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# INEQUALITIES FOR THE MODIFIED BESSEL FUNCTION OF THE SECOND KIND AND THE KERNEL OF THE KRÄTZEL INTEGRAL TRANSFORMATION

ROBERT E. GAUNT

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*Abstract.* We obtain new inequalities for the modified Bessel function of the second kind $K_\nu$ in terms of the gamma function. These bounds follow as special cases of inequalities that we derive for the kernel of the Krätzel integral transformation.

## 1. Introduction

The modified Bessel function of the second kind $K_\nu$ is an important and widely used special function. There exists a substantial literature concerning inequalities for the modified Bessel function of the second; see, for example, [7] and [1] and references therein. In a recent work, [3] derived the following simple lower bound for the function $K_0$:

$$\frac{1}{\sqrt{x+\frac{1}{2}}} < \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)} < \sqrt{\frac{2}{\pi}}\mathrm{e}^x K_0(x). \tag{1}$$

In this note, we generalise this inequality to the modified Bessel function $K_\nu$ for $\nu \geqslant 0$. In deriving our inequality, we follow the approach of [3] by exploiting the following integral representation of the modified Bessel function of the second kind ([8], formula 10.32.8):

$$K_\nu(x) = \frac{\sqrt{\pi}(\frac{1}{2}x)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_1^\infty \mathrm{e}^{-xt}(t^2-1)^{\nu-\frac{1}{2}}\,\mathrm{d}t, \quad x>0. \tag{2}$$

This integral representation of the modified Bessel function $K_\nu$ closely resembles the kernel

$$\lambda_\nu^{(n)}(x) = \frac{(2\pi)^{(n-1)/2}\sqrt{n}\left(\frac{x}{n}\right)^{n\nu}}{\Gamma(\nu+1-\frac{1}{n})}\int_1^\infty (t^n-1)^{\nu-\frac{1}{n}}\mathrm{e}^{-xt}\,\mathrm{d}t, \quad \nu > \frac{1}{n}-1,\ n=1,2,\ldots, \tag{3}$$

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of the Krätzel integral transformation [6] (see also [2] and references therein for further properties) defined by

$$\mathscr{L}_\nu^{(n)}\{f\}(z)=\int_0^\infty \lambda_\nu^{(n)}(zt)f(t)\,\mathrm{d}t,\quad \mathrm{Re}\,x>0.$$

Indeed, a simple manipulation yields the relation

$$\lambda_\nu^{(2)}(x)=2\left(\frac{x}{2}\right)^\nu K_\nu(x). \tag{4}$$

Due to the similarity between the representations (2) and (3), our approach to bounding the kernel $\lambda_\nu^{(n)}$ is no more difficult than bounding the modified Bessel function $K_\nu$. In this note, we exploit the representation (3) to derive inequalities for the kernel $\lambda_\nu^{(n)}$ and then use (4) to immediately deduce inequalities for the modified Bessel function $K_\nu$. These inequalities are a natural generalisation of the inequality (1).

## 2. Results and proofs

The following is the main result of this note.

THEOREM 1. *(i). Let $x>0$. Then, for $0\leqslant\nu\leqslant\frac{1}{n}$, $n=1,2,\ldots$, we have*

$$\lambda_\nu^{(n)}(x)\geqslant(2\pi)^{(n-1)/2}\frac{\sqrt{n}}{n-1}\left(\frac{x}{n}\right)^{n\nu}\frac{\Gamma(\frac{x}{n-1}+\frac{1}{n}-\nu)}{\Gamma(\frac{x}{n-1}+1)}\mathrm{e}^{-x} \tag{5}$$

*with equality if and only if $\nu=\frac{1}{n}$. If $\nu>\frac{1}{n}$, the strict inequality is reversed and holds for all $x>(n-1)(\nu-\frac{1}{n})$.*

*(ii). Let $x>0$. Then, for $0\leqslant\nu\leqslant\frac{1}{2}$, we have*

$$K_\nu(x)\geqslant\sqrt{\frac{\pi}{2}}\frac{x^\nu\Gamma(x+\frac{1}{2}-\nu)}{\Gamma(x+1)}\mathrm{e}^{-x} \tag{6}$$

*with equality if and only if $\nu=\frac{1}{2}$. If $\nu>\frac{1}{2}$, the strict inequality is reversed and holds for all $x>\nu-\frac{1}{2}$.*

*Proof.* We first note that part (ii) follows immediately from setting $n=2$ in part (i), due to the relation (4). In order to establish part (i), we recall the integral representation of the kernel $\lambda_\nu^{(n)}$:

$$\lambda_\nu^{(n)}(x)=\frac{(2\pi)^{(n-1)/2}\sqrt{n}\left(\frac{x}{n}\right)^{n\nu}}{\Gamma(\nu+1-\frac{1}{n})}\int_1^\infty(t^n-1)^{\nu-\frac{1}{n}}\mathrm{e}^{-xt}\,\mathrm{d}t.$$

Setting $t=\frac{2}{n-1}u+1$ gives

$$\lambda_\nu^{(n)}(x)=\frac{(2\pi)^{(n-1)/2}\sqrt{n}\left(\frac{x}{n}\right)^{n\nu}}{\Gamma(\nu+1-\frac{1}{n})}\frac{2}{n-1}\mathrm{e}^{-x}\int_0^\infty\left(\left(\frac{2u}{n-1}+1\right)^n-1\right)^{\nu-\frac{1}{n}}\mathrm{e}^{-\frac{2x}{n-1}u}\,\mathrm{d}u. \tag{7}$$

We now suppose that $0 \leqslant \nu < \frac{1}{n}$ and prove that under this condition inequality (5) is strict. For $u > 0$, we have

$$\begin{aligned}\frac{n}{n-1}\left(\mathrm{e}^{2u}-1\right) &= \frac{n}{n-1}\sum_{k=1}^{\infty}\frac{(2u)^k}{k!} \\ &> \sum_{k=1}^{n}\frac{(2u)^k}{k!}\frac{n}{n-1}\times\frac{n-1}{n-1}\times\cdots\times\frac{n-k+1}{n-1} \\ &= \sum_{k=1}^{n}\frac{(2u)^k}{k!}\frac{n!}{(n-k)!}\cdot\frac{1}{(n-1)^k} \\ &= \sum_{k=1}^{n}\binom{n}{k}\left(\frac{2u}{n-1}\right)^k = \left(1+\frac{2u}{n-1}\right)^n - 1.\end{aligned}$$

Applying this inequality to (7) yields

$$\begin{aligned}\lambda_\nu^{(n)}(x) &> \frac{(2\pi)^{(n-1)/2}\sqrt{n}\left(\frac{x}{n}\right)^{n\nu}}{\Gamma(\nu+1-\frac{1}{n})}\frac{2}{n-1}\left(\frac{n}{n-1}\right)^{\nu-\frac{1}{n}}\mathrm{e}^{-x}\int_0^\infty \mathrm{e}^{-2xu}(\mathrm{e}^{2u}-1)^{\nu-\frac{1}{n}}\,\mathrm{d}u \\ &= \frac{(2\pi)^{(n-1)/2}\sqrt{n}\left(\frac{x}{n}\right)^{n\nu}}{\Gamma(\nu+1-\frac{1}{n})}\frac{2}{n-1}\left(\frac{n}{n-1}\right)^{\nu-\frac{1}{n}} \\ &\quad \times \mathrm{e}^{-x}\int_0^\infty \mathrm{e}^{-(\frac{2x}{n-1}+\frac{2}{n}-2\nu)u}(1-\mathrm{e}^{-2u})^{\nu-\frac{1}{n}}\,\mathrm{d}u.\end{aligned} \tag{8}$$

Making the change of variables $y = \mathrm{e}^{-2u}$ gives

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-(\frac{2x}{n-1}+\frac{2}{n}-2\nu)u}(1-\mathrm{e}^{-2u})^{\nu-\frac{1}{n}}\,\mathrm{d}u &= \frac{1}{2}\int_0^1 (1-y)^{\nu-\frac{1}{n}}y^{\frac{x}{n-1}+\frac{1}{n}-\nu-1}\,\mathrm{d}y \\ &= \frac{1}{2}B(\nu+1-\tfrac{1}{n},\tfrac{x}{n-1}+\tfrac{1}{n}-\nu) \\ &= \frac{\Gamma(\nu+1-\frac{1}{n})\Gamma(\frac{x}{n-1}+\frac{1}{n}-\nu)}{2\Gamma(\frac{x}{n-1}+1)},\end{aligned}$$

where $B(a,b)$ is the beta function, and we used the standard formula $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. This completes the proof that inequality (5) holds for $0 \leqslant \nu < \frac{1}{n}$. When $\nu > \frac{1}{n}$ inequality (8) is reversed and so inequality (5) is also reversed. Note that when $\nu > \frac{1}{n}$ the integral in inequality (8) only exists if $x > (n-1)(\nu - \frac{1}{n})$. Finally, we note that we have equality when $\nu = \frac{1}{n}$, because (8) becomes an equality in this case. □

COROLLARY 1. *Let $0 \leqslant \nu < \frac{1}{2}$. Then for all $x > 0$,*

$$\left(\frac{x}{x+\frac{1}{2}-\nu}\right)^{\nu+\frac{1}{2}} < \sqrt{\frac{2}{\pi}}\mathrm{e}^x K_\nu(x) < 1.$$

*Proof.* The upper bound holds because $K_\nu(x) < K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}$ for all $\nu < \frac{1}{2}$ (see [5]). The lower bound follows from Theorem 1 and an application of the inequality $\frac{\Gamma(x+a)}{\Gamma(x+1)} > \frac{1}{(x+a)^{1-a}}$ for $0 < a < 1$ (see [4]). □

REMARK 1. The following bounds for $K_\nu(x)$ were obtained by [7]:

$$1-\frac{\frac{1}{2}(\frac{1}{4}-\nu^2)}{x+\frac{1}{2}(\frac{1}{4}-\nu^2)} < \sqrt{\frac{2x}{\pi}}\mathrm{e}^x K_\nu(x) < 1-\frac{\frac{1}{2}(\frac{1}{4}-\nu^2)}{x+\frac{1}{4}(\frac{9}{4}-\nu^2)}, \quad x>0,\ 0\leqslant \nu < \tfrac{1}{2}.$$

Despite taking a relatively simple form, numerical experiments show that, for $0 \leqslant \nu < \frac{1}{2}$, the bounds of [7] and Theorem 1, part (ii) are remarkably accurate for all but very small $x$, for which the modified Bessel function $K_\nu(x)$ has a singularity as $x \downarrow 0$. The bound (6) outperforms the lower bound of [7] for very small $x$, as it is $O(x^\nu)$ as $x \downarrow 0$, as opposed to $O(x^{1/2})$ which is the case for that bound of [7]. However, the bound of [7] performs better for large $x$.

## REFERENCES


[1] Á. BARICZ, *Bounds for modified Bessel functions of the first and second kinds*, P. Edinburgh Math. Soc. **53** (2010), 575–599.

[2] J. A. BARRIOS AND J. J. BETANCOR, *The Krätzel integral transformation of distributions*, Math Nachr. **154** (1991), 11–26.

[3] R. E. GAUNT, *Inequalities for modified Bessel functions and their integrals*, J. Math. Anal. Appl. **420** (2014), 373–386.

[4] W. GAUTSCHI, *Some elementary inequalities relating to the gamma and incomplete gamma function*, J. Math. Phys. **38** (1959), 77–81.

[5] E. K. IFANTIS AND P. D. SIAFARIKAS, *Inequalities involving Bessel and modified Bessel functions*, J. Math. Anal. Appl. **147** (1990), 214–227.

[6] E. KRÄTZEL, *Eine Verallgemeinerung der Laplace und Meijer transformation*, Wiss. Z. Univ. Jena. Math. Naturw. Reihe. **5** (1965), 369–381.

[7] Y. L. LUKE, *Inequalities for generalized hypergeometric functions*, J. Approx. Theory **5** (1972), 41–65.

[8] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT AND C. W. CLARK (Eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.


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*Robert E. Gaunt*
*School of Mathematics*
*The University of Manchester*
*Manchester, M13 9PL, UK*
*e-mail:* robert.gaunt@manchester.ac.uk