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NEW HILBERT DYNAMIC INEQUALITIES ON TIME SCALES

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Abstract. In this paper, we prove some new dynamic inequalities of Hilbert type on time scales. From these inequalities, as special cases, we will formulate some special integral and discrete inequalities. The main results are proved using some algebraic inequalities, Hölder's inequality, Jensen's inequality and a chain rule on time scales.

1. Introduction

In the early 1900's Hilbert (see [13, 19]) discovered the inequality

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \leqslant 2\pi\left(\sum_{m=1}^{\infty}a_m^2\right)^{1/2}\left(\sum_{n=1}^{\infty}b_n^2\right)^{1/2}, \tag{1.1}$$

where $\{a_m\}_{m=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are nonnegative real sequences such that

$$\sum_{m=1}^{\infty}a_m^2 < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty}a_n^2 < \infty.$$

In 1911, Schur [17] gave the best constant π in (1.1), instead of 2π that was proposed by Hilbert, and proved an integral analogue with a best constant of Hilbert's inequality (1.1), namely

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy \leqslant \pi\left(\int_0^{\infty}f^2(x)dx\right)^{1/2}\left(\int_0^{\infty}g^2(x)dx\right)^{1/2}, \tag{1.2}$$

where f and g are measurable nonnegative functions such that

$$\int_0^{\infty}f^2(x)dx < \infty, \quad \text{and} \quad \int_0^{\infty}g^2(x)dx < \infty.$$

In 1925, Hardy [8], extended (1.1) by introducing a pair of conjugate exponents $(p,\ q)$ with $1/p+1/q=1$, and proved that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \leqslant \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{m=1}^{\infty}a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}b_n^q\right)^{1/q}, \tag{1.3}$$

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where $\{a_m\}_{m=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ are nonnegative real sequences such that

$$\sum_{m=1}^{\infty} a_m^p < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} a_n^q < \infty.$$

Hardy and Reisz [9] proved an integral analogue of (1.3) of the form

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x+y}dxdy \leqslant \frac{\pi}{\sin\frac{\pi}{p}}\left(\int_0^{\infty} f^p(x)dx\right)^{1/p}\left(\int_0^{\infty} g^q(x)dx\right)^{1/q}, \tag{1.4}$$

where f and g are measurable nonnegative functions such that

$$\int_0^{\infty} f^p(x)dx < \infty, \quad \text{and} \quad \int_0^{\infty} g^q(x)dx < \infty.$$

As a special case of (1.4), we get the inequality

$$\int_0^{\infty}\left(\int_0^{\infty}\frac{f(y)}{x+y}dy\right)^p dx \leqslant \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p \int_0^{\infty} f^p(x)dx. \tag{1.5}$$

The constants $\pi/\sin(\pi/p)$ and $(\pi/\sin(\pi/p))^p$ in (1.4) and (1.5) are the best possible.

In 1926, Hardy, Littlewood and Pólya [10] proved that

$$\int_0^{\infty}\left(\int_0^{\infty}\frac{f(x)}{\max\{x,y\}}dx\right)^p dy \leqslant p^p\int_0^{\infty} f^p(x)dx, \tag{1.6}$$

where $1 < p < \infty$, and $f \in L^p(0,\ \infty)$ be a nonnegative function. The constant $(p)^p$ is the best possible.

In 1929, Hardy [11] established a new inequality with a different kernel by replacing $1/(x+y)$ by the exponential function and proved that if $f(x) \geqslant 0$, and $p > 1$, then

$$\int_0^{\infty}\left(\int_0^{\infty} e^{-xy} f(y)dy\right)^p dx \leqslant \Gamma^p(1/p)\int_0^{\infty} x^{p-2} f^p(x)dx, \tag{1.7}$$

where Γ is the gamma function.

In 1933, Hardy [12] generalized the inequality (1.7) to inequalities with a general kernel and proved that

$$\begin{aligned}&\int_0^{\infty}\int_0^{\infty} K(xy)f(x)g(y)dxdy\\ &\leqslant \phi(1/p)\left(\int_0^{\infty} x^{p-2}f^p(x)dx\right)^{1/p}\left(\int_0^{\infty} g^q(x)dx\right)^{1/q},\end{aligned} \tag{1.8}$$

and

$$\int_0^{\infty}\left(\int_0^{\infty} K(xy)f(x)dy\right)^p dx \leqslant \phi^p(1/p)\left(\int_0^{\infty} x^{p-2}f^p(x)dx\right), \tag{1.9}$$

where $q = p/(p-1)$, and

$$\phi(s) := \int_0^\infty K(u)u^{s-1}du.$$

The constants $\phi(1/p)$ and $\phi^p(1/p)$ are the best possible. The above inequalities were studied extensively and numerous variants, generalizations, and extensions appeared in the literature. We refer the reader to the survey paper [6] which discusses the development of Hilbert type inequalities.

One of the generalizations of a Hilbert type inequality was given by Pachpatte in [14]. In particular, he proved that if $p,\ q \geqslant 1,\ A_m = \sum_{s=1}^m a_s \geqslant 0$ and $B_n = \sum_{t=1}^n b_t \geqslant 0$, then

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{m+n} \leqslant C(p,\ q,\ k,\ r)\left(\sum_{m=1}^{k}(k+1-m)(A_m^{p-1}a_m)^2\right)^{\frac{1}{2}} \times \left(\sum_{n=1}^{r}(r+1-n)(B_n^{q-1}b_n)^2\right)^{\frac{1}{2}}, \tag{1.10}$$

where

$$C(p,\ q,\ k,\ r) = \frac{1}{2}pq\sqrt{kr}.$$

In the same paper [14] Pachpatte proved the integral analogue of (1.10), namely

$$\int_0^a\int_0^b \frac{F^p(s)G^q(t)}{s+t}dsdt \leqslant D(p,\ q)\left(\int_0^a (a-s)(F^{p-1}(s)f(s))^2ds\right)^{\frac{1}{2}} \times \left(\int_0^b (b-t)(G^{q-1}(t)g(t))^2dt\right)^{\frac{1}{2}}, \tag{1.11}$$

where $p,\ q \geqslant 1,\ F(s) = \int_0^s f(\tau)d\tau \geqslant 0,\ G(t) = \int_0^t g(\nu)d\nu \geqslant 0$, and

$$D(p,\ q) = \frac{1}{2}pq\sqrt{ab}.$$

In recent years the study of dynamic inequalities on time scales has received a lot of attention, and we refer the reader to [1, 18, 15, 7, 4] and the references cited therein.

The general idea is to prove a result for a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$. The cases when the time scale is equal to the reals or to the integers represent the classical theories of integral and of discrete inequalities. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R},\ \mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. For more details on time scale analysis, we refer the reader to the two books by Bohner and Peterson [2], [3] which summarize and organize much of the time scales calculus.

The natural question arises now: Is it possible to prove new delta dynamic inequalities on an arbitrary time scale $\mathbb{T}$ that resemble generalizations of both the discrete and the continuous inequalities (1.10) and (1.11)?

The main aim of this paper is to give an affirmative answer to this question. In Section 2, we present some basic concepts on the calculus of time scales. In Section 3 we state and prove the main results and formulate some special integral and discrete inequalities.

2. Preliminaries and basic lemmas

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. The mapping $\mu : \mathbb{T} \to \mathbb{R}^{+} = [0,\infty)$ such that $\mu(t) := \sigma(t) - t$ is called graininess. A function $f : [a,\ b] \to \mathbb{R}$ is said to be right–dense continuous ($rd-$continuous) if it is right continuous at each right–dense point and there exists a finite left limit at all left–dense points, and f is said to be differentiable if its derivative exists. The space of $rd-$continuous functions is denoted by $C_{rd}(\mathbb{T},\ \mathbb{R})$. A useful formula is $f^{\sigma} = f + \mu f^{\Delta}$, where $f^{\sigma} := f \circ \sigma$.

THEOREM 2.1. *Assume $f,\ g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}$, then*

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma}. \tag{2.1}$$

Integration on time scales for delta differentiable functions is defined as follows. For $a,\ b \in \mathbb{T}$, and a delta differentiable function f the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a).$$

The integration by parts formula on time scales is given by

$$\int_a^b f(t) g^{\Delta}(t)\Delta t = f(t)g(t)|_a^b - \int_a^b f^{\Delta}(t) g^{\sigma}(t)\Delta t. \tag{2.2}$$

THEOREM 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f^{'} \left[g(t) + h\mu(t) g^{\Delta}(t) \right] dh \right\} g^{\Delta}(t), \tag{2.3}$$

holds.

A special case of (2.3) is given by

$$(u^{\gamma}(t))^{\Delta} = \gamma \int_0^1 \left[h u^{\sigma} + (1-h)u \right]^{\gamma - 1} dh u^{\Delta}(t). \tag{2.4}$$

The Hölder's inequality, see [2, Theorem 6.13], on time scales is given by

$$\int_a^b |f(t)g(t)|\Delta t \leqslant \left[\int_a^b |f(t)|^\gamma \Delta t\right]^{\frac{1}{\gamma}} \left[\int_a^b |g(t)|^\nu \Delta t\right]^{\frac{1}{\nu}}, \tag{2.5}$$

where $a,\ b \in \mathbb{T}$ and $f,\ g \in C_{rd}(\mathbb{I},\ \mathbb{R})$, $\gamma > 1$ and $1/\gamma + 1/\nu = 1$.

THEOREM 2.3. (Fubini's Theorem [5, Theorem 6.13]) *Let f be bounded and delta integrable over a rectangle $R = [a,\ b) \times [c,\ d)$ and suppose that the single integral*

$$I(t) = \int_c^d f(t,\ s)\Delta_2 s,$$

exists for each $t \in [a,\ b)$. Then the iterated integral

$$\int_a^b I(t)\Delta_1 t = \int_a^b \Delta_1 t \int_c^d f(t,\ s)\Delta_2 s,$$

exists and the equality

$$\iint\limits_R f(t,\ s)\Delta_1 t\Delta_2 s = \int_a^b \Delta_1 t \int_c^d f(t,\ s)\Delta_2 s, \tag{2.6}$$

holds.

It is evident from Theorem 2.3 that we can interchange the roles t and s, that is, we may assume the existence of the double integral and existence of the single integral

$$k(s) = \int_a^b f(t,\ s)\Delta_1 t,$$

for each $s \in [c,\ d)$ and then Theorem 2.3 will guarantee the existence of the iterated iterated integral

$$\int_c^d k(s)\Delta_2 s = \int_c^d \Delta_2 s \int_a^b f(t,\ s)\Delta_1 t,$$

and the equality

$$\iint\limits_R f(t,\ s)\Delta_1 t\Delta_2 s = \int_c^d \Delta_2 s \int_a^b f(t,\ s)\Delta_1 t, \tag{2.7}$$

holds. If together with the double integral $\iint\limits_R f(t,\ s)\Delta_1 t\Delta_2 s$ there exist both single integrals, then the formulas (2.6) and (2.7) will hold simultaneously, i.e.,

$$\int_a^b \Delta_1 t \int_c^d f(t,\ s)\Delta_2 s = \int_c^d \Delta_2 s \int_a^b f(t,\ s)\Delta_1 t. \tag{2.8}$$

THEOREM 2.4. (Jensen's Inequality [1]) *Let* $a,\ b\in\mathbb{T}$ *and* $c,\ d\in\mathbb{R}$. *Suppose that* $g\in C_{rd}([a,\ b]_{\mathbb{T}},\ (c,\ d))$ *and* $h\in C_{rd}([a,\ b]_{\mathbb{T}},\ \mathbb{R})$ *are nonnegative with*

$$\int_a^b h(s)\Delta s>0.$$

If $\Phi\in C((c,\ d),\ \mathbb{R})$ *is convex, then*

$$\Phi\left(\frac{\int_a^b h(s)g(s)\Delta s}{\int_a^b h(s)\Delta s}\right)\leqslant\frac{\int_a^b h(s)\Phi(g(s))\Delta s}{\int_a^b h(s)\Delta s}. \tag{2.9}$$

3. Main results

In this section, we will prove the main results. Throughout this paper, we will assume (usually without mentioning) that the functions in the statements of the theorems are right-dense continuous nonnegative functions and the integrals considered exist. We also assume that all the constants and the boundaries of the integrals that appear in the inequalities are real numbers greater than or equal to zero. In particular, we will assume that $h,\ l\geqslant 1$ be real numbers, and $p>1,\ q>1$ with $1/p+1/q=1$.

In the following, we prove the basic lemma that will be needed in the proof of the main results. It can be considered as an extension of the power rule for integrals; see [16]. The proof uses the time scales chain rule.

LEMMA 1. *Let* $x,\ c\in\mathbb{T}$ *with* $x\geqslant c$. *If* $\alpha\geqslant 1$, *then*

$$\left(\int_c^x f(\tau)\Delta\tau\right)^{\alpha}\leqslant\alpha\int_c^x f(\eta)\left(\int_c^{\sigma(\eta)} f(\tau)\Delta\tau\right)^{\alpha-1}\Delta\eta. \tag{3.1}$$

Proof. Let

$$F(x):=\int_c^x f(\tau)\Delta\tau. \tag{3.2}$$

Using the chain rule (2.4), we see that

$$(F^{\alpha}(x))^{\Delta}=\alpha\int_0^1[hF^{\sigma}(x)+(1-h)F(x)]^{\alpha-1}dhF^{\Delta}(x). \tag{3.3}$$

Since $F^{\Delta}(x)=f(x)\geqslant 0$ and $\sigma(x)\geqslant x$, we have

$$\begin{aligned}[F^{\alpha}(x)]^{\Delta}&\leqslant\alpha\int_0^1[hF^{\sigma}(x)+(1-h)F(\sigma(x))]^{\alpha-1}dhf(x)\\&=\alpha\int_0^1[F(\sigma(x))]^{\alpha-1}dhf(x)=\alpha[F(\sigma(x))]^{\alpha-1}f(x).\end{aligned} \tag{3.4}$$

Integrating both sides of (3.4), from c to x, we have

$$\int_c^x[F^{\alpha}(\eta)]^{\Delta}\Delta\eta\leqslant\alpha\int_c^x f(\eta)[F(\sigma(\eta))]^{\alpha-1}\Delta\eta. \tag{3.5}$$

Since

$$\int_c^x [F^\alpha(\eta)]^\Delta \Delta\eta = F^\alpha(\eta)\,|_c^x = F^\alpha(x) - F^\alpha(c) = F^\alpha(x), \tag{3.6}$$

we get from (3.5), that

$$\left(\int_c^x f(\tau)\Delta\tau\right)^\alpha \leqslant \alpha \int_c^x f(\eta)\left(\int_c^{\sigma(\eta)} f(\tau)\Delta\tau\right)^{\alpha-1}\Delta\eta.$$

This completes the proof. □

Now, we are ready to state and prove the main results in this paper.

THEOREM 3.1. *Let $x,\ y,\ c \in \mathbb{T}$ with $x,\ y \geqslant c$ and define*

$$A(x) := \int_c^x a(\tau)\Delta\tau, \quad \text{and} \quad B(y) := \int_c^y b(\tau)\Delta\tau. \tag{3.7}$$

Then for $x_1,\ y_1 \in \mathbb{T}$ with $x_1,\ y_1 \geqslant c$, we have that

$$\begin{aligned}
&\int_c^{x_1}\int_c^{y_1} \frac{A^h(x)B^l(y)}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x \\
&\leqslant M_1(h,\ l,\ p,\ q)\left[\int_c^{x_1}(\sigma(x_1)-x)(a(x)A^{h-1}(\sigma(x)))^p\Delta x\right]^{\frac{1}{p}} \\
&\quad\times\left[\int_c^{y_1}(\sigma(y_1)-y)(b(y)B^{l-1}(\sigma(y)))^q\Delta y\right]^{\frac{1}{q}},
\end{aligned} \tag{3.8}$$

where

$$M_1(h,\ l,\ p,\ q) := \frac{hl}{pq}(x_1-c)^{\frac{p-1}{p}}(y_1-c)^{\frac{q-1}{q}}. \tag{3.9}$$

Proof. By using inequality (3.1), it is easy to observe that

$$A^h(x) \leqslant h\int_c^x a(\eta)A^{h-1}(\sigma(\eta))\Delta\eta. \tag{3.10}$$

$$B^l(y) \leqslant l\int_c^y b(\eta)B^{l-1}(\sigma(\eta))\Delta\eta, \tag{3.11}$$

for any $x \in (c,\ x_1]_{\mathbb{T}}$ and $y \in (c,\ y_1]_{\mathbb{T}}$. From (3.10) and (3.11), we see that

$$A^h(x)B^l(y) \leqslant hl\left(\int_c^x a(\eta)A^{h-1}(\sigma(\eta))\Delta\eta\right)\left(\int_c^y b(\eta)B^{l-1}(\sigma(\eta))\Delta\eta\right). \tag{3.12}$$

Applying Hölder's inequality (2.5) on the term

$$\int_c^x a(\eta)A^{h-1}(\sigma(\eta))\Delta\eta,$$

with indices $p,\ p/(p-1)$ with $f=1$ and $g=a(\eta)A^{h-1}(\sigma(\eta))$, we see that

$$\int_c^x a(\eta)A^{h-1}(\sigma(\eta))\Delta\eta \leqslant (x-c)^{\frac{p-1}{p}}\left(\int_c^x (a(\eta)A^{h-1}(\sigma(\eta)))^p\Delta\eta\right)^{\frac{1}{p}}. \tag{3.13}$$

Applying Hölder's inequality (2.5) on the term

$$\int_c^y b(\eta)B^{l-1}(\sigma(\eta))\Delta\eta,$$

with indices $q,\ q/(q-1)$ when $f=1$ and $g=b(\eta)B^{l-1}(\sigma(\eta))$, we see that

$$\int_c^y b(\eta)B^{l-1}(\sigma(\eta))\Delta\eta \leqslant (y-c)^{\frac{q-1}{q}}\left(\int_c^y (b(\eta)B^{l-1}(\sigma(\eta)))^q\Delta\eta\right)^{\frac{1}{q}}. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12), we have

$$A^h(x)B^l(y) \leqslant hl(x-c)^{\frac{p-1}{p}}(y-c)^{\frac{q-1}{q}}\left(\int_c^x (a(\eta)A^{h-1}(\sigma(\eta)))^p\Delta\eta\right)^{\frac{1}{p}} \times\left(\int_c^y (b(\eta)B^{l-1}(\sigma(\eta)))^q\Delta\eta\right)^{\frac{1}{q}}. \tag{3.15}$$

Applying Young's inequality

$$\alpha\beta \leqslant \frac{\alpha^p}{p}+\frac{\beta^q}{q},\ \alpha\geqslant 0,\ \beta\geqslant 0,\ \frac{1}{p}+\frac{1}{q}=1,\ p>1, \tag{3.16}$$

on the right hand side of (3.15), with $\alpha=(x-c)^{\frac{p-1}{p}}$ and $\beta=(y-c)^{\frac{q-1}{q}}$, we observe that

$$A^h(x)B^l(y) \leqslant hl\left(\frac{(x-c)^{p-1}}{p}+\frac{(y-c)^{q-1}}{q}\right) \times\left(\int_c^x (a(\eta)A^{h-1}(\sigma(\eta)))^p\Delta\eta\right)^{\frac{1}{p}}\left(\int_c^y (b(\eta)B^{l-1}(\sigma(\eta)))^q\Delta\eta\right)^{\frac{1}{q}}.$$

This implies that

$$\frac{A^h(x)B^l(y)}{q(x-c)^{p-1}+p(y-c)^{q-1}} \leqslant \frac{hl}{pq}\left(\int_c^x (a(\eta)A^{h-1}(\sigma(\eta)))^p\Delta\eta\right)^{\frac{1}{p}} \times\left(\int_c^y (b(\eta)B^{l-1}(\sigma(\eta)))^q\Delta\eta\right)^{\frac{1}{q}}. \tag{3.17}$$

Integrating both sides of (3.17) from c to y_1 and from c to x_1, we see that

$$\int_c^{x_1}\int_c^{y_1}\frac{A^h(x)B^l(y)}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x$$
$$\leqslant \frac{hl}{pq}\int_c^{x_1}\left(\int_c^x (a(\eta)A^{h-1}(\sigma(\eta)))^p\Delta\eta\right)^{\frac{1}{p}}\Delta x\int_c^{y_1}\left(\int_c^y (b(\eta)B^{l-1}(\sigma(\eta)))^q\Delta\eta\right)^{\frac{1}{q}}\Delta y.$$

Applying Hölder's inequality with indices $p,\ p/(p-1)$ on the term

$$\int_c^{x_1}\left(\int_c^x (a(\eta)A^{h-1}(\sigma(\eta)))^p\Delta\eta\right)^{\frac{1}{p}}\Delta x,$$

and the term

$$\int_c^{y_1}\left(\int_c^y (b(\eta)B^{l-1}(\sigma(\eta)))^q\Delta\eta\right)^{\frac{1}{q}}\Delta y,$$

with indices $q,\ q/(q-1)$, we see that

$$\int_c^{x_1}\left(\int_c^x (a(\eta)A^{h-1}(\sigma(\eta)))^p\Delta\eta\right)^{\frac{1}{p}}\Delta x$$
$$\leqslant (x_1-c)^{\frac{p-1}{p}}\left[\int_c^{x_1}\left(\int_c^x (a(\eta)A^{h-1}(\sigma(\eta)))^p\Delta\eta\right)\Delta x\right]^{\frac{1}{p}}, \tag{3.18}$$

and

$$\int_c^{y_1}\left(\int_c^y (b(\eta)B^{l-1}(\sigma(\eta)))^q\Delta\eta\right)^{\frac{1}{q}}\Delta y$$
$$\leqslant (y_1-c)^{\frac{q-1}{q}}\left[\int_c^{y_1}\left(\int_c^y (b(\eta)B^{l-1}(\sigma(\eta)))^q\Delta\eta\right)\Delta y\right]^{\frac{1}{q}}. \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.17), we have

$$\int_c^{x_1}\int_c^{y_1}\frac{A^h(x)B^l(y)}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x$$
$$\leqslant \frac{hl}{pq}(x_1-c)^{\frac{p-1}{p}}(y_1-c)^{\frac{q-1}{q}}\left[\int_c^{x_1}\left(\int_c^x (a(\eta)A^{h-1}(\sigma(\eta)))^p\Delta\eta\right)\Delta x\right]^{\frac{1}{p}}$$
$$\times\left[\int_c^{y_1}\left(\int_c^y (b(\eta)B^{l-1}(\sigma(\eta)))^q\Delta\eta\right)\Delta y\right]^{\frac{1}{q}}.$$

Applying Fubini's Theorem, and using $\sigma(s)\geqslant s$, we see that

$$\int_c^{x_1}\int_c^{y_1}\frac{A^h(x)B^l(y)}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x$$
$$\leqslant M_1(h,\ l,\ p,\ q)\left[\int_c^{x_1}(\sigma(x_1)-x)(a(x)A^{h-1}(\sigma(x)))^p\Delta x\right]^{\frac{1}{p}}$$
$$\times\left[\int_c^{y_1}(\sigma(y_1)-y)(b(y)B^{l-1}(\sigma(y)))^q\Delta y\right]^{\frac{1}{q}}.$$

This gives us the desired inequality (3.8). This completes the proof. □

REMARK 3.1. If we apply the inequality *(3.16) on the right-hand sides of (3.8), then we get the following inequality*

$$\int_c^{x_1}\int_c^{y_1}\frac{A^h(x)B^l(y)}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x$$
$$\leqslant M_1(h,\ l,\ p,\ q)\left\{\frac{1}{p}\left[\int_c^{x_1}(\sigma(x_1)-x)(a(x)A^{h-1}(\sigma(x)))^p\Delta x\right]\right.$$
$$\left.+\frac{1}{q}\left[\int_c^{y_1}(\sigma(y_1)-y)(b(y)B^{l-1}(\sigma(x)))^q\Delta y\right]\right\}.$$

As a special case of Theorem 3.1 when $\mathbb{T}=\mathbb{R}$ we have $\sigma(s)=s$ and then we get the following result.

COROLLARY 3.1. *Assume that $a(x)$ and $b(y)$ are nonnegative functions and define*

$$A(x)=\int_0^x a(s)ds, \quad \text{and} \quad B(y)=\int_0^y b(s)ds.$$

Then

$$\int_0^{x_1}\int_0^{y_1}\frac{A^h(x)B^l(y)}{qx^{p-1}+py^{q-1}}dydx$$
$$\leqslant C_1(h,\ l,\ p,\ q)\left[\int_0^{x_1}(x_1-x)(a(x)A^{h-1}(x))^p dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_0^{y_1}(y_1-y)(b(y)B^{l-1}(y))^q dy\right]^{\frac{1}{q}}, \tag{3.20}$$

where

$$C_1(h,\ l,\ p,\ q):=\frac{hl}{pq}x_1^{\frac{p-1}{p}}y_1^{\frac{q-1}{q}}.$$

REMARK 3.2. If we put $p=q=2$ in the inequality *(3.20),* then we get *Theorem 5 due to Pachpatte* [14].

As a special case of Theorem 3.1 when $\mathbb{T}=\mathbb{Z}$ we have $\sigma(s)=s+1$ and then we get the following result.

COROLLARY 3.2. *Assume that $a(n)$ and $b(m)$ are nonnegative sequences and define*

$$A(n)=\sum_{s=1}^{n}a(s), \quad \text{and} \quad B(m)=\sum_{k=1}^{m}b(k).$$

Then

$$\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{A^{h}(n)B^{l}(m)}{qn^{p-1}+pm^{q-1}}\leqslant C_1^{*}(h,\ l,\ p,\ q)\left(\sum_{n=1}^{N}(N+1-n)(a(n)A^{h-1}(n))^{p}\right)^{\frac{1}{p}} \times\left(\sum_{m=1}^{M}(M+1-m)(b(m)B^{l-1}(m))^{q}\right)^{\frac{1}{q}}, \tag{3.21}$$

where

$$C_1^{*}(h,\ l,\ p,\ q):=\frac{hl}{pq}(N)^{\frac{p-1}{p}}(M)^{\frac{q-1}{q}}.$$

REMARK 3.3. If we put $p=q=2$ in *the inequality (3.21), then we get Theorem 1 due to Pachpatte* [14].

COROLLARY 3.3. *If we take $h=l=1$, then the inequality (3.8) is*

$$\int_{c}^{x_1}\int_{c}^{y_1}\frac{A(x)B(y)}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x \leqslant M_1(p,\ q)\left[\int_{c}^{x_1}(\sigma(x_1)-x)(a(x))^{p}\Delta x\right]^{\frac{1}{p}}\left[\int_{c}^{y_1}(\sigma(y_1)-y)(b(y))^{q}\Delta y\right]^{\frac{1}{q}}, \tag{3.22}$$

where

$$M_1(p,\ q):=\frac{1}{pq}(x_1-c)^{\frac{p-1}{p}}(y_1-c)^{\frac{q-1}{q}}.$$

Our next result deals with further generalizations of the inequality given in (3.22). In the next theorems we assume that Φ and Ψ are two real-valued, nonnegative, convex, and submultiplicative functions defined on $[0,\ \infty)$. The function Φ is said to be a submultiplicative on $[0,\ \infty)$ if $\Phi(xy)\leqslant\Phi(x)\Phi(y)$, for $x,\ y\in[0,\ \infty)$.

THEOREM 3.2. *Let $A(x)$ and $B(y)$ be defined as in Theorem 3.1. Assume that x, $y,\ c\in\mathbb{T}$ with $x,\ y\geqslant c$ and define*

$$F(x):=\int_{c}^{x}f(\tau)\Delta\tau,\quad and\quad G(y):=\int_{c}^{y}g(\eta)\Delta\eta. \tag{3.23}$$

Then for $x_1,\ y_1\in\mathbb{T}$ with $x_1,\ y_1\geqslant c$, we have

$$\int_{c}^{x_1}\int_{c}^{y_1}\frac{\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x \leqslant K_1(p,\ q)\left\{\int_{c}^{x_1}(\sigma(x_1)-x)\left(f(x)\Phi\left[\frac{a(x)}{f(x)}\right]\right)^{p}\Delta x\right\}^{\frac{1}{p}} \times\left\{\int_{c}^{y_1}(\sigma(y_1)-y)\left(g(y)\Psi\left[\frac{b(y)}{g(y)}\right]\right)^{q}\Delta y\right\}^{\frac{1}{q}}, \tag{3.24}$$

where

$$K_1(p,\, q) = \frac{1}{pq}\left\{\int_c^{x_1}\left(\frac{\Phi(F(x))}{F(x)}\right)^{\frac{p}{p-1}}\Delta x\right\}^{\frac{p-1}{p}}\left\{\int_c^{y_1}\left(\frac{\Psi(G(y))}{G(y)}\right)^{\frac{q}{q-1}}\Delta y\right\}^{\frac{q-1}{q}}. \tag{3.25}$$

Proof. Since Φ is a convex submultiplicative function, we get by applying Jensen's inequality that

$$\begin{aligned}\Phi(A(x)) &= \Phi\left(\frac{F(x)\int_c^x f(\tau)\frac{a(\tau)}{f(\tau)}\Delta\tau}{\int_c^x f(\tau)\Delta\tau}\right) \leqslant \Phi(F(x))\Phi\left(\frac{\int_c^x f(\tau)\frac{a(\tau)}{f(\tau)}\Delta\tau}{\int_c^x f(\tau)\Delta\tau}\right)\\ &\leqslant \frac{\Phi(F(x))}{F(x)}\int_c^x f(\tau)\Phi\left[\frac{a(\tau)}{f(\tau)}\right]\Delta\tau.\end{aligned}$$

Applying Hölder's inequality with indices $p,\ p/(p-1)$, we see that

$$\Phi(A(x)) \leqslant \frac{\Phi(F(x))}{F(x)}(x-c)^{\frac{p-1}{p}}\left\{\int_c^x\left(f(\tau)\Phi\left[\frac{a(\tau)}{f(\tau)}\right]\right)^p\Delta\tau\right\}^{\frac{1}{p}}. \tag{3.26}$$

Also, since Ψ is a convex submultiplicative function, we get by applying Jensen's inequality and Hölder's with indices $q,\ q/(q-1)$ that

$$\Psi(B(y)) \leqslant \frac{\Psi(G(y))}{G(y)}(y-c)^{\frac{q-1}{q}}\left\{\int_c^y\left(g(\eta)\Psi\left[\frac{b(\eta)}{g(\eta)}\right]\right)^q\Delta\eta\right\}^{\frac{1}{q}}. \tag{3.27}$$

From (3.26) and (3.27), we have

$$\begin{aligned}&\Phi(A(x))\Psi(B(y))\\ &\leqslant (x-c)^{\frac{p-1}{p}}(y-c)^{\frac{q-1}{q}}\left(\frac{\Phi(F(x))}{F(x)}\left\{\int_c^x\left(f(\tau)\Phi\left[\frac{a(\tau)}{f(\tau)}\right]\right)^p\Delta\tau\right\}^{\frac{1}{p}}\right)\\ &\quad\times\left(\frac{\Psi(G(y))}{G(y)}\left\{\int_c^y\left(g(\eta)\Psi\left[\frac{b(\eta)}{g(\eta)}\right]\right)^q\Delta\eta\right\}^{\frac{1}{q}}\right).\end{aligned}$$

Apply Young's inequality (3.16) on the right hand side with $\alpha = (x-c)^{\frac{p-1}{p}}$ and $\beta = (y-c)^{\frac{q-1}{q}}$, and we get

$$\begin{aligned}&\Phi(A(x))\Psi(B(y))\\ &\leqslant \left[\frac{(x-c)^{p-1}}{p}+\frac{(y-c)^{q-1}}{q}\right]\left(\frac{\Phi(F(x))}{F(x)}\left\{\int_c^x\left(f(\tau)\Phi\left[\frac{a(\tau)}{f(\tau)}\right]\right)^p\Delta\tau\right\}^{\frac{1}{p}}\right)\\ &\quad\times\left(\frac{\Psi(G(y))}{G(y)}\left\{\int_c^y\left(g(\eta)\Psi\left[\frac{b(\eta)}{g(\eta)}\right]\right)^q\Delta\eta\right\}^{\frac{1}{q}}\right).\end{aligned} \tag{3.28}$$

From (3.28), we have

$$
\begin{aligned}
&\frac{\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}} \\
&\leqslant \frac{1}{pq}\left(\frac{\Phi(F(x))}{F(x)}\left\{\int_c^x\left(f(\tau)\Phi\left[\frac{a(\tau)}{f(\tau)}\right]\right)^p\Delta\tau\right\}^{\frac{1}{p}}\right) \\
&\times\left(\frac{\Psi(G(y))}{G(y)}\left\{\int_c^y\left(g(\eta)\Psi\left[\frac{b(\eta)}{g(\eta)}\right]\right)^q\Delta\eta\right\}^{\frac{1}{q}}\right).
\end{aligned}
\tag{3.29}
$$

Integrating both sides of (3.29) from c to y_1 and from c to x_1 and applying Hölder's inequality with indices $p,\ p/(p-1)$ and $q,\ q/(q-1)$, we obtain

$$
\begin{aligned}
&\int_c^{x_1}\int_c^{y_1}\frac{\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x \\
&\leqslant \frac{1}{pq}\int_c^{x_1}\left(\frac{\Phi(F(x))}{F(x)}\left\{\int_c^x\left(f(\tau)\Phi\left[\frac{a(\tau)}{f(\tau)}\right]\right)^p\Delta\tau\right\}^{\frac{1}{p}}\right)\Delta x \\
&\times\int_c^{y_1}\left(\frac{\Psi(G(y))}{G(y)}\left\{\int_c^y\left(g(\eta)\Psi\left[\frac{b(\eta)}{g(\eta)}\right]\right)^q\Delta\eta\right\}^{\frac{1}{q}}\right)\Delta y. \\
&\leqslant \frac{1}{pq}\left\{\int_c^{x_1}\left(\frac{\Phi(F(x))}{F(x)}\right)^{\frac{p}{p-1}}\Delta x\right\}^{\frac{p-1}{p}}\left\{\int_c^{x_1}\int_c^x\left(f(\tau)\Phi\left[\frac{a(\tau)}{f(\tau)}\right]\right)^p\Delta\tau\Delta x\right\}^{\frac{1}{p}} \\
&\times\left\{\int_c^{y_1}\left(\frac{\Psi(G(y))}{G(y)}\right)^{\frac{q}{q-1}}\Delta y\right\}^{\frac{q-1}{q}}\left\{\int_c^{y_1}\int_c^y\left(g(\eta)\Psi\left[\frac{b(\eta)}{g(\eta)}\right]\right)^q\Delta\eta\Delta y\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Apply Fubini's Theorem 2.3, and we obtain

$$
\begin{aligned}
&\int_c^{x_1}\int_c^{y_1}\frac{\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x \\
&\leqslant K_1(p,\ q)\left\{\int_c^{x_1}(x_1-x)\left(f(x)\Phi\left[\frac{a(x)}{f(x)}\right]\right)^p\Delta x\right\}^{\frac{1}{p}} \\
&\times\left\{\int_c^{y_1}(y_1-y)\left(g(y)\Psi\left[\frac{b(y)}{g(y)}\right]\right)^q\Delta y\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Since $\sigma(s)\geqslant s$, we have

$$
\begin{aligned}
&\int_c^{x_1}\int_c^{y_1}\frac{\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x \\
&\leqslant K_1(p,\ q)\left\{\int_c^{x_1}(\sigma(x_1)-x)\left(f(x)\Phi\left[\frac{a(x)}{f(x)}\right]\right)^p\Delta x\right\}^{\frac{1}{p}} \\
&\times\left\{\int_c^{y_1}(\sigma(y_1)-y)\left(g(y)\Psi\left[\frac{b(y)}{g(y)}\right]\right)^q\Delta y\right\}^{\frac{1}{q}},
\end{aligned}
$$

which is the desired inequality (3.24). The proof is complete. □

REMARK 3.4. If we apply the inequality *(3.16) on the right-hand sides of (3.24), then we get the following inequality*

$$\begin{aligned}
&\int_c^{x_1}\int_c^{y_1}\frac{\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x\\
&\leqslant K_1(p,\,q)\left\{\frac{1}{p}\left[\int_c^{x_1}(\sigma(x_1)-x)\left(f(x)\Phi\left[\frac{a(x)}{f(x)}\right]\right)^p\Delta x\right]\right.\\
&\left.+\frac{1}{q}\left[\int_c^{y_1}(\sigma(y_1)-y)\left(g(y)\Psi\left[\frac{b(y)}{g(y)}\right]\right)^q\Delta y\right]\right\}.
\end{aligned}$$

As a special case of Theorem 3.2 when $\mathbb{T}=\mathbb{R}$, we have $\sigma(s)=s$ and then we get the following result.

COROLLARY 3.4. *Assume that $a(x)$, $b(y)$, $f(x)$ and $g(y)$, are nonnegative functions and define*

$$A(x)=\int_0^x a(s)ds,\ B(y)=\int_0^y b(s)ds,\ F(x):=\int_0^x f(s)ds,\ \text{and } G(y):=\int_0^y g(s)ds.$$

Then

$$\begin{aligned}
&\int_0^{x_1}\int_0^{y_1}\frac{\Phi(A(x))\Psi(B(y))}{qx^{p-1}+py^{q-1}}dydx\\
&\leqslant H_1(p,\,q)\left\{\int_0^{x_1}(x_1-x)\left(f(x)\Phi\left[\frac{a(x)}{f(x)}\right]\right)^p dx\right\}^{\frac{1}{p}}\\
&\times\left\{\int_0^{y_1}(y_1-y)\left(g(y)\Psi\left[\frac{b(y)}{g(y)}\right]\right)^q dy\right\}^{\frac{1}{q}},
\end{aligned}\tag{3.30}$$

where

$$H_1(p,\,q)=\frac{1}{pq}\left\{\int_0^{x_1}\left(\frac{\Phi(F(x))}{F(x)}\right)^{\frac{p}{p-1}}dx\right\}^{\frac{p-1}{p}}\left\{\int_0^{y_1}\left(\frac{\Psi(G(y))}{G(y)}\right)^{\frac{q}{q-1}}dy\right\}^{\frac{q-1}{q}}.$$

REMARK 3.5. If we put $p=q=2$ in the inequality *(3.30)*, then we get *Theorem 6 due to Pachpatte* [14].

As a special case of Theorem 3.2 when $\mathbb{T}=\mathbb{Z}$ we have $\sigma(s)=s+1$ and then we get the following result.

COROLLARY 3.5. *Assume that $a(n)$, $b(m)$, $f(n)$ and $g(m)$ are nonnegative sequences and define*

$$A(n)=\sum_{s=1}^{n}a(s),\ B(m)=\sum_{k=1}^{m}b(k),\ F(n)=\sum_{s=1}^{n}f(s),\ \text{and } G(n)=\sum_{k=1}^{m}g(s).$$

Then

$$\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{\Phi(A(n))\Psi(B(m))}{qn^{p-1}+pm^{q-1}}$$
$$\leqslant H_1^*(p,\,q)\left\{\sum_{n=1}^{N}(N+1-n)\left(f(n)\Phi\left[\frac{a(n)}{f(n)}\right]\right)^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{m=1}^{M}(M+1-m)\left(g(m)\Psi\left[\frac{b(m)}{g(m)}\right]\right)^q\right\}^{\frac{1}{q}}, \tag{3.31}$$

where

$$H_1^*(p,\,q)=\frac{1}{pq}\left\{\sum_{n=1}^{N}\left(\frac{\Phi(F(n))}{F(n)}\right)^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\left\{\sum_{m=1}^{M}\left(\frac{\Psi(G(m))}{G(m)}\right)^{\frac{q}{q-1}}\right\}^{\frac{q-1}{q}}.$$

REMARK 3.6. If we put $p=q=2$ in *the inequality (3.31), then we get Theorem 2 due to Pachpatte* [14].

The next theorem considers different forms of the inequality given in Theorem 3.2.

THEOREM 3.3. *Assume that* $x,\ y,\ c\in\mathbb{T}$ *with* $x,\ y\geqslant c$ *and define*

$$A(x):=\frac{1}{x-c}\int_c^x a(\tau)\Delta\tau,\quad \textit{and}\quad B(y):=\frac{1}{y-c}\int_c^y b(\eta)\Delta\eta. \tag{3.32}$$

Then for $x_1,\ y_1\in\mathbb{T}$ *with* $x_1,\ y_1\geqslant c$*, we have*

$$\int_c^{x_1}\int_c^{y_1}\frac{(x-c)(y-c)\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x$$
$$\leqslant M_1^*(p,\,q)\left(\int_c^{x_1}(\sigma(x_1)-x)(\Phi[a(x)])^p\Delta x\right)^{\frac{1}{p}}$$
$$\times\left(\int_c^{y_1}(\sigma(y_1)-y)(\Psi[b(y)])^q\Delta y\right)^{\frac{1}{q}}, \tag{3.33}$$

where

$$M_1^*(p,\,q):=\frac{1}{pq}(x_1-c)^{\frac{p-1}{p}}(y_1-c)^{\frac{q-1}{q}}.$$

Proof. From (3.32), we see that

$$\Phi(A(x))=\Phi\left(\frac{1}{x-c}\int_c^x a(\tau)\Delta\tau\right). \tag{3.34}$$

Applying Jensen's inequality on the right hand side of (3.34), we observe that

$$\Phi(A(x))\leqslant\frac{1}{x-c}\int_c^x\Phi[a(\tau)]\Delta\tau.$$

Applying Hölder's inequality with indices $p,\ p/(p-1)$, we have

$$\Phi(A(x)) \leqslant \frac{(x-c)^{\frac{p-1}{p}}}{(x-c)}\left(\int_c^x (\Phi[a(\tau)])^p \Delta\tau\right)^{\frac{1}{p}}. \tag{3.35}$$

From (3.35), we get that

$$\Phi(A(x))(x-c) \leqslant (x-c)^{\frac{p-1}{p}}\left(\int_c^x (\Phi[a(\tau)])^p \Delta\tau\right)^{\frac{1}{p}}. \tag{3.36}$$

In a similar way, we obtain

$$\Psi(B(y))(y-c) \leqslant (y-c)^{\frac{q-1}{q}}\left(\int_c^y (\Psi[b(\eta)])^q \Delta\eta\right)^{\frac{1}{q}}. \tag{3.37}$$

From (3.36) and (3.37), we observe that

$$\begin{aligned}&\Phi(A(x))\Psi(B(y))(x-c)(y-c)\\ \leqslant\ & (x-c)^{\frac{p-1}{p}}(y-c)^{\frac{q-1}{q}}\left(\int_c^x (\Phi[a(\tau)])^p \Delta\tau\right)^{\frac{1}{p}}\left(\int_c^y (\Psi[b(\eta)])^q \Delta\eta\right)^{\frac{1}{q}}.\end{aligned} \tag{3.38}$$

Applying the inequality (3.16) on the right hand side with $\alpha = (x-c)^{\frac{p-1}{p}}$ and $\beta = (y-c)^{\frac{q-1}{q}}$, we get the following inequality

$$\begin{aligned}&\Phi(A(x))\Psi(B(y))(x-c)(y-c)\\ \leqslant\ & \left[\frac{(x-c)^{p-1}}{p}+\frac{(y-c)^{q-1}}{q}\right]\left(\int_c^x (\Phi[a(\tau)])^p \Delta\tau\right)^{\frac{1}{p}}\left(\int_c^y (\Psi[b(\eta)])^q \Delta\eta\right)^{\frac{1}{q}}.\end{aligned} \tag{3.39}$$

From (3.39), we have

$$\begin{aligned}&\Phi(A(x))\Psi(B(y))(x-c)(y-c)\\ \leqslant\ & \frac{1}{pq}\left[q(x-c)^{p-1}+p(y-c)^{q-1}\right]\\ &\times\left(\int_c^x (\Phi[a(\tau)])^p \Delta\tau\right)^{\frac{1}{p}}\left(\int_c^y (\Psi[b(\eta)])^q \Delta\eta\right)^{\frac{1}{q}}.\end{aligned} \tag{3.40}$$

Dividing both sides by $q(x-c)^{p-1}+p(y-c)^{q-1}$ and integrating both sides of (3.40) from c to y_1 and from c to x_1, we get that

$$\begin{aligned}&\int_c^{x_1}\int_c^{y_1}\frac{(x-c)(y-c)\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x\\ \leqslant\ & \frac{1}{pq}\left[\int_c^{x_1}\left(\int_c^x (\Phi[a(\tau)])^p \Delta\tau\right)^{\frac{1}{p}}\Delta x\right]\left[\int_c^{y_1}\left(\int_c^y (\Psi[b(\eta)])^q \Delta\eta\right)^{\frac{1}{q}}\Delta y\right].\end{aligned} \tag{3.41}$$

Applying Hölder's inequality with indices $p,\ p/(p-1)$ and $q,\ q/(q-1)$, on the right hand side of (3.41), we obtain

$$\begin{aligned}&\int_c^{x_1}\int_c^{y_1}\frac{(x-c)(y-c)\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x\\ &\leqslant \frac{1}{pq}(x_1-c)^{\frac{p-1}{p}}(y_1-c)^{\frac{q-1}{q}}\left[\int_c^{x_1}\left(\int_c^{x}(\Phi[a(\tau)])^p\Delta\tau\right)\Delta x\right]^{\frac{1}{p}}\\ &\times\left[\int_c^{y_1}\left(\int_c^{y}(\Psi[b(\eta)])^q\Delta\eta\right)\Delta y\right]^{\frac{1}{q}}.\end{aligned}$$

Applying Fubini's Theorem 2.3, we get that

$$\begin{aligned}&\int_c^{x_1}\int_c^{y_1}\frac{(x-c)(y-c)\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x\\ &\leqslant \frac{1}{pq}(x_1-c)^{\frac{p-1}{p}}(y_1-c)^{\frac{q-1}{q}}\left\{\int_c^{x_1}(x_1-x)(\Phi[a(x)])^p\Delta x\right\}^{\frac{1}{p}}\\ &\times\left\{\int_c^{y_1}(y_1-y)(\Psi[b(y)])^q\Delta y\right\}^{\frac{1}{q}}.\end{aligned}$$

Since $\sigma(s)\geqslant s,$ we have

$$\begin{aligned}&\int_c^{x_1}\int_c^{y_1}\frac{(x-c)(y-c)\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x\\ &\leqslant M_1^*(p,\,q)\left(\int_c^{x_1}(\sigma(x_1)-x)(\Phi[a(x)])^p\Delta x\right)^{\frac{1}{p}}\\ &\times\left(\int_c^{y_1}(\sigma(y_1)-y)(\Psi[b(y)])^q\Delta y\right)^{\frac{1}{q}},\end{aligned}$$

which is the desired inequality (3.33). The proof is complete. □

REMARK 3.7. If we apply the inequality *(3.16) on the right-hand sides of (3.33), then we get the following inequality*

$$\begin{aligned}&\int_c^{x_1}\int_c^{y_1}\frac{(x-c)(y-c)\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x\\ &\leqslant M_1^*(p,\,q)\left\{\frac{1}{p}\left[\int_c^{x_1}(\sigma(x_1)-x)(\Phi[a(x)])^p\Delta x\right]\right.\\ &\left.+\frac{1}{q}\left[\int_c^{y_1}(\sigma(y_1)-y)(\Psi[b(y)])^q\Delta y\right]\right\}.\end{aligned}$$

As a special case of Theorem 3.3 when $\mathbb{T}=\mathbb{R},$ we have $\sigma(s)=s$ and then we get the following result.

COROLLARY 3.6. *Assume that $a(x)$ and $b(y)$ are nonnegative functions and define*

$$A(x)=\int_0^x a(s)ds, \quad and \quad B(y)=\int_0^y b(s)ds.$$

Then

$$\begin{aligned}&\int_0^{x_1}\int_0^{y_1}\frac{(x-c)(y-c)\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}dydx\\ &\leqslant L_1(p,\,q)\left(\int_c^{x_1}(x_1-x)(\Phi[a(x)])^p dx\right)^{\frac{1}{p}}\\ &\times\left(\int_c^{y_1}(y_1-y)(\Psi[b(y)])^q dy\right)^{\frac{1}{q}},\end{aligned} \tag{3.42}$$

where

$$L_1(p,\,q):=\frac{1}{pq}x_1^{\frac{p-1}{p}}y_1^{\frac{q-1}{q}}.$$

REMARK 3.8. If we put $p=q=2$ in the inequality *(3.42)*, then we get *Theorem 7 due to Pachpatte* [14].

As a special case of Theorem 3.3 when $\mathbb{T}=\mathbb{Z}$ we have $\sigma(s)=s+1$ and then we get the following result.

COROLLARY 3.7. *Assume that $a(n)$ and $b(m)$ are nonnegative sequences and define*

$$A(n)=\sum_{s=1}^{n}a(s), \quad and \quad B(m)=\sum_{k=1}^{m}b(k).$$

Then

$$\begin{aligned}&\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{(n-c)(m-c)\Phi(A(n))\Psi(B(m))}{qn^{p-1}+pm^{q-1}}\\ &\leqslant L_1^*(p,\,q)\left\{\sum_{n=1}^{N}(N+1-n)[\Phi(a(n))]^p\right\}^{\frac{1}{p}}\\ &\times\left\{\sum_{m=1}^{M}(M+1-m)[\Psi(b(m))]^q\right\}^{\frac{1}{q}},\end{aligned} \tag{3.43}$$

where

$$L_1^*(p,\,q):=\frac{1}{pq}N^{\frac{p-1}{p}}M^{\frac{q-1}{q}}.$$

REMARK 3.9. If we put $p=q=2$ in *the inequality (3.43), then we get Theorem 3 due to Pachpatte* [14].

In the following theorem, we prove a new dynamic inequality with two different weight functions.

THEOREM 3.4. *Let $x,\ y,\ c \in \mathbb{T}$ with $x,\ y \geqslant c$, F and G be defined as in Theorem 3.2, and define*

$$A(x) := \frac{1}{F(x)} \int_c^x f(\tau)a(\tau)\Delta\tau, \quad \textit{and} \quad B(y) := \frac{1}{G(y)} \int_c^y g(\eta)b(\eta)\Delta\eta. \tag{3.44}$$

Then for $x_1,\ y_1 \in \mathbb{T}$ with $x_1,\ y_1 \geqslant c$, we have

$$\begin{aligned} &\int_c^{x_1} \int_c^{y_1} \frac{F(x)G(y)\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}} \Delta y \Delta x \\ &\leqslant N_1(p,\ q) \left\{ \int_c^{x_1} (\sigma(x_1)-x)\,(f(x)\Phi[a(x)])^p\,\Delta x \right\}^{\frac{1}{p}} \\ &\times \left\{ \int_c^{y_1} (\sigma(y_1)-y)(g(y)\Psi[b(y)])^q \Delta y \right\}^{\frac{1}{q}}, \end{aligned} \tag{3.45}$$

where

$$N_1(p,\ q) := \frac{1}{pq}(x_1-c)^{\frac{p-1}{p}}(y_1-c)^{\frac{q-1}{q}}.$$

Proof. From (3.44), we see that

$$\Phi(A(x)) = \Phi\left(\frac{1}{F(x)} \int_c^x f(\tau)a(\tau)\Delta\tau\right). \tag{3.46}$$

Applying Jensen's inequality on the right hand side of (3.46), we observe that

$$\Phi(A(x)) \leqslant \frac{1}{F(x)} \int_c^x f(\tau)\Phi[a(\tau)]\Delta\tau. \tag{3.47}$$

Applying Hölder's inequality with indices $p,\ p/(p-1)$, on the right hand side of (3.47), we obtain

$$\Phi(A(x)) \leqslant \frac{(x-c)^{\frac{p-1}{p}}}{F(x)} \left(\int_c^x (f(\tau)\Phi[a(\tau)])^p \Delta\tau \right)^{\frac{1}{p}}. \tag{3.48}$$

From (3.48), we get that

$$\Phi(A(x))F(x) \leqslant (x-c)^{\frac{p-1}{p}} \left(\int_c^x (f(\tau)\Phi[a(\tau)])^p \Delta\tau \right)^{\frac{1}{p}}. \tag{3.49}$$

In a similar way, we obtain

$$\Psi(B(y))G(y) \leqslant (y-c)^{\frac{q-1}{q}} \left(\int_c^y (g(\eta)\Psi[b(\eta)])^q \Delta\eta \right)^{\frac{1}{q}}. \tag{3.50}$$

From (3.49) and (3.50), we observe that

$$\begin{aligned} &\Phi(A(x))\Psi(B(y))F(x)G(y) \\ &\leqslant (x-c)^{\frac{p-1}{p}}(y-c)^{\frac{q-1}{q}}\left(\int_c^x (f(\tau)\Phi[a(\tau)])^p\Delta\tau\right)^{\frac{1}{p}} \\ &\quad\times\left(\int_c^y (g(\eta)\Psi[b(\eta)])^q\Delta\eta\right)^{\frac{1}{q}}. \end{aligned} \tag{3.51}$$

Applying the inequality (3.16), on the right hand side of (3.51) where $\alpha = (x-c)^{\frac{p-1}{p}}$ and $\beta = (y-c)^{\frac{q-1}{q}}$, we get that

$$\begin{aligned} &\Phi(A(x))\Psi(B(y))F(x)G(y) \\ &\leqslant \left[\frac{(x-c)^{p-1}}{p}+\frac{(y-c)^{q-1}}{q}\right]\left(\int_c^x (f(\tau)\Phi[a(\tau)])^p\Delta\tau\right)^{\frac{1}{p}} \\ &\quad\times\left(\int_c^y (g(\eta)\Psi[b(\eta)])^q\Delta\eta\right)^{\frac{1}{q}}. \end{aligned} \tag{3.52}$$

This implies that

$$\begin{aligned} &\frac{\Phi(A(x))\Psi(B(y))F(x)G(y)}{q(x-c)^{p-1}+p(y-c)^{q-1}} \\ &\leqslant \frac{1}{pq}\left(\int_c^x (f(\tau)\Phi[a(\tau)])^p\Delta\tau\right)^{\frac{1}{p}}\left(\int_c^y (g(\eta)\Psi[b(\eta)])^q\Delta\eta\right)^{\frac{1}{q}}. \end{aligned} \tag{3.53}$$

Integrating both sides of (3.53) from c to y_1 and from c to x_1, we have

$$\int_c^{x_1}\int_c^{y_1}\frac{F(x)G(y)\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x$$

$$\leqslant \frac{1}{pq}\left(\int_c^{x_1}\left(\int_c^x (f(\tau)\Phi[a(\tau)])^p\Delta\tau\right)^{\frac{1}{p}}\Delta x\right) \tag{3.54}$$

$$\times\left(\int_c^{y_1}\left(\int_c^y (g(\eta)\Psi[b(\eta)])^q\Delta\eta\right)^{\frac{1}{q}}\Delta y\right). \tag{3.55}$$

Applying Hölder's inequality with indices $p,\ p/(p-1)$ and $q,\ q/(q-1)$, on the right hand side of (3.55), we have

$$\begin{aligned} &\int_c^{x_1}\int_c^{y_1}\frac{F(x)G(y)\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x \\ &\leqslant \frac{1}{pq}(x_1-c)^{\frac{p-1}{p}}(y_1-c)^{\frac{q-1}{q}}\left[\int_c^{x_1}\left(\int_c^x (f(\tau)\Phi[a(\tau)])^p\Delta\tau\right)\Delta x\right]^{\frac{1}{p}} \\ &\quad\times\left[\int_c^{y_1}\left(\int_c^y (g(\eta)\Psi[b(\eta)])^q\Delta\eta\right)\Delta y\right]^{\frac{1}{q}}. \end{aligned}$$

Applying Fubini's Theorem, and using $\sigma(s) \geqslant s$, we see that

$$\int_c^{x_1}\int_c^{y_1}\frac{F(x)G(y)\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x$$
$$\leqslant N_1(p,\,q)\left\{\int_c^{x_1}(\sigma(x_1)-x)\,(f(x)\Phi[a(x)])^p\,\Delta x\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_c^{y_1}(\sigma(y_1)-y)(g(y)\Psi[b(y)])^q\Delta y\right\}^{\frac{1}{q}},$$

which is (3.45). This completes the proof. □

REMARK 3.10. If we apply the inequality *(3.16) on the right-hand sides of (3.45), then we get the following inequality*

$$\int_c^{x_1}\int_c^{y_1}\frac{F(x)G(y)\Phi(A(x))\Psi(B(y))}{q(x-c)^{p-1}+p(y-c)^{q-1}}\Delta y\Delta x$$
$$\leqslant N_1(p,\,q)\left\{\frac{1}{p}\left[\int_c^{x_1}(\sigma(x_1)-x)\,(f(x)\Phi[a(x)])^p\,\Delta x\right]\right.$$
$$\left.+\frac{1}{q}\left[\int_c^{y_1}(\sigma(y_1)-y)(g(y)\Psi[b(y)])^q\Delta y\right]\right\}.$$

As a special case of Theorem 3.4 when $\mathbb{T}=\mathbb{R}$, we have $\sigma(s)=s$ and then we get the following result.

COROLLARY 3.8. *Assume that $a(x)$, $b(y)$, $f(x)$ and $g(y)$ are nonnegative functions and define*

$$A(x)=\int_0^x a(s)ds,\ B(y)=\int_0^y b(s)ds,\ F(x):=\int_0^x f(s)ds,\ \text{and } G(y):=\int_0^y g(s)ds.$$

Then

$$\int_0^{x_1}\int_0^{y_1}\frac{F(x)G(y)\Phi(A(x))\Psi(B(y))}{qx^{p-1}+py^{q-1}}dydx$$
$$\leqslant D_1(p,\,q)\left\{\int_0^{x_1}(x_1-x)\,(f(x)\Phi[a(x)])^p\,dx\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_0^{y_1}(y_1-y)(g(y)\Psi[b(y)])^q dy\right\}^{\frac{1}{q}}, \tag{3.56}$$

where

$$D_1(p,\,q):=\frac{1}{pq}x_1^{\frac{p-1}{p}}y_1^{\frac{q-1}{q}}.$$

REMARK 3.11. If we put $p=q=2$ in *the inequality (3.56), then we get Theorem 8 due to Pachpatte* [14].

As a special case of Theorem 3.3 when $\mathbb{T}=\mathbb{Z}$ we have $\sigma(s)=s+1$ and then we get the following result.

COROLLARY 3.9. *Assume that $a(n)$, $b(m)$, $f(n)$ and $g(m)$ are nonnegative sequences and define*

$$A(n)=\sum_{s=1}^{n}a(s),\ B(m)=\sum_{k=1}^{m}b(k),\ F(n)=\sum_{s=1}^{n}f(s),\ \text{and } G(n)=\sum_{k=1}^{m}g(s).$$

Then

$$\begin{aligned}\sum_{n=1}^{N}\sum_{m=1}^{M}&\frac{F(n)G(m)\Phi(A(n))\Psi(B(m))}{qn^{p-1}+pm^{q-1}}\\ &\leqslant D_1^*(p,\,q)\left\{\sum_{n=1}^{N}(N+1-n)[f(n)\Phi(A(n))]^p\right\}^{\frac{1}{p}}\\ &\times\left\{\sum_{m=1}^{M}(M+1-m)[g(m)\Psi(B(m))]^q\right\}^{\frac{1}{q}},\end{aligned}\tag{3.57}$$

where

$$D_1^*(p,\,q):=\frac{1}{pq}(N)^{\frac{p-1}{p}}(M)^{\frac{q-1}{q}}.$$

REMARK 3.12. If we put $p=q=2$ in *the inequality (3.57), then we get Theorem 4 due to Pachpatte* [14].

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