Mathematical
Inequalities
& Applications
Volume 20, Number 4 (2017), 1051–1066 doi:10.7153/mia-20-67

ALMOST EVERYWHERE STRONG SUMMABILITY OF CUBIC PARTIAL SUMS OF D–DIMENSIONAL WALSH–FOURIER SERIES

USHANGI GOGINAVA

(*Communicated by I. Perić*)

Abstract. In this paper we study the a. e. strong summability of the cubic partial sums of the d-dimensional Walsh-Fourier series of the functions belonging to $L\left(\log^{+}L\right)^{d-1}$.

1. Introduction

We shall denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $\mathbb{I}:=[0,1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p,n\in\mathbb{N}$, $0\leqslant p\leqslant 2^n$. Denote $I_N:=[0,2^{-N})$, $I_N(x):=I_N\dotplus x$, where by $\dotplus$ we denote dyadic addition (see [37], [14]).

Let $r_0(x)$ be the function defined by

$$r_0(x)=\begin{cases}1, & \text{if } x\in[0,1/2)\\ -1, & \text{if } x\in[1/2,1)\end{cases}, \qquad r_0(x+1)=r_0(x).$$

The Rademacher system is defined by

$$r_n(x)=r_0(2^n x), \quad n\geqslant 1.$$

Let $w_0,w_1,\dots$ represent the Walsh functions, i.e. $w_0(x)=1$ and if $k=2^{n_1}+\dots+2^{n_s}$ is a positive integer with $n_1>n_2>\dots>n_s$ then

$$w_k(x)=r_{n_1}(x)\cdots r_{n_s}(x).$$

Given $x\in\mathbb{I}$, the expansion

$$x=\sum_{k=0}^{\infty}x_k 2^{-(k+1)}, \tag{1}$$

where each $x_k=0$ or 1, will be called a dyadic expansion of x. If $x\in\mathbb{I}\setminus\mathbb{Q}$, then (1) is uniquely determined. For the dyadic expansion $x\in\mathbb{Q}$ we choose the one for which $\lim\limits_{k\to\infty}x_k=0$.

Mathematics subject classification (2010): 42C10.
Keywords and phrases: Walsh function, strong summability, cubic partial sums, Marcinkiewicz means.
The research was supported by Shota Rustaveli National Science Foundation grant 217282 (Operators of Fourier analysis in some classical and new function spaces.

The dyadic addition of $x,y \in \mathbb{I}$ in terms of the dyadic expansion of x and y is defined by

$$\rho(x,y) := x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

If $f \in L^1(\mathbb{I})$, then

$$\hat{f}(n) = \int_{\mathbb{I}} f(x) w_n(x) dx$$

is the n-th Fourier coefficient of f.

The partial sums of Fourier series with respect to the Walsh system are defined by

$$S_M(x;f) = \sum_{m=0}^{M-1} \hat{f}(m) w_m(x).$$

For $n \in \mathbb{N}$ let us itroduce the projections

$$E_n(x;f) := S_{2^n}(x;f) = 2^n \int_{I_n(x)} f(s)\, ds \quad (f \in L_1(\mathbb{I}), x \in \mathbb{I}),$$

$$E^*(x;f) := \sup_{n \in \mathbb{N}} E_n(x;|f|).$$

Let $\mathbb{I}^d = [0,1)^d$ denote the cube in the d-dimensional Euclidean space R^d. The elements of R^d will be denoted by $\overrightarrow{x} = (x_1, \ldots, x_d)$. For any $\overrightarrow{x} = (x_1, \ldots, x_d)$ and $\overrightarrow{y} = (y_1, \ldots, y_d)$ the vector $(x_1 \dotplus y_1, \ldots, x_d \dotplus y_d)$ of the space R^d is denoted by $\overrightarrow{x} \dotplus \overrightarrow{y}$.

Let $M = \{1, 2, \ldots, d\}$, $B = \{l_1, \ldots, l_r\}$, $l_i < l_{i+1}$, $i = 1, \ldots, r-1$, $B \subset M$, $B' = M \setminus B$, $|B| = card(B)$.

For any $x \in R^d$ and $B \subset M$, the symbol $\overrightarrow{x}_B$ will stand for the point of R^d whose coordinates with indices from B coincide with the corresponding coordinates of $\overrightarrow{x}$, and those with indices from B' are zero. Besides, $\overrightarrow{x}_M := \overrightarrow{x}$.

If $\overrightarrow{x} \in \mathbb{I}^d$ and $I_k(x_i)$, $i = 1, 2, \ldots, d$, are dyadic intervals containing x_i, then the set

$$I_k(\overrightarrow{x}) = I_k(x_1) \times I_k(x_2) \times \cdots \times I_k(x_d)$$

is a dyadic d-dimensional cube.

We denote by $L\left(\log^+ L\right)^{\alpha}\left(\mathbb{I}^d\right)$ the class of measurable functions f, with

$$\int_{\mathbb{I}^d} |f| \left(\log^+ |f|\right)^{\alpha} < \infty, \quad \alpha > 0,$$

where $\log^+ u := \mathbb{I}_{(1,\infty)} \log u$ and $\mathbb{I}_E$ is character function of the set E.

The rectangular partial sums of d-dimensional Walsh-Fourier series are defined as follows:

$$S_{m_1,\ldots,m_d}(x_1, \ldots, x_d; f) = \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_d=0}^{m_d-1} \widehat{f}(j_1, \ldots, j_d) \prod_{i=1}^{d} w_{j_i}(x_i),$$

where the number

$$\widehat{f}(j_1,\ldots,j_d)=\int\limits_{\mathbb{I}^d} f(x_1,\ldots,x_d)\prod_{i=1}^{d} w_{j_i}(x_i)dx_1\cdots dx_d$$

is said to be the $(j_1,\ldots,j_d)$th Walsh-Fourier coefficient of f.

Set

$$\begin{aligned}
E_n^{(i)}(x_1,\ldots,x_d;f) &:= S_{2^n}^{(i)}(x_1,\ldots,x_d;f)\\
&:= 2^n \int\limits_{I_n(x_i)} f(x_1,\ldots,x_{i-1},s,x_{i+1},\ldots,x_d)\,ds,\ i=1,\ldots,d,\\
E_n(x_1,\ldots,x_d;f) &:= S_{2^n,\ldots,2^n}(x_1,\ldots,x_d;f)\\
&= 2^{nd} \int\limits_{I_n(x_1)\times\cdots\times I_n(x_d)} f(s_1,\ldots,s_d)\,ds_1\cdots ds_d,\\
E_n^{(B)}(x_1,\ldots,x_d;f) &:= S_{2^n,\ldots,2^n}^{(B)}(x_1,\ldots,x_d;f)\\
&:= 2^{n|B|} \int\limits_{I_n(x_{l_1})\times\cdots\times I_n(x_{l_r})} f\left(\overrightarrow{s}_B \dotplus \overrightarrow{x}_{B'}\right) ds_{l_1}\cdots ds_{l_r},\ B\subset M,\ B\neq\varnothing.
\end{aligned}$$

The dyadic maximal functions is given by

$$E_*(x_1,\ldots,x_d;f) := \sup_{n\in\mathbb{N}} E_n(x_1,\ldots,x_d;|f|),$$

$$E_*^{(B)}(x_1,\ldots,x_d;f) := \sup_{n\in\mathbb{N}} E_n^{(B)}(x_1,\ldots,x_d;|f|).$$

We denote by $L_0\left(\mathbb{I}^d\right)$ the Lebesgue space of functions that are measurable and finite almost everywhere on $\mathbb{I}^d$, $|A|$ is the Lebesgue measure of the set $A\subset\mathbb{I}^d$.

Let $B\subset M$, $B\neq\varnothing$ and $f\in L\left(\log^+ L\right)^{|B|}\left(\mathbb{I}^d\right)$. Then it is well-known that (see [43])

$$\begin{aligned}
&\int\limits_{\mathbb{I}^d} E_*^{(B)}(f)(x_1,\ldots,x_d;f)\,dx_1\cdots dx_d \\
&\leqslant \int\limits_{\mathbb{I}^d} |f(x_1,\ldots,x_d)|\left(\log^+|f(x_1,\ldots,x_d)|\right)^{|B|} dx_1\cdots dx_d + c_d
\end{aligned}\tag{2}$$

and for $B\subset M$, $B\neq\varnothing$ and $f\in L\left(\log^+ L\right)^{|B|-1}\left(\mathbb{I}^d\right)$,

$$\begin{aligned}
&\left|\left\{(x_1,\ldots,x_d)\in\mathbb{I}^d : E_*^{(B)}(f)(x_1,\ldots,x_d;f) > c\lambda\right\}\right| \\
&\leqslant \frac{c}{\lambda}\left(\int\limits_{\mathbb{I}^d} |f((x_1,\ldots,x_d))|\left(\log^+|f((x_1,\ldots,x_d))|\right)^{|B|-1} dx_1\cdots dx_d + c_d\right)
\end{aligned}\tag{3}$$

Denote by $S_n^T(x,f)$ the partial sums of the trigonometric Fourier series of f and let

$$\sigma_n^T(x,f) = \frac{1}{n+1}\sum_{k=0}^{n} S_k^T(x,f)$$

be the $(C,1)$ means. Fejér [3] proved that $\sigma_n^T(f)$ converges to f uniformly for any 2π-periodic continuous function. Lebesgue in [23] established almost everywhere convergence of $(C,1)$ means if $f \in L_1(\mathbb{T}), \mathbb{T} := [-\pi,\pi)$. The strong summability problem, i.e. the convergence of the strong means

$$\frac{1}{n+1}\sum_{k=0}^{n} \left|S_k^T(x,f) - f(x)\right|^p, \quad x \in \mathbb{T}, \quad p > 0, \tag{4}$$

was first considered by Hardy and Littlewood in [20]. They showed that for any $f \in L_r(\mathbb{T})$ $(1 < r < \infty)$ the strong means tend to 0 a.e., if $n \to \infty$. The Fourier series of $f \in L_1(\mathbb{T})$ is said to be (H,p)-summable at $x \in T$, if the values (4) converge to 0 as $n \to \infty$. The (H,p)-summability problem in $L_1(\mathbb{T})$ has been investigated by Marcinkiewicz [28] for $p = 2$, and later by Zygmund [48] for the general case $1 \leqslant p < \infty$. Almost everywhere (H,p)-summability of Walsh-Fourier series with $p > 0$ was proved by F. Schipp in [35].

Almost everywhere Φ-summability of Fourier series with respect to trigonometric and Walsh systems with condition was proved by Oskolkov in [29], Rodin [32], Schipp [33], Karagulyan [21], Gát, Goginava, Karagulyan [8] .

For quadratic partial sums of two-dimensional trigonometric Fourier series Marcinkiewicz [28] has proved, that if $f \in L\log^+ L\left(\mathbb{T}^2\right)$, $\mathbb{T} := [-\pi,\pi)^2$, then

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} \left(S_{kk}^T(x,y,f) - f(x,y)\right) = 0$$

for a. e. $(x,y) \in \mathbb{T}^2$. Zhizhiashvili [47] improved this result showing that class $L\log^+ L\left(\mathbb{T}^2\right)$ can be replaced by $L_1\left(\mathbb{T}^2\right)$ (for d-dimensional trigonometric Fourier series see Dyachenko [2], for two-dimensional Walsh system see Weisz [42], and in case of d-dimensional Walsh-Fourier series see Goginava [9]). Almost Everywhere strong summability of quadratical partial sums of two-dimensional trigonometric Forier series was proved by Goginava, Gogoladze, Karagulyan in [13] and in case of two-dimensional Walsh system by Gát, Goginava and karagulyan in [7]. In this paper we study the a. e. strong summability of the cubic partial sums of the d-dimensional Walsh-Fourier series.

The results on strong summation and approximation of trigonometric Fourier series have been extended for several other orthogonal systems, see Schipp [34, 35, 36], Fridli and Schipp [4, 5], Leindler [24, 25, 26, 27], Totik [39, 40, 41], Rodin [30, 31, 32], Weisz [45, 44, 46], Gabisonia [6], Goginava, Gogoladze [12, 11], Gogoladze [16, 17], Glukhov [18], Goginava [9, 10].

2. Main results

THEOREM 1. *Let* $f \in L\left(\log^{+} L\right)^{d-1}\left(\mathbb{I}^d\right)$, $p>0$ *and* $d \geqslant 1$. *Then*

$$\left|\left\{(x_1,\ldots,x_d) \in \mathbb{I}^d : \sup_{n\in\mathbb{N}}\left(\frac{1}{2^n}\sum_{m=0}^{2^n-1}\left|S_{m,\ldots,m}(x_1,\ldots,x_d;f)\right|^p\right)^{1/p} > \lambda\right\}\right|$$

$$\leqslant \frac{c(d,p)}{\lambda}\left(1+\int_{\mathbb{I}^d}\left|f(x_1,\ldots,x_d)\right|\left(\left(\log^{+}\left|f(x_1,\ldots,x_d)\right|\right)\right)^{d-1}dx_1\cdots dx_d\right)$$

By making use the well-known density argument due to Marcinkiewicz and Zygmund we can show that Theorem 2 follows from Theorem 1.

THEOREM 2. *Let* $f \in L\left(\log^{+} L\right)^{d-1}\left(\mathbb{I}^d\right)$ *and* $d \geqslant 1$. *Then for any* $p>0$

$$\lim_{n\to\infty}\frac{1}{2^n}\sum_{m=0}^{2^n-1}\left|S_{m,\ldots,m}(x_1,\ldots,x_d;f)-f(x_1,\ldots,x_d)\right|^p = 0$$

a.e. on $\mathbb{I}^d$.

We note that from the theorem of Getsadze [15] and Konyagin [22] it follows that the class $L(\log L)^{d-1}\left(\mathbb{I}^d\right)$ in the last theorem is necessary in the context of strong summability question. That is, it is not possible to give a larger convergence space than $L(\log L)^{d-1}\left(\mathbb{I}^d\right)$.

We also note that in the case of two-dimensional trigonometric system Sjölin [38] (for d-dimensional case see Antonov [1]) proved that for every $r>1$ and d- variable function $f \in L_r(\mathbb{T}^d)$ the almost everywhere convergence $S_{n,\ldots,n}f \to f$ $(n\to\infty)$ holds. Since this issue with respect to the Walsh system is still open, then in this point of view Theorem 2 may seem more interesting.

3. Proofs of main results

In [33] Schipp introduced the following operator

$$V_n^{(p)}(x;f) := \left(\frac{1}{2^{n(q-1)}}\int_{\mathbb{I}}\left|\sum_{k=0}^{n-1}2^k\mathbb{I}_{I_k}(t)\,S_{2^n}(x\dotplus t\dotplus e_k;f)\right|^q dt\right)^{1/q},$$

where

$$p>1, \quad 1/p+1/q=1, \quad e_j := 2^{-j-1}.$$

In order to prove Theorem 2we need the following lemma (for $p=2$, see [33], for $p>2$ see [7]).

LEMMA 1. *Let* $p \geqslant 2$. *Then*

$$\sup_{\lambda>0} \lambda \left| \left\{ x \in \mathbb{I} : V_*^{(p)}(x;|f|) > \lambda \right\} \right| \leqslant c(p) \|f\|_1 .$$

Proof of Theorem 1. It is easy to show that

$$\left(\sum_{m=0}^{2^n-1} |S_{m,\ldots,m}(x_1,\ldots,x_d;f)|^p \right)^{1/p} \tag{5}$$

$$= \left(\sum_{m=0}^{2^n-1} |S_{m,\ldots,m}(x_1,\ldots,x_d;S_{2^n,\ldots,2^n}(f))|^p \right)^{1/p}$$

$$= \sup_{\{\alpha_{mn}(x_1,\ldots,x_d)\}} \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1,\ldots,x_d) S_{m,\ldots,m}(x_1,\ldots,x_d;S_{2^n,\ldots,2^n}(f)) \right|$$

by taking the supremum over all $\{\alpha_{mn}(x_1,\ldots,x_d)\}$ for which

$$\left(\sum_{m=0}^{2^n-1} |\alpha_{mn}(x_1,\ldots,x_d)|^q \right)^{1/q} \leqslant 1, \quad 1/p+1/q=1. \tag{6}$$

Let

$$\varepsilon_{ji} := \begin{cases} -1, & \text{if } j=0,1,\ldots,i-1 \\ 1, & \text{if } j=i \end{cases}.$$

In ([33]) Schipp proved that

$$D_m(t) = \sum_{k=0}^{n-1} \mathbb{I}_{I_k \setminus I_{k+1}}(t) \sum_{j=0}^{k} \varepsilon_{kj} 2^{j-1} w_m(t+e_j) - \frac{1}{2} w_m(t) + \left(m + \frac{1}{2} \right) \mathbb{I}_{I_n}(t), \quad m < 2^n. \tag{7}$$

Then we can write

$$J_n^{(p)}(x_1,\ldots,x_d) \tag{8}$$

$$:= \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1,\ldots,x_d) S_{m,\ldots,m}(x_1,\ldots,x_d;S_{2^n,\ldots,2^n}(f)) \right|$$

$$= \left| \int_{\mathbb{I}^d} S_{2^n,\ldots,2^n}(x_1 \dotplus t_1,\ldots,x_d \dotplus t_d;f) \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1,\ldots,x_d) \prod_{j=1}^{d} D_m(t_j)\, dt_1 \cdots dt_d \right|$$

$$\leqslant \sum_{B \subset M, B \neq \varnothing} \left| \int_{\mathbb{I}^d} S_{2^n,\ldots,2^n}(x_1 \dotplus t_1,\ldots,x_d \dotplus t_d;f) \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1,\ldots,x_d) \right.$$

$$\left. \times \prod_{i \in B} \sum_{k=0}^{n-1} \mathbb{I}_{I_k \setminus I_{k+1}}(t_i) \sum_{j=0}^{k} \varepsilon_{kj} 2^j w_m(t+e_j) \prod_{s \in B'} \frac{(-1)^{|B'|}}{2} w_m(t_s)\, dt_1 \cdots dt_d \right|$$

$$+ \sum_{B \subset M, B \neq M} \left| \int_{\mathbb{I}^d} S_{2^n,\ldots,2^n}(x_1 \dotplus t_1,\ldots,x_d \dotplus t_d;f) \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1,\ldots,x_d) \right.$$

$$\times \prod_{i\in B}\sum_{k=0}^{n-1}\mathbb{I}_{I_k\setminus I_{k+1}}(t_i)\sum_{j=0}^{k}\varepsilon_{kj}2^j w_m(t+e_j)\prod_{s\in B'}\left(m+\frac{1}{2}\right)\mathbb{I}_{I_n}(t_s)\,dt_1\cdots dt_d\Bigg|$$

$$+\sum_{B\subset M,B\neq\varnothing}\Bigg|\int_{\mathbb{I}^d} S_{2^n,\dots,2^n}(x_1\dotplus t_1,\dots,x_d\dotplus t_d;f)\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\dots,x_d)$$

$$\times\prod_{i\in B}\frac{(-1)^{|B|}}{2}w_m(t_i)\prod_{s\in B'}\left(m+\frac{1}{2}\right)\mathbb{I}_{I_n}(t_s)\,dt_1\cdots dt_d\Bigg|$$

$$=\sum_{B\subset M,B\neq\varnothing}J_{nB1}^{(p)}(x_1,\dots,x_d)+\sum_{B\subset M,B\neq M}J_{nB2}^{(p)}(x_1,\dots,x_d)+\sum_{B\subset M,B\neq\varnothing}J_{nB3}^{(p)}(x_1,\dots,x_d).$$

Let $B=M$. Since

$$\sum_{k=j}^{n-1}\mathbb{I}_{I_k\setminus I_{k+1}}(t)\leqslant\mathbb{I}_{I_j}(t)$$

we can write

$$J_{nM1}^{(p)}(x_1,\dots,x_d) \tag{9}$$

$$=\Bigg|\int_{\mathbb{I}^d} S_{2^n,\dots,2^n}(x_1\dotplus t_1,\dots,x_d\dotplus t_d;f)\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\dots,x_d)$$

$$\times\prod_{i=1}^{d}\sum_{k=0}^{n-1}\mathbb{I}_{I_k\setminus I_{k+1}}(t_i)\sum_{j=0}^{k}\varepsilon_{kj}2^j w_m(t_i\dotplus e_j)\,dt_1\cdots dt_d\Bigg|$$

$$\leqslant\sum_{k_1=0}^{n-1}\cdots\sum_{k_d=0}^{n-1}\sum_{j_1=0}^{k_1}\cdots\sum_{j_d=0}^{k_d}2^{j_1+\cdots+j_d}\int_{I_{k_1}\setminus I_{k_1+1}\times\cdots\times I_{k_d}\setminus I_{k_d+1}}S_{2^n,\dots,2^n}(x_1\dotplus t_1,\dots,x_d\dotplus t_d;|f|)$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\dots,x_d)w_m\left(t_1\dotplus\cdots\dotplus t_d\dotplus e_{j_1}\dotplus\cdots\dotplus e_{j_d}\right)\right|dt_1\cdots dt_d$$

$$\leqslant\sum_{j_1=0}^{n-1}\cdots\sum_{j_d=0}^{n-1}2^{j_1+\cdots+j_d}\int_{I_{j_1}\times\cdots\times I_{j_d}}S_{2^n,\dots,2^n}(x_1\dotplus t_1,\dots,x_d\dotplus t_d;|f|)$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\dots,x_d)w_m\left(t_1\dotplus\cdots\dotplus t_d\dotplus e_{j_1}\dotplus\cdots\dotplus e_{j_d}\right)\right|dt_1\cdots dt_d$$

$$=\sum_{\sigma}\sum_{0\leqslant j_{\sigma(1)}\leqslant\cdots\leqslant j_{\sigma(d)}<n}2^{j_1+\cdots+j_d}\int_{I_{j_1}\times\cdots\times I_{j_d}}S_{2^n,\dots,2^n}(x_1\dotplus t_1,\dots,x_d\dotplus t_d;|f|)$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\dots,x_d)w_m\left(t_1\dotplus\cdots\dotplus t_d\dotplus e_{j_1}\dotplus\cdots\dotplus e_{j_d}\right)\right|dt_1\cdots dt_d$$

$$:=\sum_{\sigma}J_{nM1\sigma}^{(p)}(x_1,\dots,x_d),$$

where the sum is taken over all rearrangements $\sigma := \{\sigma(k)\}_{k=1}^{d}$ of the set $\{1,2,\ldots,d\}$.

Let $\sigma(k) = d+1-k$ for some $k = 1,\ldots,d$. Then we have

$$J_{nM1\sigma}^{(p)}(x_1,\ldots,x_d) \tag{10}$$
$$= \sum_{0\leqslant j_d\leqslant\cdots\leqslant j_1<n} 2^{j_1+\cdots+j_d} \int\limits_{I_{j_1}\times\cdots\times I_{j_d}} S_{2^n,\ldots,2^n}(x_1 \dotplus t_1,\ldots,x_d \dotplus t_d;|f|)$$
$$\times \left|\sum_{m=0}^{2^n-1} \alpha_{mn}(x_1,\ldots,x_d)\, w_m\left(t_1 \dotplus \cdots \dotplus t_d \dotplus e_{j_1} \dotplus \cdots \dotplus e_{j_d}\right)\right| dt_1\cdots dt_d.$$

Denote

$$u_{d-1} := t_{d-1} \dotplus t_d \dotplus e_{j_d};$$
$$u_{d-2} := t_{d-2} \dotplus u_{d-1};$$
$$u_{d-3} := t_{d-3} \dotplus u_{d-2};$$
$$\vdots$$
$$u_1 := t_1 \dotplus u_2.$$

Then it is easy to see that

$$u_k \in I_{j_d}, \quad k = 1,2,\ldots,d-1.$$

Hence, we can write

$$J_{nM1\sigma}^{(p)}(x_1,\ldots,x_d) \tag{11}$$
$$= \sum_{0\leqslant j_d\leqslant\cdots\leqslant j_1<n} 2^{j_1+\cdots+j_d}$$
$$\times \int\limits_{I_{j_1}\times\cdots\times I_{j_d}} S_{2^n,\ldots,2^n}\left(x_1 \dotplus t_1,\ldots,x_{d-1} \dotplus t_{d-1}, x_d \dotplus u_{d-1} \dotplus t_{d-1} \dotplus e_{j_d};|f|\right)$$
$$\times \left|\sum_{m=0}^{2^n-1} \alpha_{mn}(x_1,\ldots,x_d)\right.$$
$$\left.\times w_m\left(t_1 \dotplus \cdots \dotplus t_{d-2} \dotplus u_{d-1} \dotplus e_{j_1} \dotplus \cdots \dotplus e_{j_{d-1}}\right)\right| dt_1\cdots dt_{d-1}du_{d-1}$$
$$= \sum_{0\leqslant j_d\leqslant\cdots\leqslant j_1<n} 2^{j_1+\cdots+j_d}$$
$$\times \int\limits_{I_{j_1}\times\cdots\times I_{j_d}} S_{2^n,\ldots,2^n}\left(x_1 \dotplus t_1,\ldots,x_{d-1} \dotplus t_{d-1}, x_d \dotplus u_{d-2} \dotplus t_{d-1} \dotplus t_{d-2} \dotplus e_{j_d};|f|\right)$$
$$\times \left|\sum_{m=0}^{2^n-1} \alpha_{mn}(x_1,\ldots,x_d)\right.$$
$$\left.\times w_m\left(t_1 \dotplus \cdots \dotplus t_{d-3} \dotplus u_{d-2} \dotplus e_{j_1} \dotplus \cdots \dotplus e_{j_{d-1}}\right)\right| dt_1\cdots dt_{d-1}du_{d-2}$$

$$= \cdots =$$
$$= \sum_{0 \leqslant j_d \leqslant \cdots \leqslant j_1 < n} 2^{j_1 + \cdots + j_d}$$
$$\times \int_{I_{j_1} \times \cdots \times I_{j_d}} S_{2^n,\ldots,2^n}\left(x_1 \dotplus t_1, \ldots, x_{d-1} \dotplus t_{d-1}, x_d \dotplus t_1 \dotplus \cdots \dotplus t_{d-1} \dotplus u_1 \dotplus e_{j_d}; |f|\right)$$
$$\times \left| \sum_{m=0}^{2^n - 1} \alpha_{mn}(x_1, \ldots, x_d) w_m\left(u_1 \dotplus e_{j_1} \dotplus \cdots \dotplus e_{j_{d-1}}\right) \right| dt_1 \cdots dt_{d-1} du_1.$$

Set

$$F(x_1, \ldots, x_d) := f(x_1, \ldots, x_{d-1}, x_1 \dotplus \cdots \dotplus x_d)$$

and

$$A_{j_1,\ldots,j_{d-1}}(x_1, \ldots, x_d; f)$$
$$:= 2^{j_1 + \cdots + j_{d-1}} \int_{I_{j_1} \times \cdots \times I_{j_{d-1}}} |f(x_1 \dotplus t_1, \ldots, x_{d-1} \dotplus t_{d-1}, x_d \dotplus t_1 \dotplus \cdots \dotplus t_{d-1})| dt_1 \cdots dt_{d-1}.$$

It is evident that

$$A_{j_1,\ldots,j_{d-1}}(x_1, \ldots, x_{d-1}, x_1 \dotplus \cdots \dotplus x_d; f) \tag{12}$$
$$:= 2^{j_1 + \cdots + j_{d-1}}$$
$$\times \int_{I_{j_1} \times \cdots \times I_{j_{d-1}}} |f(x_1 \dotplus t_1, \ldots, x_{d-1} \dotplus t_{d-1}, x_1 \dotplus \cdots \dotplus x_d \dotplus t_1 \dotplus \cdots \dotplus t_{d-1})| dt_1 \cdots dt_{d-1}$$
$$= 2^{j_1 + \cdots + j_{d-1}} \int_{I_{j_1} \times \cdots \times I_{j_{d-1}}} F(x_1 \dotplus t_1, \ldots, x_{d-1} \dotplus t_{d-1}, x_d) dt_1 \cdots dt_{d-1}$$
$$\leqslant E_*^{(1)}\left(E_*^{(2)}\left(\cdots\left(E_*^{(d-1)}(x_1, \ldots, x_d; F)\right)\cdots\right)\right).$$

Denote

$$A(x_1, \ldots, x_d; f) := \sup_{j_1,\ldots,j_{d-1} \in \mathbb{N}} A_{j_1,\ldots,j_{d-1}}(x_1, \ldots, x_d; f).$$

Then from (2) and (12) for $f \in L\left(\log^+ L\right)^{d-1}\left(\mathbb{I}^d\right)$ we obtain

$$\int_{\mathbb{I}^d} A(x_1, \ldots, x_d; f) dx_1 \cdots dx_d \tag{13}$$
$$= \int_{\mathbb{I}^d} A(x_1, \ldots, x_{d-1}, x_1 \dotplus \cdots \dotplus x_d; f) dx_1 \cdots dx_d$$
$$\leqslant \int_{\mathbb{I}^d} E_*^{(1)}\left(E_*^{(2)}\left(\cdots\left(E_*^{(d-1)}(x_1, \ldots, x_d; F)\right)\cdots\right)\right) dx_1 \cdots dx_d$$

$$\leqslant c_1(d)+c_2(d)\int\limits_{\mathbb{I}^d}|F(x_1,\ldots,x_d)|\left(\log^+|F(x_1,\ldots,x_d)|\right)^{d-1}dx_1\cdots dx_d$$
$$=c_1(d)+c_2(d)\int\limits_{\mathbb{I}^d}|f(x_1,\ldots,x_d)|\left(\log^+|f(x_1,\ldots,x_d)|\right)^{d-1}dx_1\cdots dx_d.$$

From (11) we can write

$$J^{(p)}_{nM1\sigma}(x_1,\ldots,x_d) \tag{14}$$
$$=\sum_{0\leqslant j_d\leqslant\cdots\leqslant j_1<n}2^{j_d}\int\limits_{I_{j_d}}2^{j_1+\cdots+j_{d-1}}\Bigg(\int\limits_{I_{j_1}\times\cdots\times I_{j_{d-1}}}S_{2^n,\ldots,2^n}(x_1\dotplus t_1,\ldots,x_{d-1}\dotplus t_{d-1},$$
$$x_d\dotplus t_1\dotplus\cdots\dotplus t_{d-1}\dotplus u\dotplus e_{j_d};|f|)dt_1\cdots dt_{d-1}\Bigg)$$
$$\times\Bigg|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\ldots,x_d)w_m(u\dotplus e_{j_1}\dotplus\cdots\dotplus e_{j_{d-1}})\Bigg|du$$
$$=\sum_{0\leqslant j_d\leqslant\cdots\leqslant j_1<n}2^{j_d}\Bigg(\int\limits_{I_{j_d}}2^{j_1+\cdots+j_{d-1}}\Bigg(\int\limits_{I_{j_1}\times\cdots\times I_{j_{d-1}}}2^{nd}\int\limits_{I_n\times\cdots\times I_n}|f(x_1\dotplus t_1\dotplus s_1,\ldots,$$
$$x_{d-1}\dotplus t_{d-1}\dotplus s_{d-1},x_d\dotplus t_1\dotplus\cdots\dotplus t_{d-1}\dotplus u\dotplus e_{j_d}\dotplus s_d)|ds_1\cdots ds_d\Bigg)dt_1\cdots dt_{d-1}\Bigg)$$
$$\times\Bigg|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\ldots,x_d)w_m(u\dotplus e_{j_1}\dotplus\cdots\dotplus e_{j_{d-1}})\Bigg|du$$
$$\leqslant\sum_{0\leqslant j_d\leqslant\cdots\leqslant j_1<n}2^{j_d}\int\limits_{I_{j_d}}2^{nd}\Bigg(\int\limits_{I_n\times\cdots\times I_n}(2^{j_1+\cdots+j_{d-1}}\int\limits_{I_{j_1}\times\cdots\times I_{j_{d-1}}}|f(x_1\dotplus t_1\dotplus s_1,\ldots,$$
$$x_{d-1}\dotplus t_{d-1}\dotplus s_{d-1},x_d\dotplus t_1\dotplus\cdots\dotplus t_{d-1}\dotplus u\dotplus e_{j_d}\dotplus s_d)|dt_1\cdots dt_{d-1})ds_1\cdots ds_d\Bigg)$$
$$\times\Bigg|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\ldots,x_d)w_m(u\dotplus e_{j_1}\dotplus\cdots\dotplus e_{j_{d-1}})\Bigg|du$$
$$=\sum_{0\leqslant j_d\leqslant\cdots\leqslant j_1<n}2^{j_d}\int\limits_{I_{j_d}}2^{n}\Bigg(\int\limits_{I_n}(2^{j_1+\cdots+j_{d-1}}\int\limits_{I_{j_1}\times\cdots\times I_{j_{d-1}}}|f(x_1\dotplus t_1,\ldots,x_{d-1}\dotplus t_{d-1},$$
$$x_d\dotplus t_1\dotplus\cdots\dotplus t_{d-1}\dotplus u\dotplus e_{j_d}\dotplus s_d)|dt_1\cdots dt_{d-1})ds_d\Bigg)$$
$$\times\Bigg|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\ldots,x_d)w_m(u\dotplus e_{j_1}\dotplus\cdots\dotplus e_{j_{d-1}})\Bigg|du$$

$$\leqslant \sum_{0\leqslant j_d\leqslant\cdots\leqslant j_1<n} 2^{j_d}\int\limits_{I_{j_d}} 2^n\left(\int\limits_{I_n} A(x_1,\ldots,x_{d-1},x_d\dotplus u\dotplus e_{j_d}\dotplus s_d)ds_d\right)$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\ldots,x_d)w_m(u\dotplus e_{j_1}\dotplus\cdots\dotplus e_{j_{d-1}})\right|du$$

$$=\sum_{0\leqslant j_d\leqslant\cdots\leqslant j_1<n} 2^{j_d}\int\limits_{I_{j_d}} S_{2^n}^{(d)}(x_1,\ldots,x_{d-1},x_d\dotplus u\dotplus e_{j_d};A)$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\ldots,x_d)w_m(u\dotplus e_{j_1}\dotplus\cdots\dotplus e_{j_{d-1}})\right|du$$

$$\leqslant \sum_{j_{d-1}\leqslant\cdots\leqslant j_1<n}\int\limits_{\mathbb{I}}\sum_{j_d\leqslant j_{d-1}} 2^{j_d}\mathbb{I}_{I_{j_d}}(u)S_{2^n}^{(d)}(x_1,\ldots,x_{d-1},x_d\dotplus u\dotplus e_{j_d};A)$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\ldots,x_d)w_m(u\dotplus e_{j_1}\dotplus\cdots\dotplus e_{j_{d-1}})\right|du$$

$$\leqslant \sum_{j_{d-1}\leqslant\cdots\leqslant j_1<n}\left(\int\limits_{\mathbb{I}}\left(\sum_{j_d\leqslant j_{d-1}} 2^{j_d}\mathbb{I}_{I_{j_d}}(u)S_{2^n}^{(d)}(x_1,\ldots,x_{d-1},x_d\dotplus u\dotplus e_{j_d};A)\right)^q du\right)^{1/q}$$

$$\times\left(\int\limits_{\mathbb{I}}\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\ldots,x_d)w_m(u\dotplus e_{j_1}\dotplus\cdots\dotplus e_{j_{d-1}})\right|^p du\right)^{1/p}.$$

First use Hölder's inequality and Hausdorf-Young inequality from (6) we have $(p\geqslant 2, 1/p+1/q=1)$

$$\left(\int\limits_{\mathbb{I}}\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\ldots,x_d)\,w_m\left(u\dotplus e_{j_1}\dotplus\cdots\dotplus e_{j_{d-1}}\right)\right|^p du\right)^{1/p}$$

$$=\sup_{\|h\|_q\leqslant 1}\left|\int\limits_{\mathbb{I}}\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\ldots,x_d)\,w_m\left(u\dotplus e_{j_1}\dotplus\cdots\dotplus e_{j_{d-1}}\right)h(u)du\right|$$

$$=\sup_{\|h\|_q\leqslant 1}\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\ldots,x_d)\,w_m\left(e_{j_1}\right)\cdots w_m\left(e_{j_{d-1}}\right)\widehat{h}(m)\right|$$

$$\leqslant\sup_{\|h\|_q\leqslant 1}\left(\sum_{m=0}^{2^n-1}|\alpha_{mn}(x_1,\ldots,x_d)|^q\right)^{1/q}\left(\sum_{m=0}^{2^n-1}\left|\widehat{h}(m)\right|^p\right)^{1/p}\leqslant c\sup_{\|h\|_q\leqslant 1}\|h\|_q<\infty.$$

Hence, by (14) we obtain

$$J_{nM1\sigma}^{(p)}(x_1,\ldots,x_d)\leqslant c(d)\sum_{j_1=0}^{n-1}\sum_{j_2=0}^{j_1}\cdots\sum_{j_{d-1}=0}^{j_{d-2}}2^{j_{d-1}/p}V_d^{(p)}(x_1,\ldots,x_d;A) \tag{15}$$
$$\leqslant c2^{n/p}p^{d-1}V_d^{(p)}(x_1,\ldots,x_d;A).$$

Consequently,

$$\sup_{n\in\mathbb{N}} \frac{J_{nM1\sigma}^{(p)}(x_1,\ldots,x_d)}{2^{n/p}} \leqslant c(d,p)\, V_d^{(p)}(x_1,\ldots,x_d;A). \tag{16}$$

Set

$$\Omega := \left\{(x_1,\ldots,x_d)\in\mathbb{I}^d : V_d^{(p)}(x_1,\ldots,x_d;A) > c_1\lambda\right\}.$$

Then from Lemma 1 we can write

$$\begin{aligned}
|\Omega| &= \int_{\mathbb{I}^d} \mathbb{I}_\Omega(x_1,\ldots,x_d)\,dx_1\cdots dx_d = \int_{\mathbb{I}^{d-1}} \left(\int_{\mathbb{I}} \mathbb{I}_\Omega(x_1,\ldots,x_d)\,dx_d\right) dx_2\cdots dx_{d-1} \\
&\leqslant \int_{\mathbb{I}^{d-1}} \left(\frac{c}{\lambda}\int_{\mathbb{I}} |A(x_1,\ldots,x_d)|\,dx_d\right) dx_2\cdots dx_{d-1} \\
&= \left(\frac{c}{\lambda}\int_{\mathbb{I}^d} |A(x_1,\ldots,x_d)|\,dx_1\cdots dx_d\right)^{1/2}.
\end{aligned}$$

Hence, from (13) we obtain $\left(f\in L\left(\log^+ L\right)^{d-1}\left(\mathbb{I}^d\right)\right)$

$$\begin{aligned}
&\left|\left\{(x_1,\ldots,x_d)\in\mathbb{I}^d : V_d^{(p)}(x_1,\ldots,x_d;A) > \lambda\right\}\right| \\
&\leqslant \frac{c(d,p)}{\lambda}\left(1+\int_{\mathbb{I}^d} |f(x_1,\ldots,x_d)|\left(\log^+|f(x_1,\ldots,x_d)|\right)^{d-1} dx_1\cdots dx_d\right).
\end{aligned}$$

From (16) we conclude that

$$\begin{aligned}
&\left|\left\{(x_1,\ldots,x_d)\in\mathbb{I}^d : \sup_{n\in\mathbb{N}} \frac{J_{nM1\sigma}^{(p)}(x_1,\ldots,x_d)}{2^{n/p}} > c\lambda\right\}\right| \\
&\leqslant \frac{c(d,p)}{\lambda}\left(1+\int_{\mathbb{I}^d} |f(x_1,\ldots,x_d)|\left(\log^+|f(x_1,\ldots,x_d)|\right)^{d-1} dx_1\cdots dx_d\right)
\end{aligned} \tag{17}$$

for $\sigma(k) = d+1-k$.

Similarly we can prove that for all other summands in (9) the estimation (17) holds. So, we have

$$\begin{aligned}
&\left|\left\{(x_1,\ldots,x_d)\in\mathbb{I}^d : \sup_{n\in\mathbb{N}} \frac{J_{nM1}^{(p)}(x_1,\ldots,x_d)}{2^{n/p}} > c\lambda\right\}\right| \\
&\leqslant \frac{c(d,p)}{\lambda}\left(1+\int_{\mathbb{I}^d} |f(x_1,\ldots,x_d)|\left(\log^+|f(x_1,\ldots,x_d)|\right)^{d-1} dx_1\cdots dx_d\right).
\end{aligned} \tag{18}$$

Let $B\subset M$, $B\neq M$. Then analogously, we can prove that

$$\left|\left\{(x_1,\ldots,x_d)\in\mathbb{I}^d:\sup_{n\in\mathbb{N}}\frac{J_{nB1}^{(p)}(x_1,\ldots,x_d)}{2^{n/p}}>c\lambda\right\}\right| \tag{19}$$
$$\leqslant\frac{c(d,p)}{\lambda}\left(1+\int_{\mathbb{I}^d}|f(x_1,\ldots,x_d)|\left(\log^+|f(x_1,\ldots,x_d)|\right)^{d-1}dx_1\cdots dx_d\right).$$

Now, we estimate $J_{nB2}^{(p)}(x_1,\ldots,x_d)$. Let $B=\varnothing$. Then we have

$$J_{nB2}^{(p)}(x_1,\ldots,x_d)$$
$$=\left|\int_{\mathbb{I}^d}S_{2^n,\ldots,2^n}(x_1\dot{+}t_1,\ldots,x_d\dot{+}t_d;f)\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\ldots,x_d)\prod_{s\in M}\left(m+\frac{1}{2}\right)\mathbb{I}_{I_n}(t_s)\,dt_1\cdots dt_d\right|$$
$$\leqslant\int_{\mathbb{I}^d}S_{2^n,\ldots,2^n}(x_1\dot{+}t_1,\ldots,x_d\dot{+}t_d;|f|)\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x_1,\ldots,x_d)\left(m+\frac{1}{2}\right)^d\right|\prod_{s\in M}\mathbb{I}_{I_n}(t_s)\,dt_1\cdots dt_d$$
$$\leqslant c(d)2^{nd+n/p}\int_{I_n\times\cdots\times I_n}|f(x_1\dot{+}t_1,\ldots,x_d\dot{+}t_d)|\,dt_1\cdots dt_d\left(\sum_{m=0}^{2^n-1}|\alpha_{mn}(x_1,\ldots,x_d)|^q\right)^{1/q}$$
$$\leqslant c(d)2^{n/p}E_*(x_1,\ldots,x_d;f).$$

Hence, from (2), we have

$$\left|\left\{(x_1,\ldots,x_d)\in\mathbb{I}^d:\sup_{n\in\mathbb{N}}\frac{J_{nB2}^{(p)}(x_1,\ldots,x_d)}{2^{n/p}}>c_1\lambda\right\}\right| \tag{20}$$
$$\leqslant\frac{c(d,p)}{\lambda}\left(1+\int_{\mathbb{I}^d}|f(x_1,\ldots,x_d)|\left(\log^+|f(x_1,\ldots,x_d)|\right)^{d-1}dx_1\cdots dx_d\right).$$

Let $B:=\{l_1,\ldots,l_r\}\subset M, l_i<l_{i+1}$, $i=1,\ldots,r-1$, $B\neq\varnothing$, $B'=M\setminus B:=\{k_1,\ldots,k_s\}$, $r+s=d$. Since

$$\int_{[0,2^{-n}]^{|B'|}}S_{2^n,\ldots,2^n}(x_1\dot{+}t_1,\ldots,x_d\dot{+}t_d;|f|)\,dt_{k_1}\cdots dt_{k_s}=2^{-n|B'|}S_{2^n,\ldots,2^n}\left(\overrightarrow{x}_M\dot{+}\overrightarrow{t}_B;|f|\right). \tag{21}$$

Repeating the arguments in the same way as in estimation $J_{nB1}^{(p)}(x_1,\ldots,x_d)$ we get

$$J_{nB2}^{(p)}(x_1,\ldots,x_d) \tag{22}$$
$$\leqslant\sum_{j_{l_1}=0}^{n-1}\cdots\sum_{j_{l_r}=0}^{n-1}2^{j_1+\cdots+j_r}\int_{I_{j_{l_1}}\times\cdots\times I_{j_{l_r}}}\int_{[0,2^{-n}]^{|B'|}}S_{2^n,\ldots,2^n}(x_1\dot{+}t_1,\ldots,x_d\dot{+}t_d;|f|)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1,\ldots,x_d)\, w_m\left(t_{l_1} \dotplus \cdots \dotplus t_{l_r} \dotplus e_{j_{l_1}} \dotplus \cdots \dotplus e_{j_{l_r}}\right)\left(m+\frac{1}{2}\right)^{|B'|} \right| dt_1 \cdots dt_d$$

$$\leqslant \sum_{j_{l_1}=0}^{n-1} \cdots \sum_{j_{l_r}=0}^{n-1} 2^{j_1+\cdots+j_r} 2^{-n|B'|} \int\limits_{I_{j_{l_1}} \times \cdots \times I_{j_{l_r}}} S^{(B)}_{2^n,\ldots,2^n}\left(\overrightarrow{x}_M \dotplus \overrightarrow{t}_B; E_*^{(B')}(|f|)\right)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1,\ldots,x_d)\, w_m\left(t_{l_1} \dotplus \cdots \dotplus t_{l_r} \dotplus e_{j_{l_1}} \dotplus \cdots \dotplus e_{j_{l_r}}\right)\left(m+\frac{1}{2}\right)^{|B'|} \right| dt_{l_1} \cdots dt_{l_r}.$$

Hence, from (2) and (3) we have

$$\left| \left\{ (x_1,\ldots,x_d) \in \mathbb{I}^d : \sup_{n\in\mathbb{N}} \frac{J^{(p)}_{nB2}(x_1,\ldots,x_d)}{2^{n/p}} > c\lambda \right\} \right| \tag{23}$$

$$\leqslant \frac{c(d,p)}{\lambda}\left(1+\int\limits_{\mathbb{I}^d} \left|E_*^{B'}(x_1,\ldots,x_d;f)\right| \left(\log^+ \left|E_*^{B'}(x_1,\ldots,x_d;f)\right|\right)^{|B|-1} dx_1\cdots dx_d\right)$$

$$\leqslant \frac{c(d,p)}{\lambda}\left(1+\int\limits_{\mathbb{I}^d} |f(x_1,\ldots,x_d)| \left(\log^+ |f(x_1,\ldots,x_d)|\right)^{d-1} dx_1\cdots dx_d\right).$$

Analogously, we can write

$$J^{(p)}_{nB3}(x_1,\ldots,x_d) \leqslant \int\limits_{\mathbb{I}^{|B|}} \int\limits_{[0,2^{-n}]^{|B'|}} S_{2^n,\ldots,2^n}(x_1 \dotplus t_1,\ldots,x_d \dotplus t_d;|f|) \tag{24}$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1,\ldots,x_d) w_m(t_{l_1} \dotplus \cdots \dotplus t_{l_r})\left(m+\frac{1}{2}\right)^{|B'|} \right| dt_1\cdots dt_d$$

$$\leqslant \frac{1}{2^{|B'|}} \int\limits_{\mathbb{I}^{|B|}} S^{(B)}_{2^n,\ldots,2^n}\left(x_M \dotplus t_B; E_*^{(B')}(|f|)\right)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1,\ldots,x_d)\, w_m\left(t_{l_1} \dotplus \cdots \dotplus t_{l_r}\right)\left(m+\frac{1}{2}\right)^{|B'|} \right| dt_{l_1}\cdots dt_{l_r},$$

$$\left| \left\{ (x_1,\ldots,x_d) \in \mathbb{I}^d : \sup_{n\in\mathbb{N}} \frac{J^{(p)}_{nB3}(x_1,\ldots,x_d)}{2^{n/p}} > c_1\lambda \right\} \right|$$

$$\leqslant \frac{c(d,p)}{\lambda}\left(1+\int\limits_{\mathbb{I}^d} |f(x_1,\ldots,x_d)| \left(\log^+ |f(x_1,\ldots,x_d)|\right)^{d-1} dx_1\cdots dx_d\right).$$

Combining (8), (17)–(24) we complete the proof of Theorem 1. □

REFERENCES

[1] N. YU. ANTONOV, *On the convergence almost everywhere of multiple trigonometric Fourier series over cubes* (Russian), Izv. Ross. Akad. Nauk Ser. Mat. **68**, 2 (2004), 3–22; translation in Izv. Math. **68**, 2 (2004), 223–241.

[2] M. I. DYACHENKO, *On the (C,α)-summability of multiple trigonometric Fourier series*, Soobshch. Akad. Grusin **131**, 2 (1988), 261–263 (Russian).

[3] L. FEJÉR, *Untersuchungen über Fouriersche Reihen*, Math. Annalen **58** (1904), 501–569.

[4] S. FRIDLI AND F. SCHIPP, *Strong summability and Sidon type inequalities*, Acta Sci. Math. (Szeged) **60**, 1–2 (1995), 277–289.

[5] S. FRIDLI AND F. SCHIPP, *Strong approximation via Sidon type inequalities*, J. Approx. Theory **94** (1998), 263–284.

[6] O. D. GABISONIA, *On strong summability points for Fourier series*, Mat. Zametki **14**, 5 (1973), 615–626.

[7] G. GÁT, U. GOGINAVA AND G. KARAGULYAN, *Almost everywhere strong summability of Marcinkiewicz means of double Walsh-Fourier series*, Anal. Math. **40**, 4 (2014), 243–266.

[8] G. GÁT, U. GOGINAVA AND G. KARAGULYAN, *On everywhere divergence of the strong Φ-means of Walsh-Fourier series*, J. Math. Anal. Appl. **421**, 1 (2015), 206–214.

[9] U. GOGINAVA, *Uniform convergence of Cesáro means of negative order of double Walsh-Fourier series*, J. Approx. Theory **124**, 1 (2003), 96–108.

[10] U. GOGINAVA, *The weak type inequality for the Walsh system*, Studia Math. **185**, 1 (2008), 35–48.

[11] U. GOGINAVA AND L. GOGOLADZE, *Strong approximation by Marcinkiewicz means of two-dimensional Walsh-Fourier series*, Constr. Approx. **35**, 1 (2012), 1–19.

[12] U. GOGINAVA AND L. GOGOLADZE, *Convergence in measure of strong logarithmic means of double Fourier series*, Izv. Nats. Akad. Nauk Armenii Mat. **49**, 3 (2014), 39–49; translation in J. Contemp. Math. Anal. **49**, 3 (2014), 109–116.

[13] U. GOGINAVA, L. GOGOLADZE AND G. KARAGULYAN, *BMO-estimation and almost everywhere exponential summability of quadratic partial sums of double Fourier series*, Constr. Approx. **40**, 1 (2014), 105–120.

[14] B. I. GOLUBOV, A. V. EFIMOV AND V. A. SKVORTSOV, *Series and transformations of Walsh*, Moscow, 1987 (Russian); English translation, Kluwer Academic, Dordrecht, 1991.

[15] R. GETSADZE, *On the boundedness in measure of sequences of superlinear operators in classes $L\phi(L)$*, Acta Sci. Math. (Szeged) **71**, 1–2 (2005), 195–226.

[16] L. GOGOLADZE, *On the exponential uniform strong summability of multiple trigonometric Fourier series*, Georgian Math. J. **16** (2009), 517–532.

[17] L. GOGOLADZE, *Strong means of Marcinkiewicz type*, (Russian), Soobshch. Akad. Nauk Gruzin. SSR **102**, 2 (1981), 293–295.

[18] V. A. GLUKHOV, *Summation of multiple Fourier series in multiplicative systems*, (Russian), Mat. Zametki **39**, 5 (1986), 665–673.

[19] R. E. EDWARDS, *Fourier series: a Modern Introduction*, vol. 1, Springer-Verlang, New-Yourk, Heidelberg, Berlin 1982.

[20] G. H. HARDY AND J. E. LITTLEWOOD, *Sur la series de Fourier d'une fonction a carre sommable*, Comptes Rendus (Paris) **156** (1913), 1307–1309.

[21] G. A. KARAGULYAN, *Everywhere divergent Φ-means of Fourier series*, (Russian), Mat. Zametki **80**, 1 (2006), 50–59; translation in Math. Notes **80**, 1–2 (2006), 47–56.

[22] S. V. KONYAGIN, *On the divergence of subsequences of partial sums of multiple trigonometric Fourier series*, Trudy MIAN **190** (1989), 102–116.

[23] H. LEBESGUE, *Recherches sur la sommabilite forte des series de Fourier*, Math. Annalen **61** (1905), 251–280.

[24] L. LEINDLER, *Über die Approximation im starken Sinne*, Acta Math. Acad. Hungar. **16** (1965), 255–262.

[25] L. LEINDLER, *On the strong approximation of Fourier series*, Acta Sci. Math. (Szeged) **38** (1976), 317–324.

[26] L. LEINDLER, *Strong approximation and classes of functions*, Mitteilungen Math. Seminar Giessen **132** (1978), 29–38.

[27] L. LEINDLER, *Strong approximation by Fourier series*, Akadémiai Kiadó, Budapest, 1985.

[28] J. MARCINKIEWICZ, *Sur la sommabilité forte de séries de Fourier*, (French), J. London Math. Soc. **14** (1939), 162–168.
[29] K. I. OSKOLKOV, *Strong summability of Fourier series*, (Russian), Studies in the theory of functions of several real variables and the approximation of functions, Trudy Mat. Inst. Steklov **172** (1985), 280–290, 355.
[30] V. A. RODIN, *The space BMO and strong means of Fourier series*, Anal. Math. **16**, 4 (1990), 291–302.
[31] V. A. RODIN, *BMO-strong means of Fourier series*, Funct. anal. Appl. **23** (1989), 73–74 (Russian).
[32] V. A. RODIN, *The space BMO and strong means of Fourier-Walsh series* (Russian), Mat. Sb. **182**, 10 (1991), 1463–1478; translation in Math. USSR-Sb **74**, 1 (1993), 203–218.
[33] F. SCHIPP, *On the strong summability of Walsh series*, Publ. Math. Debrecen **52**, 3–4 (1998), 611–633.
[34] F. SCHIPP, *Über die starke Summation von Walsh-Fourier Reihen*, Acta Sci. Math (Szeged) **30** (1969), 77–87.
[35] F. SCHIPP, *On strong approximation of Walsh-Fourier series*, MTA III. Oszt. Kozl. **19**, (1969), 101–111 (Hungarian).
[36] F. SCHIPP AND N. X. KY, *On strong summability of polynomial expansions*, Anal. Math. **12** (1986), 115–128.
[37] F. SCHIPP, W. WADE, P. SIMON AND P. PÁL, *Walsh Series, an Introduction to Dyadic Harmonic Analysis*, Adam Hilger, Bristol, New York, 1990.
[38] P. SJÖLIN, *Convergence almost everywhere of certain singular integrals and multiple Fourier series*, Ark. Mat. **9**, (1971), 65–90.
[39] V. TOTIK, *On the strong approximation of Fourier series*, Acta Math. Sci. Hungar. **35** (1980), 151–172.
[40] V. TOTIK, *On the generalization of Fejér's summation theorem, Functions, Series, Operators*, Coll. Math. Soc. J. Bolyai (Budapest) Hungary, **35**, North Holland, Amsterdam-Oxford-New-York, 1980, 1195–1199.
[41] V. TOTIK, *Notes on Fourier series: Strong approximation*, J. Approx. Theory **43** (1985), 105–111.
[42] F. WEISZ, *Convergence of double Walsh-Fourier series and Hardy spaces*, Approx. Theory & its. Appl. **17**, 2 (2001), 32–44.
[43] F. WEISZ, *Summability of multi-dimensional Fourier series and Hardy space*, Kluwer Academic, Dordrecht, 2002.
[44] F. WEISZ, *Strong summability of more-dimensional Ciesielski-Fourier series*, East J. Approx. **10**, 3 (2004), 333–354.
[45] F. WEISZ, *Strong Marcinkiewicz summability of multi-dimensional Fourier series*, Ann. Univ. Sci. Budapest. Sect. Comput. **29** (2008), 297–317.
[46] F. WEISZ, *Lebesgue points of double Fourier series and strong summability*, J. Math. Anal. Appl. **432**, 1 (2015), 441–462.
[47] L. V. ZHIZHIASHVILI, *Generalization of a theorem of Marcinkiwicz*, Izvest.AN USSR, ser. matem. **32** (1968), 1112–1122 (Russian).
[48] A. ZYGMUND, *Trigonometric series*, Cambridge University Press, Cambridge, 1959.

(Received September 13, 2016)

Ushangi Goginava
Department of Mathematics, Faculty of Exact and Natural Sciences
Ivane Javakhishvili Tbilisi State University
Chavchavadze str. 1, Tbilisi 0128, Georgia
e-mail: zazagoginava@gmail.com