Mathematical
Inequalities
& Applications
Volume 20, Number 4 (2017), 1003–1016 doi:10.7153/mia-20-64

SERIES REPRESENTATIONS OF THE REMAINDERS IN THE EXPANSIONS FOR CERTAIN TRIGONOMETRIC FUNCTIONS AND SOME RELATED INEQUALITIES, I

CHAO-PING CHEN AND RICHARD B. PARIS

(*Communicated by J. Pečarić*)

Abstract. We present series representations of the remainders in the expansions for certain trigonometric and hyperbolic functions. From these results, we establish some inequalities for trigonometric and hyperbolic functions.

1. Introduction

The Bernoulli numbers B_n and Euler numbers E_n are defined, respectively, by the following generating functions:

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}\quad (|t|<2\pi)\quad\text{and}\quad \sec t=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}\quad (|t|<\pi).$$

Series representations of the remainders in the expansions for $2/(e^t+1)$, $\operatorname{sech} t$ and $\coth t$ can be found in [5, 11]. For example, for $t>0$ and $N\in\mathbb{N}:=\{1,2,\ldots\}$,

$$\operatorname{sech} t=\sum_{j=0}^{N-1}\frac{E_{2j}}{(2j)!}t^{2j}+R_N(t)$$

with

$$R_N(t)=\frac{(-1)^N 2t^{2N}}{\pi^{2N-1}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+\frac{1}{2})^{2N-1}\left(t^2+\pi^2(k+\frac{1}{2})^2\right)},$$

and, in addition,

$$\operatorname{sech} t=\sum_{j=0}^{N-1}\frac{E_{2j}}{(2j)!}t^{2j}+\Theta(t,N)\frac{E_{2N}}{(2N)!}t^{2N}$$

Mathematics subject classification (2010): 11B68, 26D05.
Keywords and phrases: Bernoulli numbers, Euler numbers, trigonometric function, hyperbolic function, inequalities.

with a suitable $0 < \Theta(t,N) < 1$. By using the obtained results, Chen and Paris [5] deduced some inequalities and completely monotonic functions associated with the ratio of gamma functions.

This paper is a continuation of our earlier work [5]. In Part I we present series representations of the remainders in the expansions for certain trigonometric and hyperbolic functions. From these results, we establish some inequalities for trigonometric and hyperbolic functions.

2. Series representations of the remainders

In this section we present expansions for several trigonometric and hyperbolic functions together with expressions for their remainders. Here, and throughout this paper, an empty sum is understood to be zero.

THEOREM 1. *Let $N \geqslant 0$ be an integer. Then for $|t| < \pi/2$, we have*

$$\tan t = \sum_{j=1}^{N} \frac{2^{2j}(2^{2j}-1)|B_{2j}|}{(2j)!} t^{2j-1} + \vartheta_N(t), \tag{1}$$

where

$$\vartheta_N(t) = \frac{2^{2N+3}t^{2N+1}}{\pi^{2N}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2N}\left(\pi^2(2k-1)^2-4t^2\right)}. \tag{2}$$

Proof. It follows from [10, p. 44] that

$$\tan\frac{\pi x}{2} = \frac{4x}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2-x^2}.$$

Replacement of x by $2t/\pi$ yields

$$\tan t = \sum_{k=1}^{\infty}\frac{8t}{\pi^2(2k-1)^2-4t^2}, \tag{3}$$

which can be written as

$$\tan t = \frac{8t}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2\left(1-\left(\frac{2t}{\pi(2k-1)}\right)^2\right)}. \tag{4}$$

Using the following identities:

$$\frac{1}{1-q} = \sum_{j=0}^{N-1} q^j + \frac{q^N}{1-q} \qquad (q \neq 1) \tag{5}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n}} = \frac{(2^{2n}-1)\pi^{2n}|B_{2n}|}{2\cdot(2n)!} \tag{6}$$

(see [10, p. 8]), we obtain from (4) that

$$\begin{aligned}\tan t &= \frac{8t}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\left(\sum_{j=0}^{N-1}\left(\frac{2t}{\pi(2k-1)}\right)^{2j} + \frac{\left(\frac{2t}{\pi(2k-1)}\right)^{2N}}{1-\left(\frac{2t}{\pi(2k-1)}\right)^2}\right)\\ &= \sum_{j=1}^{N}\frac{2^{2j}(2^{2j}-1)|B_{2j}|}{(2j)!}t^{2j-1} + \vartheta_N(t),\end{aligned}$$

where

$$\vartheta_N(t) = \frac{2^{2N+3}t^{2N+1}}{\pi^{2N}}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^{2N}\left(\pi^2(2k-1)^2-4t^2\right)}.$$

The proof of Theorem 1 is complete. □

Becker and Stark [3] showed that for $0 < x < \pi/2$,

$$\frac{8}{\pi^2-4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2-4x^2}. \tag{7}$$

The constant 8 and π^2 are the best possible. The Becker–Stark inequality (7) has attracted much interest of many mathematicians and has motivated a large number of research papers (cf. [2, 4, 8, 9, 12, 14, 16, 17, 18] and the references cited therein). For example, Banjac et al. [2, Theorem 2.7] proved recently that for $0 < x < \pi/2$,

$$\frac{\pi^2+\left(\frac{\pi^2}{3}-4\right)x^2+\left(\frac{\pi^2}{18}-\frac{2}{3}\right)x^4}{\pi^2-4x^2} < \frac{\tan x}{x} < \frac{\pi^2-\frac{\pi^2}{16}x^2+\frac{1}{2}x^4-\frac{1}{\pi^2}x^6}{\pi^2-4x^2}. \tag{8}$$

There is no strict comparison between the two lower bounds in (7) and (8). The upper bound in (8) is sharper than that in (7).

Here we shall improve on the above inequalities. Write (1) as

$$\begin{aligned}\frac{\tan t}{t} &= \sum_{j=1}^{N}\frac{2^{2j}(2^{2j}-1)|B_{2j}|}{(2j)!}t^{2j-2}\\ &\quad + \frac{2^{2N+3}t^{2N}}{\pi^{2N}}\left\{\frac{1}{\pi^2-4t^2} + \sum_{k=2}^{\infty}\frac{1}{(2k-1)^{2N}\left(\pi^2(2k-1)^2-4t^2\right)}\right\}.\end{aligned} \tag{9}$$

Noting that the function

$$F(t) := \sum_{k=2}^{\infty}\frac{1}{(2k-1)^{2N}\left(\pi^2(2k-1)^2-4t^2\right)}$$

is strictly increasing for $0 < t < \pi/2$, we then obtain from (9) that for $0 < t < \pi/2$,

$$\frac{2^{2N+3}t^{2N}}{\pi^{2N+2}} \sum_{k=2}^{\infty} \frac{1}{(2k-1)^{2N+2}} < \frac{\tan t}{t} - \sum_{j=1}^{N} \frac{2^{2j}(2^{2j}-1)|B_{2j}|}{(2j)!} t^{2j-2} - \frac{2^{2N+3}t^{2N}}{\pi^{2N}(\pi^2-4t^2)}$$
$$< \frac{2^{2N+1}t^{2N}}{\pi^{2N+2}} \sum_{k=2}^{\infty} \frac{1}{k(k-1)(2k-1)^{2N}}. \tag{10}$$

Direct computations yield

$$\sum_{k=2}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96} - 1, \quad \sum_{k=2}^{\infty} \frac{1}{k(k-1)(2k-1)^2} = 5 - \frac{\pi^2}{2}.$$

The choice $N = 1$ in (10) therefore yields

$$\frac{32t^2}{\pi^4}\left(\frac{\pi^4}{96} - 1\right) < \frac{\tan t}{t} - 1 - \frac{32t^2}{\pi^2(\pi^2-4t^2)} < \frac{8t^2}{\pi^4}\left(5 - \frac{\pi^2}{2}\right), \qquad 0 < t < \frac{\pi}{2},$$

which can be rearranged for $0 < x < \pi/2$ as

$$\frac{\pi^2 + \frac{\pi^2-12}{3}x^2 + \frac{384-4\pi^4}{3\pi^4}x^4}{\pi^2-4x^2} < \frac{\tan x}{x} < \frac{\pi^2 + \frac{72-8\pi^2}{\pi^2}x^2 + \frac{16\pi^2-160}{\pi^4}x^4}{\pi^2-4x^2}. \tag{11}$$

The inequality (11) improves the inequalities (7) and (8).

THEOREM 2. *Let $N \geqslant 0$ be an integer. Then for all $t \in \mathbb{R}$, we have*

$$\tanh t = \sum_{j=1}^{N} \frac{2^{2j}(2^{2j}-1)B_{2j}}{(2j)!} t^{2j-1} + \tau_N(t), \tag{12}$$

where

$$\tau_N(t) = (-1)^N \frac{2^{2N+3}t^{2N+1}}{\pi^{2N}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2N}\left(\pi^2(2k-1)^2 + 4t^2\right)}, \tag{13}$$

and

$$\tanh t = \sum_{j=1}^{N} \frac{2^{2j}(2^{2j}-1)B_{2j}}{(2j)!} t^{2j-1} + \xi(t,N) \frac{2^{2N+2}(2^{2N+2}-1)B_{2N+2}}{(2N+2)!} t^{2N+1}, \tag{14}$$

where $0 < \xi(t,N) < 1$.

Proof. It follows from [10, p. 44] that

$$\tanh \frac{\pi x}{2} = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 + x^2}. \tag{15}$$

Replacement of x by $2t/\pi$ yields

$$\tanh t = \sum_{k=1}^{\infty} \frac{8t}{\pi^2(2k-1)^2+4t^2} = \frac{8t}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2\left(1+\left(\frac{2t}{\pi(2k-1)}\right)^2\right)}. \tag{16}$$

Using the identity

$$\frac{1}{1+q} = \sum_{j=0}^{N-1}(-1)^j q^j + (-1)^N \frac{q^N}{1+q} \qquad (q \neq -1) \tag{17}$$

and (6), we obtain from (16) that

$$\begin{aligned} \tanh t &= \frac{8t}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \left(\sum_{j=0}^{N-1} (-1)^j \left(\frac{2t}{\pi(2k-1)}\right)^{2j} + (-1)^N \frac{\left(\frac{2t}{\pi(2k-1)}\right)^{2N}}{1+\left(\frac{2t}{\pi(2k-1)}\right)^2} \right) \\ &= \sum_{j=1}^{N} \frac{2^{2j}(2^{2j}-1)B_{2j}}{(2j)!} t^{2j-1} + \tau_N(t), \end{aligned}$$

where

$$\tau_N(t) = (-1)^N \frac{2^{2N+3} t^{2N+1}}{\pi^{2N}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2N}\left(\pi^2(2k-1)^2+4t^2\right)}.$$

Noting that (6) holds, we can rewrite $\tau_N(t)$ as

$$\tau_N(t) = \xi(t,N) \frac{2^{2N+2}(2^{2N+2}-1)B_{2N+2}}{(2N+2)!} t^{2N+1},$$

where

$$\xi(t,N) := \frac{g(t)}{g(0)}, \qquad g(t) := \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2N}\left(\pi^2(2k-1)^2+4t^2\right)}.$$

Obviously, the even function $g(t) > 0$ and is strictly decreasing for $t > 0$. Hence, for $t \neq 0$, $0 < g(t) < g(0)$ and thus $0 < \xi(t,N) < 1$. The proof of Theorem 2 is complete. □

From (12), we obtain the following

COROLLARY 1. *For $t \neq 0$, we have*

$$(-1)^N \left(\frac{\tanh t}{t} - \sum_{j=1}^{N} \frac{2^{2j}(2^{2j}-1)B_{2j}}{(2j)!} t^{2j-2} \right) > 0,$$

that is,

$$\sum_{j=1}^{2m} \frac{2^{2j}(2^{2j}-1)B_{2j}}{(2j)!} t^{2j-2} < \frac{\tanh t}{t} < \sum_{j=1}^{2m-1} \frac{2^{2j}(2^{2j}-1)B_{2j}}{(2j)!} t^{2j-2}. \tag{18}$$

We now establish an inequality for $\tanh t/t$ analogous to that in (8). Write (12) as

$$(-1)^N\left(\frac{\tanh t}{t}-\sum_{j=1}^{N}\frac{2^{2j}(2^{2j}-1)B_{2j}}{(2j)!}t^{2j-2}\right)$$
$$=\frac{2^{2N+3}t^{2N}}{\pi^{2N}}\left\{\frac{1}{\pi^2+4t^2}+\sum_{k=2}^{\infty}\frac{1}{(2k-1)^{2N}\left(\pi^2(2k-1)^2+4t^2\right)}\right\}. \tag{19}$$

Noting that the even function

$$G(t):=\sum_{k=2}^{\infty}\frac{1}{(2k-1)^{2N}\left(\pi^2(2k-1)^2+4t^2\right)}$$

is strictly decreasing for $t>0$, we then obtain from (19) that for $t\neq 0$,

$$(-1)^N\left(\frac{\tanh t}{t}-\sum_{j=1}^{N}\frac{2^{2j}(2^{2j}-1)B_{2j}}{(2j)!}t^{2j-2}\right)$$
$$<\frac{2^{2N+3}t^{2N}}{\pi^{2N}}\left\{\frac{1}{\pi^2+4t^2}+\sum_{k=2}^{\infty}\frac{1}{\pi^2(2k-1)^{2N+2}}\right\}. \tag{20}$$

The choice $N=1$ and $N=2$ in (20), respectively, yields

$$1-\frac{32t^2}{\pi^2(\pi^2+4t^2)}-\frac{32t^2}{\pi^4}\sum_{k=2}^{\infty}\frac{1}{(2k-1)^4}<\frac{\tanh t}{t} \tag{21}$$

and

$$\frac{\tanh t}{t}<1-\frac{1}{3}t^2+\frac{128t^4}{\pi^4(\pi^2+4t^2)}+\frac{128t^4}{\pi^6}\sum_{k=2}^{\infty}\frac{1}{(2k-1)^6}. \tag{22}$$

Noting that

$$\sum_{k=2}^{\infty}\frac{1}{(2k-1)^4}=\frac{\pi^4}{96}-1,\quad \sum_{k=2}^{\infty}\frac{1}{(2k-1)^6}=\frac{\pi^6}{960}-1,$$

we obtain from (21) and (22) that for $t\neq 0$,

$$\frac{\pi^2+\left(4-\frac{\pi^2}{3}\right)t^2-\left(\frac{4}{3}-\frac{128}{\pi^4}\right)t^4}{\pi^2+4t^2}<\frac{\tanh t}{t}$$
$$<\frac{\pi^2+\left(4-\frac{\pi^2}{3}\right)t^2-\left(\frac{4}{3}-\frac{2\pi^2}{15}\right)t^4+\left(\frac{8}{15}-\frac{512}{\pi^6}\right)t^6}{\pi^2+4t^2}, \tag{23}$$

which is an analogous result to (8).

THEOREM 3. *Let $N \geqslant 0$ be an integer. Then for $|t| < \pi/2$, we have*

$$\sec t = \sum_{j=0}^{N-1} \frac{|E_{2j}|}{(2j)!} t^{2j} + \omega_N(t), \tag{24}$$

where

$$\omega_N(t) = \frac{2^{2N+2} t^{2N}}{\pi^{2N-1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^{2N-1}\left(\pi^2(2k-1)^2 - 4t^2\right)}. \tag{25}$$

Proof. It follows from [10, p. 44] that

$$\sec \frac{\pi x}{2} = \frac{4}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k-1}{(2k-1)^2 - x^2}.$$

Replacement of x by $2t/\pi$ yields

$$\sec t = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)\left(1 - \left(\frac{2t}{\pi(2k-1)}\right)^2\right)}. \tag{26}$$

Using (5) and the following identity:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+1}(2n)!} |E_{2n}| \tag{27}$$

(see [10, p. 8]), we obtain from (26) that

$$\begin{aligned}
\sec t &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \left(\sum_{j=0}^{N-1} \left(\frac{2t}{\pi(2k-1)} \right)^{2j} + \frac{\left(\frac{2t}{\pi(2k-1)}\right)^{2N}}{1 - \left(\frac{2t}{\pi(2k-1)}\right)^2} \right) \\
&= \sum_{j=0}^{N-1} \frac{|E_{2j}|}{(2j)!} t^{2j} + \omega_N(t),
\end{aligned}$$

where

$$\omega_N(t) = \frac{2^{2N+2} t^{2N}}{\pi^{2N-1}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k-1)^{2N-1}\left(\pi^2(2k-1)^2 - 4t^2\right)}.$$

The proof of Theorem 3 is complete. □

Chen and Sandor [7, Theorem 3.1(i)] proved that for $0 < |t| < \pi/2$,

$$\frac{\pi^2}{\pi^2 - 4t^2} < \sec t < \frac{4\pi}{\pi^2 - 4t^2}. \tag{28}$$

The constants π^2 and 4π are best possible. To improve on this inequality write (24) as

$$\sec t = \sum_{j=0}^{N-1}\frac{|E_{2j}|}{(2j)!}t^{2j} + \frac{2^{2N+2}t^{2N}}{\pi^{2N-1}}\left\{\frac{1}{\pi^2-4t^2} + \sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^{2N-1}\left(\pi^2(2k-1)^2-4t^2\right)}\right\}$$
$$= \sum_{j=0}^{N-1}\frac{|E_{2j}|}{(2j)!}t^{2j} + \frac{2^{2N+2}t^{2N}}{\pi^{2N-1}(\pi^2-4t^2)}$$
$$+ \frac{2^{2N+2}t^{2N}}{\pi^{2N-1}}\sum_{k=2}^{\infty}(-1)^{k+1}\frac{1}{(2k-1)^{2N-1}\left(\pi^2(2k-1)^2-4t^2\right)}. \tag{29}$$

Let

$$H(t) = \sum_{k=2}^{\infty}(-1)^{k+1}\frac{1}{(2k-1)^{2N-1}\left(\pi^2(2k-1)^2-4t^2\right)}.$$

Differentiation yields

$$H'(t) = -8t\sum_{k=2}^{\infty}(-1)^k\eta_k, \quad \eta_k = \frac{1}{(2k-1)^{2N-1}\left(\pi^2(2k-1)^2-4t^2\right)^2}.$$

Then it is easily seen that $\eta_{2k} > \eta_{2k+1}$ for $k \in \mathbb{N}$, $0 < t < \pi/2$ and $N \in \mathbb{N}$; thus $H'(t) < 0$ for $0 < t < \pi/2$. Hence, for all $0 < t < \pi/2$ and $N \in \mathbb{N}$, we have $H(\pi/2) < H(t) < H(0)$. We then obtain from (29) that for $0 < |t| < \pi/2$,

$$\sum_{j=0}^{N-1}\frac{|E_{2j}|}{(2j)!}t^{2j} + \frac{2^{2N+2}t^{2N}}{\pi^{2N-1}(\pi^2-4t^2)} + \frac{2^{2N}t^{2N}}{\pi^{2N+1}}\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k(k-1)(2k-1)^{2N-1}}$$
$$< \sec t < \sum_{j=0}^{N-1}\frac{|E_{2j}|}{(2j)!}t^{2j} + \frac{2^{2N+2}t^{2N}}{\pi^{2N-1}(\pi^2-4t^2)} + \frac{2^{2N+2}t^{2N}}{\pi^{2N+1}}\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^{2N+1}}. \tag{30}$$

Direct computations yield

$$\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k(k-1)(2k-1)} = 3-\pi, \qquad \sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}-1.$$

The choice $N = 1$ in (30) then yields, for $0 < |t| < \pi/2$,

$$\frac{\pi^2 + \frac{28-8\pi}{\pi}t^2 + \frac{-48+16\pi}{\pi^3}t^4}{\pi^2-4t^2} < \sec t < \frac{\pi^2 - \frac{8-\pi^2}{2}t^2 - \frac{4\pi^3-128}{2\pi^3}t^4}{\pi^2-4t^2}, \tag{31}$$

which improves the inequality (28).

THEOREM 4. *For $0 < |t| < \pi$, we have*

$$\cot t = \frac{1}{t} - \sum_{j=1}^{N}\frac{2^{2j}|B_{2j}|}{(2j)!}t^{2j-1} + \theta_N(t), \tag{32}$$

where

$$\theta_N(t) = \frac{2t^{2N+1}}{\pi^{2N}} \sum_{k=1}^{\infty} \frac{1}{k^{2N}\left(t^2 - \pi^2 k^2\right)}. \tag{33}$$

Proof. It follows from [13, p. 118] that

$$\cot t = \frac{1}{t} + 2t \sum_{k=1}^{\infty} \frac{1}{t^2 - \pi^2 k^2}, \tag{34}$$

which can be written as

$$\cot t = \frac{1}{t} - 2t \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2 \left(1 - \left(\frac{t}{k\pi}\right)^2\right)}. \tag{35}$$

Using (5) and the following identity:

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{2^{2n-1}\pi^{2n}}{(2n)!} |B_{2n}| \tag{36}$$

(see [10, p. 8]), we obtain from (35) that

$$\begin{aligned} \cot t &= \frac{1}{t} - 2t \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} \left(\sum_{j=0}^{N-1} \left(\frac{t}{k\pi}\right)^{2j} + \frac{\left(\frac{t}{k\pi}\right)^{2N}}{1 - \left(\frac{t}{k\pi}\right)^2} \right) \\ &= \frac{1}{t} - 2 \sum_{j=1}^{N} \frac{2^{2j-1}|B_{2j}|}{(2j)!} t^{2j-1} + \theta_N(t), \end{aligned}$$

where

$$\theta_N(t) = \frac{2t^{2N+1}}{\pi^{2N}} \sum_{k=1}^{\infty} \frac{1}{k^{2N}\left(t^2 - \pi^2 k^2\right)}.$$

The proof of Theorem 4 is complete. □

THEOREM 5. *For* $0 < |t| < \pi$, *we have*

$$\csc t = \frac{1}{t} + \sum_{j=1}^{N} \frac{(2^{2j} - 2)|B_{2j}|}{(2j)!} t^{2j-1} + r_N(t), \tag{37}$$

where

$$r_N(t) = \frac{2t^{2N+1}}{\pi^{2N}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2N}\left(\pi^2 k^2 - t^2\right)}. \tag{38}$$

Proof. It follows from [13, p. 118] that

$$\csc t = \frac{1}{t} + 2t\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k\pi)^2 - t^2}, \tag{39}$$

which can be written as

$$\csc t = \frac{1}{t} + 2t\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k\pi)^2\left(1-\left(\frac{t}{k\pi}\right)^2\right)}. \tag{40}$$

Using (5) and the following identity:

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{2n}} = \frac{(2^{2n-1}-1)\pi^{2n}}{(2n)!}|B_{2n}| \tag{41}$$

(see [10, p. 8]), we obtain from (40) that

$$\begin{aligned}\csc t &= \frac{1}{t} + 2t\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k\pi)^2}\left(\sum_{j=0}^{N-1}\left(\frac{t}{k\pi}\right)^{2j} + \frac{\left(\frac{t}{k\pi}\right)^{2N}}{1-\left(\frac{t}{k\pi}\right)^2}\right)\\ &= \frac{1}{t} + \sum_{j=1}^{N}\frac{(2^{2j}-2)|B_{2j}|}{(2j)!}t^{2j-1} + r_N(t),\end{aligned}$$

where

$$r_N(t) = \frac{2t^{2N+1}}{\pi^{2N}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{2N}\left(\pi^2k^2 - t^2\right)}.$$

The proof of Theorem 5 is complete. □

Theorems 4 and 5 will be used in Part II.

3. A double inequality for the remainder in the expansion for $\sec x$

Let $S_n(x)$ denote

$$S_n(x) = \sum_{k=1}^{n}\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}x^{2k-1}, \qquad |x| < \frac{\pi}{2}.$$

By using induction, Chen and Qi [6] (see also [15]) established a double inequality for the difference $\tan x - S_n(x)$:

$$\frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!}x^{2n}\tan x < \tan x - S_n(x) < \left(\frac{2}{\pi}\right)^{2n}x^{2n}\tan x \tag{42}$$

for $0 < x < \pi/2$ and $n \in \mathbb{N}$, where the the constants

$$\frac{2^{2n+2}(2^{2n+2}-1)|B_{2n+2}|}{(2n+2)!} \quad\text{and}\quad \left(\frac{2}{\pi}\right)^{2n}$$

are the best possible.

It is well known [10, p. 43] that

$$\sec x = \sum_{j=0}^{\infty} \frac{|E_{2j}|}{(2j)!} x^{2j}, \qquad |x| < \frac{\pi}{2}. \tag{43}$$

Let $s_N(x)$ denote

$$s_N(x) = \sum_{j=0}^{N-1} \frac{|E_{2j}|}{(2j)!} x^{2j}, \qquad |x| < \frac{\pi}{2}. \tag{44}$$

In this section, we establish a double inequality for the difference $\sec x - s_N(x)$, which is an analogous result to (42) given by Theorem 6.

THEOREM 6. *Let $N \geqslant 0$ be an integer. Then for $0 < x < \pi/2$, we have*

$$\frac{|E_{2N}|}{(2N)!} x^{2N-1} \tan x < \sec x - s_N(x) < \left(\frac{2}{\pi}\right)^{2N-1} x^{2N-1} \tan x, \tag{45}$$

where the constants $|E_{2N}|/(2N)!$ and $(2/\pi)^{2N-1}$ are the best possible.

Proof. From the expansion [10, p. 42]

$$\tan x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!} x^{2k-1}, \qquad |x| < \frac{\pi}{2}$$

and (43), the left-hand side inequality (45) can be written for $0 < x < \pi/2$ as

$$\sum_{j=N}^{\infty} \frac{|E_{2N}|}{(2N)!} \frac{2^{2j-2N+2}(2^{2j-2N+2}-1)|B_{2j-2N+2}|}{(2j-2N+2)!} x^{2j} < \sum_{j=N}^{\infty} \frac{|E_{2j}|}{(2j)!} x^{2j},$$

or

$$\sum_{j=N+1}^{\infty} \left\{ \frac{|E_{2N}|}{(2N)!} \frac{2^{2j-2N+2}(2^{2j-2N+2}-1)|B_{2j-2N+2}|}{(2j-2N+2)!} - \frac{|E_{2j}|}{(2j)!} \right\} x^{2j} < 0.$$

We now prove that

$$\frac{|E_{2N}|}{(2N)!} \frac{2^{2j-2N+2}(2^{2j-2N+2}-1)|B_{2j-2N+2}|}{(2j-2N+2)!} < \frac{|E_{2j}|}{(2j)!}, \qquad j \geqslant N+1. \tag{46}$$

Using the following inequalities (see [1, p. 805])

$$\frac{2}{(2\pi)^{2n}(1-2^{1-2n})} > \frac{|B_{2n}|}{(2n)!} > \frac{2}{(2\pi)^{2n}}, \qquad n \geqslant 1,$$

$$\frac{4^{n+1}}{\pi^{2n+1}}\left(\frac{1}{1+3^{-1-2n}}\right) < \frac{|E_{2n}|}{(2n)!} < \frac{4^{n+1}}{\pi^{2n+1}}, \qquad n = 0,1,2,\ldots,$$

it suffices to show that

$$\frac{4^{N+1}}{\pi^{2N+1}} \frac{2^{2j-2N+2}(2^{2j-2N+2}-1)2}{(2\pi)^{2j-2N+2}\left(1-2^{1-2(j-N+1)}\right)} < \frac{4^{j+1}}{\pi^{2j+1}}\left(\frac{1}{1+3^{-1-2j}}\right), \qquad j \geqslant N+1,$$

which can be rearranged as

$$\frac{8}{\pi^2}\frac{4^{j-N+1}-1}{4^{j-N+1}-2} < \frac{3^{2j+1}}{3^{2j+1}+1},$$

$$\frac{8}{\pi^2}\left(1+\frac{1}{4^{j-N+1}-2}\right) < 1-\frac{1}{3^{2j+1}+1},$$

$$\frac{8}{\pi^2(4^{j-N+1}-2)} + \frac{1}{3^{2j+1}+1} < 1-\frac{8}{\pi^2}.$$

Noting that the sequence

$$\frac{8}{\pi^2(4^{j-N+1}-2)} + \frac{1}{3^{2j+1}+1}$$

is strictly decreasing for $j \geqslant N+1$, it is enough to prove the following inequality:

$$\frac{4}{7\pi^2} + \frac{1}{3^{2N+3}+1} < 1-\frac{8}{\pi^2},$$

which can be rearranged as

$$3^{2N+3} > \frac{60}{7\pi^2-60} = 6.60267151\ldots. \tag{47}$$

Obviously, (47) holds for all integers $N \geqslant 0$. This proves (46). Hence, the left-hand side inequality (45) holds.

By Theorem 3 and (3), the right-hand side inequality (45) can be rearranged for $0 < x < \pi/2$ as

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^{2N-1}\left(\pi^2(2k-1)^2-4x^2\right)} < \sum_{k=1}^{\infty} \frac{1}{\pi^2(2k-1)^2-4x^2},$$

or

$$\sum_{k=2}^{\infty}\left\{1-\frac{(-1)^{k+1}}{(2k-1)^{2N-1}}\right\}\frac{1}{\pi^2(2k-1)^2-4x^2} > 0. \tag{48}$$

Obviously, (48) holds. Hence, the right-hand side inequality (45) holds.

Write (45) as

$$\frac{|E_{2N}|}{(2N)!} < \frac{\sec x - \sum_{j=0}^{N-1} \frac{|E_{2j}|}{(2j)!} x^{2j}}{x^{2N-1} \tan x} < \left(\frac{2}{\pi}\right)^{2N-1}.$$

We find

$$\lim_{x \to 0} \frac{\sec x - \sum_{j=0}^{N-1} \frac{|E_{2j}|}{(2j)!} x^{2j}}{x^{2N-1} \tan x} = \frac{|E_{2N}|}{(2N)!}$$

and

$$\lim_{x \to \frac{\pi}{2}} \frac{\sec x - \sum_{j=0}^{N-1} \frac{|E_{2j}|}{(2j)!} x^{2j}}{x^{2N-1} \tan x} = \left(\frac{2}{\pi}\right)^{2N-1}.$$

Hence, the inequality (45) holds, where the constants $|E_{2N}|/(2N)!$ and $(2/\pi)^{2N-1}$ are the best possible. The proof of Theorem 6 is complete. □

Acknowledgements. The authors thank the referees for their careful reading of the manuscript and insightful comments. Some computations in this paper were performed using Maple software.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 9th printing, Washington, 1970.

[2] B. BANJAC, M. MAKRAGIĆ AND B. MALEŠEVIĆ, *Some notes on a method for proving inequalities by computer*, Results. Math. **69**, 1 (2016), 161–176.

[3] M. BECKER AND E. L. STRAK, *On a hierarchy of quolynomial inequalities for* $\tan x$, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **602–633** (1978), 133–138.

[4] C.-P. CHEN AND W.-S. CHEUNG, *Sharp Cusa and Becker–Stark inequalities*, J. Inequal. Appl. **2011** (2011), 136, http://www.journalofinequalitiesandapplications.com/content/2011/1/136.

[5] C.-P. CHEN AND R. B. PARIS, *Series representations of the remainders in the expansions for certain functions with applications*, Results. Math. 2016, DOI: 10.1007/s00025-016-0612-1.

[6] C.-P. CHEN AND F. QI, *A double inequality for remainder of power series of tangent function*, Tamkang J. Math. **34**, 4 (2003), 351–355.

[7] C.-P. CHEN AND J. SÁNDOR, *Sharp inequalities for trigonometric and hyperbolic functions*, J. Math. Inequal. **9**, 1 (2015), 203–217.

[8] L. DEBNATH, C. MORTICI AND L. ZHU, *Refinements of Jordan–Stečkin and Becker–Stark inequalities*, Results Math. **67** (2015), 207–215.

[9] H.-F. GE, *New sharp bounds for the Bernoulli numbers and refinement of Becker–Stark inequalities*, J. Appl. Math. **2012**, Article ID 137507, 7 pages.

[10] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of integrals, series, and products*, translated from the Russian, Sixth edition, translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, Academic Press, Inc., San Diego, CA, 2000.

[11] S. KOUMANDOS, *On completely monotonic and related functions*, Mathematics Without Boundaries, pp. 285–321. Springer, New York, 2014.

[12] Y. NISHIZAWA, *Sharp Becker–Stark's type inequalities with power exponential functions*, J. Inequal. Appl. 2015 (2015) 402, http://rd.springer.com/article/10.1186/s13660-015-0932-9/fulltext.html.
[13] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT, C. W. CLARKS (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010.
[14] Z.-J. SUN AND L. ZHU, *Simple proofs of the Cusa–Huygens–type and Becker–Stark–type inequalities*, J. Math. Inequal. **7** (2013), 563–567.
[15] J.-L. ZHAO, Q.-M. LUO, B.-N. GUO AND F. QI, *Remarks on inequalities for the tangent function*, Hacet. J. Math. Stat. **41**, 4 (2012), 499–506.
[16] L. ZHU, *Sharp Becker-Stark-type inequalities for Bessel functions*, J. Inequal. Appl. **2010**, Article ID 838740, 4 pages.
[17] L. ZHU, *A refinement of the Becker-Stark inequalities*, Math. Notes **93** (2013), 421–425.
[18] L. ZHU AND J. K. HUA, *Sharpening the Becker-Stark inequalities*, J. Inequal. Appl. **2010**, Article ID 931275, 4 pages.

(Received June 2, 2016)

Chao-Ping Chen
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454000, Henan Province, China
e-mail: chenchaoping@sohu.com

Richard B. Paris
Division of Computing and Mathematics
University of Abertay
Dundee, DD1 1HG, UK
e-mail: r.paris@abertay.ac.uk