w,(0,\infty)} \leqslant c\|h\|_{p,V^p v^{1-p},(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty). \tag{5.1}$$

and

$$\|T_{u,b}\mathbf{1}\|_{q,w,(0,\infty)} \leqslant c\|\mathbf{1}\|_{p,v,(0,\infty)} \tag{5.2}$$

hold.

Note that

$$\begin{aligned} T_{u,b}\left(\int_t^\infty h\right)(x) &= \sup_{x\leqslant\tau<\infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau \left(\int_s^\infty h(y)\,dy\right) b(s)\,ds \\ &\approx \sup_{x\leqslant\tau<\infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau h(y)B(y)\,dy + \sup_{x\leqslant\tau<\infty} u(\tau) \int_\tau^\infty h(s)\,ds \\ &= S^*_{u/B}\left(\int_0^\tau hB\right) + S^*_u\left(\int_\tau^\infty h\right). \end{aligned}$$

Hence, inequality (1.12) holds iff inequalities

$$\left\| S^*_{u/B}\left(\int_0^\tau h\right)\right\|_{q,w,(0,\infty)} \leqslant c\|h\|_{p,B^{-p}V^p v^{1-p},(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty), \tag{5.3}$$

$$\left\| S^*_{u}\left(\int_\tau^\infty h\right)\right\|_{q,w,(0,\infty)} \leqslant c\|h\|_{p,V^p v^{1-p},(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty), \tag{5.4}$$

and (5.2) hold.

Again by Theorem 2.2, (5.4) with (5.2) is equivalent to

$$\|S^*_u f\|_{q,w,(0,\infty)} \leqslant c\|f\|_{p,v,(0,\infty)}, \quad f \in \mathfrak{M}^+((0,\infty);\downarrow). \tag{5.5}$$

Now by Theorem 2.3, (5.5) is equivalent to

$$\left\| S^*_{u/V^2}\left(\int_0^x h\right)\right\|_{q,w,(0,\infty)} \leqslant c\|h\|_{p,V^{-p}v^{1-p},(0,\infty)}, \quad h \in \mathfrak{M}^+(0,\infty). \tag{5.6}$$

Consequently, (1.12) holds iff inequalities (5.3) and (5.6) hold.

(i) and (ii). Let $p \leqslant q$. By Theorem 3.1, (5.3) and (5.6) hold iff both $A_i < \infty$, $i = 1,2$, when $p > 1$, and $B_i < \infty$, $i = 1,2$, when $p = 1$, respectively.

(iii) and (iv). Let $q < p$. By Theorem 3.1, (5.3) and (5.6) hold iff $C_i < \infty$, $i = 1,2,3,4$, when $p > 1$, and $D_i < \infty$, $i = 1,2,3,4$, when $p = 1$, respectively.

### 5.2. The case $0 < p < 1$

We start with a simple observation. If $0 < p \leqslant 1$ and $t \in (0,\infty)$, then

$$\begin{aligned}\sup_{0<\tau\leqslant t} f(\tau)B(\tau) &\leqslant \int_0^t f(y)b(y)\,dy \\ &\lesssim \left(\int_0^t f(y)^p B(y)^{p-1} b(y)\,dy\right)^{1/p}, \quad f \in \mathfrak{M}^+((0,\infty);\downarrow). \end{aligned} \tag{5.7}$$

Since $f$ is non-increasing, the first inequality in (5.7) is obvious. The second one follows, for instance, from the fact that (see, for instance, [5, Theorem 3.2], cf. also [28])

$$\sup_{f\in\mathfrak{M}^+((0,\infty);\downarrow):\, f\not\sim 0} \frac{\int_0^\infty f(x)g(x)\,dx}{\left(\int_0^\infty f(x)^p v(x)\,dx\right)^{1/p}} \approx \sup_{t>0}\left(\int_0^t g(x)\,dx \left(\int_0^t v(x)\,dx\right)^{-1/p}\right).$$

Our first aim is to prove a reduction theorem for the operator $T_{u,b}$. We first note that, using the monotonicity of $\int_0^t fb$ and interchanging the suprema, we get that

$$\begin{aligned}(T_{u,b}g)(t) &= \sup_{t\leqslant\tau<\infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(y)b(y)\,dy \\ &= \sup_{t\leqslant\tau<\infty}\left(\sup_{\tau\leqslant x<\infty} \frac{u(x)}{B(x)}\right)\int_0^\tau g(y)b(y)\,dy, \qquad t\in(0,\infty).\end{aligned}$$

As a consequence, we can safely assume that $u(x)/B(x)$ is non-increasing on $(0,\infty)$, since otherwise we would just replace $u(x)/B(x)$ by $\sup_{\tau\leqslant x<\infty} u(x)/B(x)$.

THEOREM 5.2. *Let $0 < p \leqslant 1$, $0 < q < \infty$. Assume that $u \in \mathscr{W}(0,\infty)\cap C(0,\infty)$ and $b, v, w \in \mathscr{W}(0,\infty)$ are such that $0 < B(t) < \infty$ for all $x > 0$. Then the following three statements are equivalent:*

$$\begin{aligned}\left(\int_0^\infty \left(\sup_{t\leqslant\tau<\infty} \frac{u(\tau)}{B(\tau)}\int_0^\tau f(y)b(y)\,dy\right)^q w(t)\,dt\right)^{1/q} \\ \lesssim \left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau\right)^{1/p}, \ f \in \mathfrak{M}^+((0,\infty);\downarrow);\end{aligned} \tag{5.8}$$

$$\begin{aligned}\left(\int_0^\infty \left(\sup_{t\leqslant\tau<\infty} \left(\frac{u(\tau)}{B(\tau)}\right)^p \int_0^\tau f(y)B(y)^{p-1}b(y)\,dy\right)^{q/p} w(t)\,dt\right)^{1/q} \\ \lesssim \left(\int_0^\infty f(\tau)v(\tau)\,d\tau\right)^{1/p}, \ f \in \mathfrak{M}^+((0,\infty);\downarrow);\end{aligned} \tag{5.9}$$

$$\begin{aligned}\left(\int_0^\infty \left(\sup_{t\leqslant\tau<\infty} \frac{u(\tau)}{B(\tau)} \sup_{0<y\leqslant\tau} f(y)B(y)\right)^q w(t)\,dt\right)^{1/q} \\ \lesssim \left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau\right)^{1/p}, \ f \in \mathfrak{M}^+((0,\infty);\downarrow).\end{aligned} \tag{5.10}$$

*Proof.* Again, in view of (5.7), the implications (5.9) $\Rightarrow$ (5.8) $\Rightarrow$ (5.10) are obvious, and it just remains to show that (5.10) implies (5.9).

Suppose that (5.10) holds. Since $u(x)/B(x)$ is non-increasing, we have

$$\begin{aligned}
&\sup_{t\leqslant\tau<\infty}\frac{u(\tau)}{B(\tau)}\sup_{0<y\leqslant\tau} f(y)B(y)\\
&\qquad=\max\left\{\sup_{t\leqslant\tau<\infty}\frac{u(\tau)}{B(\tau)}\sup_{0<y\leqslant t} f(y)B(y),\ \sup_{t\leqslant\tau<\infty}\frac{u(\tau)}{B(\tau)}\sup_{t<y\leqslant\tau} f(y)B(y)\right\}\\
&\qquad=\max\left\{\frac{u(t)}{B(t)}\sup_{0<y\leqslant t} f(y)B(y),\ \sup_{t\leqslant y<\infty} f(y)B(y)\sup_{y\leqslant\tau<\infty}\frac{u(\tau)}{B(\tau)}\right\}\\
&\qquad=\max\left\{\frac{u(t)}{B(t)}\sup_{0<y\leqslant t} f(y)B(y),\ \sup_{t\leqslant y<\infty} f(y)u(y)\right\}.
\end{aligned}$$

Hence, (5.10) breaks down into the following two inequalities:

$$\begin{aligned}
&\left(\int_0^\infty\left(\sup_{0<y\leqslant t} f(y)B(y)\right)^q w(t)\left(\frac{u(t)}{B(t)}\right)^q dt\right)^{1/q}\\
&\qquad\lesssim\left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau\right)^{1/p},\ f\in\mathfrak{M}^+((0,\infty);\downarrow), \qquad (5.11)
\end{aligned}$$

$$\begin{aligned}
&\left(\int_0^\infty\left(\sup_{t\leqslant y<\infty} f(y)u(y)\right)^q w(t)\,dt\right)^{1/q}\\
&\qquad\lesssim\left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau\right)^{1/p},\ f\in\mathfrak{M}^+((0,\infty);\downarrow). \qquad (5.12)
\end{aligned}$$

Obviously, (5.11) and (5.12) are equivalent to

$$\begin{aligned}
&\left(\int_0^\infty\left(\sup_{0<y\leqslant t} f(y)B(y)^p\right)^{q/p} w(t)\left(\frac{u(t)}{B(t)}\right)^q dt\right)^{p/q}\\
&\qquad\lesssim\int_0^\infty f(\tau)v(\tau)\,d\tau,\ f\in\mathfrak{M}^+((0,\infty);\downarrow), \qquad (5.13)
\end{aligned}$$

$$\begin{aligned}
&\left(\int_0^\infty\left(\sup_{t\leqslant y<\infty} f(y)u(y)^p\right)^{q/p} w(t)\,dt\right)^{p/q}\\
&\qquad\lesssim\int_0^\infty f(\tau)v(\tau)\,d\tau,\ f\in\mathfrak{M}^+((0,\infty);\downarrow). \qquad (5.14)
\end{aligned}$$

(i) Let $p\leqslant q$. By Theorem 3.3, (5.13) holds iff both

$$\sup_{x>0}\left(\int_0^x u(t)^q w(t)\,dt+B(x)^q\int_x^\infty\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q} V^{-1/p}(x)<\infty \qquad (5.15)$$

and

$$\left(\int_0^\infty u(t)^q w(t)\,dt\right)^{1/q}\lesssim\left(\int_0^\infty v(\tau)\,d\tau\right)^{1/p} \qquad (5.16)$$

hold.

By Theorem 3.6, (5.14) holds iff both

$$\sup_{x>0}\left(\left[\sup_{x\leqslant y<\infty}\frac{u(y)^p}{V^2(y)}\right]^{q/p}\int_0^x w(t)\,dt+\int_x^\infty\left[\sup_{t\leqslant y<\infty}\frac{u(y)^p}{V^2(y)}\right]^{q/p}w(t)\,dt\right)^{1/q}V^{1/p}(x)<\infty \tag{5.17}$$

and

$$\left(\int_0^\infty\left(\sup_{t\leqslant\tau<\infty}u(\tau)^p\right)^{q/p}w(t)\,dt\right)^{1/q}\lesssim\left(\int_0^\infty v(\tau)\,d\tau\right)^{1/p} \tag{5.18}$$

hold.

On the other hand, by Theorem 5.1, (5.9) holds iff inequalities

$$\sup_{x>0}\left(\left[\frac{u(x)}{B(x)}\right]^q\int_0^x w(t)\,dt+\int_x^\infty\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q}\sup_{0<\tau\leqslant x}\frac{B(\tau)}{V^{1/p}(\tau)}<\infty \tag{5.19}$$

$$\sup_{x>0}\left(\left[\sup_{x\leqslant\tau<\infty}\frac{u(\tau)^p}{V^2(\tau)}\right]^{q/p}\int_0^x w(t)\,dt+\int_x^\infty\left[\sup_{t\leqslant\tau<\infty}\frac{u(\tau)^p}{V^2(\tau)}\right]^{q/p}w(t)\,dt\right)^{1/q}V^{1/p}(x)<\infty \tag{5.20}$$

hold.

We will thus be done if we can show that (5.15) together with (5.17) imply (5.19). The latter can be proved as follows:

Since

$$\sup_{x>0}\left(\int_x^\infty\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q}\sup_{0<\tau\leqslant x}\frac{B(\tau)}{V^{1/p}(\tau)}=\sup_{x>0}\left(\int_x^\infty\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q}\frac{B(x)}{V^{1/p}(x)},$$

it remains to show that

$$\begin{aligned}&\sup_{x>0}\frac{u(x)}{B(x)}\left(\int_0^x w(t)\,dt\right)^{1/q}\sup_{0<\tau\leqslant x}\frac{B(\tau)}{V^{1/p}(\tau)}\\&\qquad\lesssim\sup_{x>0}\frac{u(x)}{V^{1/p}(x)}\left(\int_0^x w(t)\,dt\right)^{1/q}+\sup_{x>0}\frac{B(x)}{V^{1/p}(x)}\left(\int_x^\infty\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q}.\end{aligned}$$

Interchanging the suprema, using the monotonicity of $u/B$, we get that

$$\begin{aligned}&\sup_{x>0}\frac{u(x)}{B(x)}\left(\int_0^x w(t)\,dt\right)^{1/q}\sup_{0<\tau\leqslant x}\frac{B(\tau)}{V^{1/p}(\tau)}\\&=\sup_{\tau>0}\frac{B(\tau)}{V^{1/p}(\tau)}\sup_{\tau\leqslant x<\infty}\frac{u(x)}{B(x)}\left(\int_0^x w(t)\,dt\right)^{1/q}\\&\lesssim\sup_{\tau>0}\frac{B(\tau)}{V^{1/p}(\tau)}\left(\sup_{\tau\leqslant x<\infty}\frac{u(x)}{B(x)}\right)\left(\int_0^\tau w(t)\,dt\right)^{1/q}\end{aligned}$$

$$+\sup_{\tau>0}\frac{B(\tau)}{V^{1/p}(\tau)}\sup_{\tau\leqslant x<\infty}\frac{u(x)}{B(x)}\left(\int_\tau^x w(t)\,dt\right)^{1/q}$$

$$\lesssim \sup_{\tau>0}\frac{u(\tau)}{V^{1/p}(\tau)}\left(\int_0^\tau w(t)\,dt\right)^{1/q}+\sup_{\tau>0}\frac{B(\tau)}{V^{1/p}(\tau)}\sup_{\tau\leqslant x<\infty}\left(\int_\tau^x\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q}$$

$$=\sup_{\tau>0}\frac{u(\tau)}{V^{1/p}(\tau)}\left(\int_0^\tau w(t)\,dt\right)^{1/q}+\sup_{\tau>0}\frac{B(\tau)}{V^{1/p}(\tau)}\left(\int_\tau^\infty\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q}.$$

(ii) Let $q<p$. By Theorem 3.3, (5.13) holds iff

$$\int_0^\infty\left(\int_0^x u(t)^q w(t)\,dt\right)^{r/p} u(x)^q V^{-r/p}(x) w(x)\,dx<\infty,$$

$$\int_0^\infty\left(\int_x^\infty\left(\frac{u(t)}{B(t)}\right)^q w(t)\,dt\right)^{r/p}\left[\sup_{0<\tau\leqslant x}\frac{B(\tau)}{V^{1/p}(\tau)}\right]^r w(x)\left(\frac{u(x)}{B(x)}\right)^q dx<\infty.$$

By Theorem 3.6, (5.14) holds iff

$$\int_0^\infty\left(\int_x^\infty\left[\sup_{t\leqslant y<\infty}\frac{u(y)^p}{V^2(y)}\right]^{q/p} w(t)\,dt\right)^{r/p}\left[\sup_{x\leqslant y<\infty}\frac{u(y)^p}{V^2(y)}\right]^{q/p} V^{r/p}(x) w(x)\,dx<\infty,$$

$$\int_0^\infty\left(\int_0^x w(t)\,dt\right)^{r/p}\left(\sup_{x\leqslant\tau<\infty}\left[\sup_{\tau\leqslant y<\infty}\frac{u(y)^p}{V^2(y)}\right]V(\tau)\right)^{r/p} w(x)\,dx<\infty.$$

On the other hand, by Theorem 5.1, (5.9) holds iff

$$\int_0^\infty\left(\int_x^\infty\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{r/p}\left[\sup_{0<t\leqslant\tau}\frac{B(t)^p}{V(t)}\right]^{r/p} w(x)\left[\frac{u(x)}{B(x)}\right]^q dx<\infty,$$

$$\int_0^\infty\left(\int_0^x w(t)\,dt\right)^{r/p}\left(\sup_{x\leqslant\tau<\infty}\left[\frac{u(\tau)}{B(\tau)}\right]^p\left[\sup_{0<t\leqslant\tau}\frac{B(t)^p}{V(t)}\right]\right)^{r/p} w(x)\,dx<\infty,$$

$$\int_0^\infty\left(\int_x^\infty\left[\sup_{t\leqslant y<\infty}\frac{u(y)^p}{V^2(y)}\right]^{q/p} w(t)\,dt\right)^{r/p}\left[\sup_{x\leqslant y<\infty}\frac{u(y)^p}{V^2(y)}\right]^{q/p} V^{r/p}(x) w(x)\,dx<\infty,$$

$$\int_0^\infty\left(\int_0^x w(t)\,dt\right)^{r/p}\left(\sup_{x\leqslant\tau<\infty}\left[\sup_{\tau\leqslant y<\infty}\frac{u(y)^p}{V^2(y)}\right]V(\tau)\right)^{r/p} w(x)\,dx$$

$$=\int_0^\infty\left(\int_0^x w(t)\,dt\right)^{r/p}\left(\sup_{x\leqslant\tau<\infty}\frac{u(\tau)^p}{V(\tau)}\right)^{r/p} w(x)\,dx<\infty.$$

Obviously, it remains to show that

$$\int_0^\infty\left(\int_0^x w(t)\,dt\right)^{r/p}\left(\sup_{x\leqslant\tau<\infty}\left[\frac{u(\tau)}{B(\tau)}\right]^p\left[\sup_{0<t\leqslant\tau}\frac{B(t)^p}{V(t)}\right]\right)^{r/p} w(x)\,dx$$

$$\lesssim\int_0^\infty\left(\int_x^\infty\left(\frac{u(t)}{B(t)}\right)^q w(t)\,dt\right)^{r/p}\left[\sup_{0<\tau\leqslant x}\frac{B(\tau)}{V^{1/p}(\tau)}\right]^r w(x)\left(\frac{u(x)}{B(x)}\right)^q dx$$

$$+\int_0^\infty \left(\int_0^x w(t)\,dt\right)^{r/p} \left(\sup_{x\leqslant\tau<\infty} \frac{u(\tau)^p}{V(\tau)}\right)^{r/p} w(x)\,dx.$$

We will prove the assertion only in the case when $\int_0^\infty w(\tau)\,d\tau=\infty$. Let $\{x_k\}$ be such that $\int_0^{x_k} w(\tau)\,d\tau = 2^k$. Then

$$\int_0^\infty \left(\int_0^x w(t)\,dt\right)^{r/p} \left(\sup_{x\leqslant\tau<\infty} \left[\frac{u(\tau)}{B(\tau)}\right]^p \left[\sup_{0<t\leqslant\tau} \frac{B(t)^p}{V(t)}\right]\right)^{r/p} w(x)\,dx$$
$$\approx \sum_{k\in\mathbb{Z}} 2^{kr/q} \left(\sup_{x_k\leqslant\tau<\infty} \left[\frac{u(\tau)}{B(\tau)}\right]^p \left[\sup_{0<t\leqslant\tau} \frac{B(t)^p}{V(t)}\right]\right)^{r/p}.$$

Note that

$$\sup_{x_k\leqslant\tau<\infty} \left[\frac{u(\tau)}{B(\tau)}\right]^p \left[\sup_{0<t\leqslant\tau} \frac{B(t)^p}{V(t)}\right]$$
$$\approx \sup_{x_k\leqslant\tau<\infty} \left[\frac{u(\tau)}{B(\tau)}\right]^p \left[\sup_{0<t\leqslant x_k} \frac{B(t)^p}{V(t)}\right] + \sup_{x_k\leqslant\tau<\infty} \left[\frac{u(\tau)}{B(\tau)}\right]^p \left[\sup_{x_k\leqslant t\leqslant\tau} \frac{B(t)^p}{V(t)}\right]$$
$$= \left[\frac{u(x_k)}{B(x_k)}\right]^p \left[\sup_{0<t\leqslant x_k} \frac{B(t)^p}{V(t)}\right] + \sup_{x_k\leqslant t<\infty} \frac{B(t)^p}{V(t)} \sup_{t\leqslant\tau<\infty} \left[\frac{u(\tau)}{B(\tau)}\right]^p$$
$$= \left[\frac{u(x_k)}{B(x_k)}\right]^p \left[\sup_{0<t\leqslant x_k} \frac{B(t)^p}{V(t)}\right] + \sup_{x_k\leqslant t<\infty} \frac{u(t)^p}{V(t)}$$
$$\approx \left[\frac{u(x_k)}{B(x_k)}\right]^p \left[\sup_{0<t\leqslant x_{k-1}} \frac{B(t)^p}{V(t)}\right] + \left[\frac{u(x_k)}{B(x_k)}\right]^p \left[\sup_{x_{k-1}<t\leqslant x_k} \frac{B(t)^p}{V(t)}\right] + \sup_{x_k\leqslant t<\infty} \frac{u(t)^p}{V(t)}$$
$$\leqslant \left[\frac{u(x_k)}{B(x_k)}\right]^p \left[\sup_{0<t\leqslant x_{k-1}} \frac{B(t)^p}{V(t)}\right] + \sup_{x_{k-1}\leqslant t<x_k} \frac{u(t)^p}{V(t)} + \sup_{x_k\leqslant t<\infty} \frac{u(t)^p}{V(t)}$$
$$\leqslant \left[\frac{u(x_k)}{B(x_k)}\right]^p \left[\sup_{0<t\leqslant x_{k-1}} \frac{B(t)^p}{V(t)}\right] + \sup_{x_{k-1}\leqslant t<\infty} \frac{u(t)^p}{V(t)}.$$

Hence,

$$\int_0^\infty \left(\int_0^x w(t)\,dt\right)^{r/p} \left(\sup_{x\leqslant\tau<\infty} \left[\frac{u(\tau)}{B(\tau)}\right]^p \left[\sup_{0<t\leqslant\tau} \frac{B(t)^p}{V(t)}\right]\right)^{r/p} w(x)\,dx$$
$$\lesssim \sum_{k\in\mathbb{Z}} 2^{kr/q} \left(\left[\frac{u(x_k)}{B(x_k)}\right]^p \left[\sup_{0<t\leqslant x_{k-1}} \frac{B(t)^p}{V(t)}\right]\right)^{r/p} + \sum_{k\in\mathbb{Z}} 2^{kr/q} \left(\sup_{x_{k-1}\leqslant t<\infty} \frac{u(t)^p}{V(t)}\right)^{r/p}$$
$$\approx \sum_{k\in\mathbb{Z}} \left(\int_{x_{k-1}}^{x_k} \left(\int_x^{x_k} w\right)^{r/p} w(x)\,dx\right) \left(\left[\frac{u(x_k)}{B(x_k)}\right]^p \left[\sup_{0<t\leqslant x_{k-1}} \frac{B(t)^p}{V(t)}\right]\right)^{r/p}$$
$$+ \sum_{k\in\mathbb{Z}} \left(\int_{x_{k-2}}^{x_{k-1}} \left(\int_{x_k}^x w\right)^{r/p} w(x)\,dx\right) \left(\sup_{x_{k-1}\leqslant t<\infty} \frac{u(t)^p}{V(t)}\right)^{r/p}$$
$$\lesssim \sum_{k\in\mathbb{Z}} \int_{x_{k-1}}^{x_k} \left(\int_x^\infty \left(\frac{u(t)}{B(t)}\right)^q w(t)\,dt\right)^{r/p} \left[\sup_{0<\tau\leqslant x} \frac{B(\tau)}{V^{1/p}(\tau)}\right]^r \left[\frac{u(x)}{B(x)}\right]^q w(x)\,dx$$

$$+\sum_{k\in\mathbb{Z}}\int_{x_{k-1}}^{x_{k+1}}\left(\int_0^x w(t)\,dt\right)^{r/p}\left(\sup_{x\leqslant\tau<\infty}\frac{u(\tau)^p}{V(\tau)}\right)^{r/p} w(x)\,dx$$

$$\lesssim\int_0^\infty\left(\int_x^\infty\left(\frac{u(t)}{B(t)}\right)^q w(t)\,dt\right)^{r/p}\left[\sup_{0<\tau\leqslant x}\frac{B(\tau)}{V^{1/p}(\tau)}\right]^r w(x)\left(\frac{u(x)}{B(x)}\right)^q dx$$

$$+\int_0^\infty\left(\int_0^x w(t)\,dt\right)^{r/p}\left(\sup_{x\leqslant\tau<\infty}\frac{u(\tau)^p}{V(\tau)}\right)^{r/p} w(x)\,dx. \quad \square$$

REMARK 5.3. Note that Theorem 5.2, namely the fact that (5.9) $\Leftrightarrow$ (5.8) $\Leftrightarrow$ (5.10), when $b\equiv 1$, was proved in [16].

As a corollary we obtain that for all the three operators mentioned in (5.7), the corresponding weighted inequalities are equivalent. It is worth noticing that this is not so when $p>1$.

COROLLARY 5.4. *Assume that* $0<p\leqslant 1$, $0<q<\infty$, *and* $v,w\in\mathscr{W}(0,\infty)$. *Let* $b$ *be a weight on* $(0,\infty)$ *such that* $0<B(t)<\infty$ *for every* $t\in(0,\infty)$. *Then the following three statements are equivalent:*

$$\left(\int_0^\infty\left(\int_0^t f(\tau)b(\tau)\,d\tau\right)^q w(t)\,dt\right)^{1/q} \lesssim\left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau\right)^{1/p},\ f\in\mathfrak{M}^+((0,\infty);\downarrow); \tag{5.21}$$

$$\left(\int_0^\infty\left(\int_0^t f(\tau)^p B(\tau)^{p-1}b(\tau)\,d\tau\right)^{q/p} w(t)\,dt\right)^{1/q} \lesssim\left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau\right)^{1/p},\ f\in\mathfrak{M}^+((0,\infty);\downarrow); \tag{5.22}$$

$$\left(\int_0^\infty\left(\sup_{0<\tau\leqslant t} f(\tau)B(\tau)\right)^q w(t)\,dt\right)^{1/q} \lesssim\left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau\right)^{1/p},\ f\in\mathfrak{M}^+((0,\infty);\downarrow). \tag{5.23}$$

This fact was proved in [16, Theorem 2.1], when $b\equiv 1$. Recently, in [17, Theorem 3.9], it was proved that (5.21) $\Leftrightarrow$ (5.23) for more general Volterra operators with continuous Oinarov kernels in the case when $0<q<p\leqslant 1$.

*Proof.* The proof immediately follows from Theorem 5.2 by taking $u\equiv 1$.
By the way, we have proved the following statement.

THEOREM 5.5. *Let* $0<p\leqslant 1$, $0<q<\infty$. *Assume that* $u\in\mathscr{W}(0,\infty)\cap C(0,\infty)$ *and* $b,v,w\in\mathscr{W}(0,\infty)$ *are such that* $0<V(t)<\infty$ *and* $0<B(t)<\infty$ *for all* $x>0$. *Then inequality* (1.12) *with the best constant* $c$ *holds if and only if the following holds:*

(i) *$p \leqslant q$ and $E_1 + E_2 < \infty$, where*

$$E_1 := \sup_{x>0} \left( \left[ \sup_{x \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q \int_0^x w(t)\,dt + \int_x^\infty \left[ \sup_{t \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t)\,dt \right)^{1/q} \sup_{0<y\leqslant x} \frac{B(y)}{V^{1/p}(y)};$$

$$E_2 := \sup_{x>0} \left( \left[ \sup_{x \leqslant y < \infty} \frac{u(y)^p}{V^2(y)} \right]^{q/p} \int_0^x w(t)\,dt + \int_x^\infty \left[ \sup_{t \leqslant y < \infty} \frac{u(y)^p}{V^2(y)} \right]^{q/p} w(t)\,dt \right)^{1/q} V^{1/p}(x),$$

*and in this case* $c \approx E_1 + E_2$;

(ii) *$q < p$ and $F_1 + F_2 + F_3 + F_4 < \infty$, where*

$$F_1 := \left( \int_0^\infty \left( \int_0^x w(t)\,dt \right)^{r/p} \left[ \sup_{x \leqslant \tau < \infty} \left[ \sup_{\tau \leqslant y < \infty} \frac{u(y)}{B(y)} \right]^p \left( \sup_{0<y\leqslant \tau} \frac{B(y)^p}{V(y)} \right) \right]^{r/p} w(x)\,dx \right)^{1/r},$$

$$F_2 := \left( \int_0^\infty \left( \int_x^\infty \left[ \sup_{t \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t)\,dt \right)^{r/p} \left[ \sup_{0<\tau\leqslant x} \frac{B^p(\tau)}{V(\tau)} \right]^{r/p} \times \left[ \sup_{x \leqslant \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(x)\,dx \right)^{1/r},$$

$$F_3 := \left( \int_0^\infty \left( \int_0^x w(t)\,dt \right)^{r/p} \left( \sup_{x \leqslant \tau < \infty} \left[ \sup_{\tau \leqslant y < \infty} \frac{u(y)^p}{V^2(y)} \right] V(\tau) \right)^{r/p} w(x)\,dx \right)^{1/r},$$

$$F_4 := \left( \int_0^\infty \left( \int_x^\infty \left[ \sup_{t \leqslant y < \infty} \frac{u(y)^p}{V^2(y)} \right]^{q/p} w(t)\,dt \right)^{r/p} \left[ \sup_{x \leqslant y < \infty} \frac{u(y)^p}{V^2(y)} \right]^{q/p} V^{r/p}(x) w(x)\,dx \right)^{1/r},$$

*and in this case* $c \approx F_1 + F_2 + F_3 + F_4$.

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*Amiran Gogatishvili*
*Institute of Mathematics, Academy of Sciences of the Czech Republic*
*Žitná 25 115 67 Praha 1, Czech Republic*
*e-mail:* gogatish@math.cas.cz

*Rza Mustafayev*
*Institute of Mathematics and Mechanics*
*Academy of Sciences of Azerbaijan*
*B. Vahabzade St. 9, Baku, AZ 1141, Azerbaijan*
and
*Department of Mathematics*
*Faculty of Science and Arts, Kirikkale University*
*71450 Yahsihan, Kirikkale, Turkey*
*e-mail:* rzamustafayev@gmail.com