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SHARP GAUTSCHI INEQUALITY FOR PARAMETER $0 < p < 1$ WITH APPLICATIONS

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Abstract. In the article, we present the best possible parameters a, b on the interval $(0,\infty)$ such that the Gautschi double inequality $[(x^p+a)^{1/p}-x]/a < e^{x^p}\int_x^\infty e^{-t^p}dt < [(x^p+b)^{1/p}-x]/b$ holds for all $x>0$ and $p\in(0,1)$. As applications, we find new bounds for the incomplete gamma function $\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt$.

1. Introduction

Let $a>0$ and $x>0$. Then the classical gamma function $\Gamma(x)$, incomplete gamma function $\Gamma(a,x)$ and psi function $\psi(x)$ are defined by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt, \quad \Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt, \quad \psi(x)=\frac{\Gamma'(x)}{\Gamma(x)},$$

respectively. It is well known that the identities

$$\int_x^\infty e^{-t^p}dt=\frac{1}{p}\Gamma\left(\frac{1}{p},x^p\right), \quad \int_0^x e^{-t^p}dt=\frac{1}{p}\Gamma\left(\frac{1}{p}\right)-\frac{1}{p}\Gamma\left(\frac{1}{p},x^p\right) \tag{1.1}$$

hold for all $x,p>0$.

Recently, the bounds and asymptotic expansions for the integral $\int_x^\infty e^{-t^p}dt$ or $\int_0^x e^{-t^p}dt$ have attracted the interest of many researchers. In particular, many remarkable inequalities and asymptotic formulas for both integrals can be found in the literature [2, 4, 6, 9, 11, 12, 13, 14, 18, 22, 23, 24, 25, 27, 28, 29, 31]. Let

$$I_p(x)=e^{x^p}\int_x^\infty e^{-t^p}dt. \tag{1.2}$$

Then we clearly see that

$$I_1(x)=1, \quad I_{1/2}(x)=2(\sqrt{x}+1), \tag{1.3}$$

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and $I_p(x)$ is divergent if $p \leqslant 0$. The functions $I_3(x)$ and $I_4(x)$ can be used to research the heat transfer problem [36] and electrical discharge in gases [30], respectively.

Komatu [17] and Pollak [26] proved that the double inequality

$$\frac{1}{\sqrt{x^2+2}+x} < I_2(x) < \frac{1}{\sqrt{x^2+\frac{4}{\pi}}+x}$$

holds for all $x > 0$.

In [8], Gautschi proved that the double inequality

$$\frac{1}{a}\left[(x^p+a)^{1/p}-x\right] < I_p(x) < \frac{1}{b}\left[(x^p+b)^{1/p}-x\right] \tag{1.4}$$

holds for all $x > 0$ and $p > 1$ if and only if $a \geqslant 2$ and

$$b \leqslant \lambda_0 = \Gamma^{p/(1-p)}\left(1+\frac{1}{p}\right) \tag{1.5}$$

by use of the monotonicity of the difference of the functions $I_p(x)$ and $[(x^p+a)-x]/a$.

An application of inequality (1.4) in radio propagation mode was given in [7].

Alzer [1] presented the best possible parameters α and β such that the double inequality

$$\left(1-e^{-\alpha x^p}\right)^{1/p} < \frac{1}{\Gamma\left(1+\frac{1}{p}\right)}\int_0^x e^{-t^p}dt < \left(1-e^{-\beta x^p}\right)^{1/p}$$

holds for all $x > 0$ and $p > 0$ with $p \neq 1$.

Motivated by the Gautschi double inequality (1.4), it is natural to ask what are the best possible parameters a and b on the interval $(0,\infty)$ such that the Gautschi double inequality (1.4) takes place for all $x > 0$ and $p \in (0,1)$? The main purpose of this paper is to answer this question and present new bounds for the incomplete gamma function $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$.

2. Lemmas

In order to prove our main results, we first need to introduce an auxiliary function.

Let $-\infty \leqslant a < b \leqslant \infty$, f and g be differentiable on (a,b), and $g' \neq 0$ on (a,b). Then the function $H_{f,g}$ [37, 39] is defined by

$$H_{f,g}(x) = \frac{f'(x)}{g'(x)}g(x) - f(x). \tag{2.1}$$

LEMMA 2.1. (See [37, Theorem 9]) *Let $\infty \leqslant a < b \leqslant \infty$, f and g be differentiable on (a,b) with $f(b^-) = g(b^-) = 0$ and $g'(x) < 0$ on (a,b), $H_{f,g}$ be defined by (2.1). Then the following statements are true:*

(1) *If* $H_{f,g}(a^+) > 0$ *and there exists* $\lambda \in (a,b)$ *such that* $f'(x)/g'(x)$ *is strictly decreasing on* (a,λ) *and strictly increasing on* (λ,b), *then there exists* $\mu \in (a,b)$ *such that* $f(x)/g(x)$ *is strictly decreasing on* (a,μ) *and strictly increasing on* (μ,b);

(2) *If* $H_{f,g}(a^+) < 0$ *and there exists* $\lambda^* \in (a,b)$ *such that* $f'(x)/g'(x)$ *is strictly increasing on* (a,λ^*) *and strictly decreasing on* (λ^*,b), *then there exists* $\mu^* \in (a,b)$ *such that* $f(x)/g(x)$ *is strictly increasing on* (a,μ^*) *and strictly decreasing on* (μ^*,b).

LEMMA 2.2. (See [3, Theorem 1.25]) *Let* $-\infty < a < b < \infty$, $f,g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b), *and* $g'(x) \neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b), *then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

LEMMA 2.3. *The double inequality*

$$x < \Gamma^{1/(x-1)}(1+x) < 2 \tag{2.2}$$

holds for all $x \in (1,2)$, *and inequality (2.2) is reversed for all* $x \in (2,\infty)$.

Proof. Let $J_1(x) = \log\Gamma(x+1)$, $J_2(x) = x-1$ and $J(x) = \log\left[\Gamma^{1/(x-1)}(1+x)\right]$. Then we clearly see that

$$J_1(1) = J_2(1) = 0, \quad J(x) = \frac{J_1(x)}{J_2(x)} \tag{2.3}$$

and $J_1'(x)/J_2'(x) = \psi(x+1)$ is strict increasing on the interval $(1,\infty)$.

It follows from Lemma 2.2 and (2.3) together with the monotonicity of the function $J_1'(x)/J_2'(x)$ on the interval $(1,\infty)$ that the function $\Gamma^{1/(x-1)}(1+x)$ is strictly increasing on $(1,\infty)$. Therefore, $\Gamma^{1/(x-1)}(1+x) < 2$ for $x \in (1,2)$ and $\Gamma^{1/(x-1)}(1+x) > 2$ for $x \in (2,\infty)$ follow easily from the monotonicity of the function $\Gamma^{1/(x-1)}(1+x)$ on the interval $(1,\infty)$.

Next, we prove that the inequality

$$\Gamma^{1/(x-1)}(1+x) > (<) x \tag{2.4}$$

holds for all $x \in (1,2)$ $(x \in (2,\infty))$. Let

$$\varphi(x) = \log\Gamma(x+1) - (x-1)\log x. \tag{2.5}$$

Then we clearly see that

$$\varphi(1) = \varphi(2) = 0 \tag{2.6}$$

and

$$\varphi''(x) = \psi'(x) - \frac{1}{x} - \frac{1}{x^2} < 0 \tag{2.7}$$

for $x \in (1,\infty)$.

Inequality (2.7) implies that $\varphi(x)$ is strictly concave on $(1,\infty)$. Then equation (2.6) leads to the conclusion that

$$\varphi(x) > (2-x)\varphi(1) + (x-1)\varphi(2) = 0 \tag{2.8}$$

for all $x \in (1,2)$, and

$$0 = \varphi(2) > \frac{x-2}{x-1}\varphi(1) + \frac{1}{x-1}\varphi(x) = \frac{1}{x-1}\varphi(x) \tag{2.9}$$

for $x > 2$.

Therefore, inequality (2.4) follows easily from (2.5), (2.8) and (2.9). □

LEMMA 2.4. *Let $p \in (0,1)$ and $a,x \in (0,\infty)$. Then the function $a \to [(x^p + a)^{1/p} - x]/a$ is strictly increasing on $(0,\infty)$.*

Proof. Let

$$\omega_1(a) = (x^p + a)^{1/p} - x, \quad \omega_2(a) = a, \quad \omega(a) = \frac{\omega_1(a)}{\omega_2(a)} = \frac{(x^p + a)^{1/p} - x}{a}. \tag{2.10}$$

Then we clearly see that

$$\omega_1(0) = \omega_2(0) = 0, \tag{2.11}$$

$$\left[\frac{\omega_1'(a)}{\omega_2'(a)}\right]' = \frac{1-p}{p^2\,(x^p + a)^{(2p-1)/p}} > 0 \tag{2.12}$$

for all $p \in (0,1)$ and $a,x \in (0,\infty)$.

Therefore, Lemma 2.4 follows easily from Lemma 2.2 and (2.10)–(2.12). □

LEMMA 2.5. *Let $p \in (0,1)$ and $a,x \in (0,\infty)$, $H_{f,g}(x)$ be defined by (2.1), and $f_1(x)$ and $g_1(x)$ be defined by*

$$f_1(x) = \left[(x^p + a)^{1/p} - x\right] e^{-x^p}, \quad g_1(x) = \int_x^\infty e^{-t^p}\,dt, \tag{2.13}$$

respectively. Then the following statements are true:

(1) $H_{f_1,g_1}(0^+) = +\infty$ *for* $a > 1/p$;

(2) $H_{f_1,g_1}(0^+) = -\infty$ *for* $a < 1/p$.

Proof. Let

$$u = u(x) = \left(\frac{x^p + a}{x^p}\right)^{1/p} \in (1,\infty). \tag{2.14}$$

Then from (2.13) and (2.14) one has

$$f_1(0) = a^{1/p}, \quad g_1(0) = \frac{1}{p}\Gamma\left(\frac{1}{p}\right) = \Gamma\left(1 + \frac{1}{p}\right) > 0, \tag{2.15}$$

$$\begin{aligned}\frac{f_1'(x)}{g_1'(x)} &= -\left(\frac{x^p+a}{x^p}\right)^{1/p-1} + px^p\left[\left(\frac{x^p+a}{x^p}\right)^{1/p}-1\right]+1 \qquad (2.16)\\ &= 1+\frac{(pa-1)u+u^{1-p}-pa}{u^p-1}.\end{aligned}$$

It follows from (2.1), (2.15) and (2.16) that

$$\begin{aligned}H_{f_1,g_1}(0^+) &= \lim_{x\to 0^+}\frac{f_1'(x)}{g_1'(x)}\lim_{x\to 0^+}g_1(x)-\lim_{x\to 0^+}f_1(x) \qquad (2.17)\\ &= \Gamma\left(1+\frac{1}{p}\right)\left[1+\lim_{u\to\infty}\frac{(pa-1)u+u^{1-p}-pa}{u^p-1}\right]-a^{1/p}.\end{aligned}$$

Note that

$$\lim_{u\to\infty}\frac{(pa-1)u+u^{1-p}-pa}{u^p-1}=\begin{cases}+\infty, & a>\frac{1}{p},\\ -\infty, & a<\frac{1}{p}.\end{cases} \qquad (2.18)$$

Therefore, Lemma 2.5 follows from (2.17) and (2.18). □

LEMMA 2.6. *Let* $p\in(0,1/2)\cup(1/2,1)$*,* $a,x\in(0,\infty)$*,* $I_p(x)$ *be defined by (1.2) and the function* $x\to R_p(a,x)$ *be defined by*

$$R_p(a,x)=\frac{(x^p+a)^{1/p}-x}{I_p(x)}. \qquad (2.19)$$

Then the following statements are true:

(1) *The function* $x\to R_p(a,x)$ *is strictly decreasing on* $(0,\infty)$ *if* $a\geqslant\max\{1/p,2\}$*;*

(2) *The function* $x\to R_p(a,x)$ *is strictly increasing on* $(0,\infty)$ *if* $a\leqslant\min\{1/p,2\}$*;*

(3) *There exists* $x_0\in(0,\infty)$ *such that the function* $x\to R_p(a,x)$ *is strictly increasing on* $(0,x_0)$ *and strictly decreasing on* (x_0,∞) *if* $p\in(0,1/2)$ *and* $2=\min\{1/p,2\}<a<\max\{1/p,2\}=1/p$*;*

(4) *There exists* $x^*\in(0,\infty)$ *such that the function* $x\to R_p(a,x)$ *is strictly decreasing on* $(0,x^*)$ *and strictly increasing on* (x^*,∞) *if* $p\in(1/2,1)$ *and* $1/p=\min\{1/p,2\}<a<\max\{1/p,2\}=2$.

Proof. Let $f_1(x)$ and $g_1(x)$ be defined by (2.13), $u=u(x)\in(1,\infty)$ be defined by (2.14), and and $h(u)$ and $h_1(u)$ be defined by

$$h(u)=(p-1)(ap-1)u^{2p}-ap^2u^{2p-1}+(2p+ap-2)u^p+1-p, \qquad (2.20)$$

$$h_1(u)=2(p-1)(ap-1)u^p-ap(2p-1)u^{p-1}+2p+ap-2. \qquad (2.21)$$

Then we clearly see that

$$f_1(\infty)=g_1(\infty)=0, \quad g_1'(x)=-e^{x^p}<0, \qquad (2.22)$$

$$R_p(a,x)=\frac{f_1(x)}{g_1(x)}. \qquad (2.23)$$

It follows from (2.14), (2.16), (2.20) and (2.21) that

$$h(1) = h_1(1) = 0, \tag{2.24}$$

$$\left[\frac{f_1'(x)}{g_1'(x)}\right]' = \frac{\frac{d}{du}\left[1+\frac{(pa-1)u+u^{1-p}-pa}{u^p-1}\right]}{\frac{dx}{du}} = \frac{(u^p-1)^{1/p-1}}{a^{1/p}u^{2p-1}}h(u), \tag{2.25}$$

$$h'(u) = pu^{p-1}h_1(u), \tag{2.26}$$

$$h_1'(u) = p(p-1)u^{p-2}[2(ap-1)(u-1)+(a-2)]. \tag{2.27}$$

We divide the proof into six cases.

Case 1 $p \in (1/2,1)$ and $a \geqslant \max\{1/p,2\}$. Then we clearly see that $a \geqslant 2 > 1/p$ and (2.24)–(2.27) lead to the conclusion that $f_1'(x)/g_1'(x)$ is strictly decreasing on $(0,\infty)$. Therefore, the function $x \to R_p(a,x)$ is strictly decreasing on $(0,\infty)$ follows from (2.22), (2.23) and Lemma 2.2 together with the monotonicity of $f_1'(x)/g_1'(x)$ on the interval $(0,\infty)$.

Case 2 $p \in (1/2,1)$ and $a \leqslant \min\{1/p,2\}$. Then we clearly see that $a \leqslant 1/p < 2$ and (2.24)–(2.27) lead to the conclusion that $f_1'(x)/g_1'(x)$ is strictly increasing on $(0,\infty)$. Therefore, the function $x \to R_p(a,x)$ is strictly increasing on $(0,\infty)$ follows from (2.22), (2.23) and Lemma 2.2 together with the monotonicity of $f_1'(x)/g_1'(x)$ on the interval $(0,\infty)$.

Case 3 $p \in (1/2,1)$ and $\min\{1/p,2\} < a < \max\{1/p,2\}$. Then we clearly see that

$$\frac{1}{p} < a < 2 \tag{2.28}$$

and (2.27) can be rewritten as

$$h_1'(u) = 2p(ap-1)(p-1)u^{p-2}(u-u_0) \tag{2.29}$$

with $u_0 = 1+(2-a)/[2(ap-1)] \in (1,\infty)$.

It follows from (2.20), (2.21), (2.28) and (2.29) that

$$h(\infty) = -\infty, \quad h_1(\infty) = -\infty \tag{2.30}$$

and $h_1(u)$ is strictly increasing on $(1,u_0)$ and strictly decreasing on (u_0,∞). Then (2.24), (2.26) and (2.30) lead to the conclusion that there exists $u_1 \in (1,\infty)$ such that $h(u)$ is strictly increasing on $(1,u_1)$ and strictly decreasing on (u_1,∞).

From (2.14) we clearly see that the function $x \to u = u(x)$ is strictly decreasing from $(0,\infty)$ onto $(1,\infty)$. Then from (2.24), (2.25) and (2.30) together with the piecewise monotonicity of $h(u)$ on the interval $(0,\infty)$ we know that there exists $x_1 \in (0,\infty)$ such that $f_1'(x)/g_1'(x)$ is strictly decreasing on $(0,x_1)$ and strictly increasing on (x_1,∞).

Therefore, part (4) follows from Lemma 2.5(1), (2.22) and (2.28) together with the piecewise monotonicity of $f_1'(x)/g_1'(x)$ on the interval $(0,\infty)$ and Lemma 2.1(1).

Case 4 $p \in (0,1/2)$ and $a \geqslant \max\{1/p,2\}$. Then we clearly see that $a \geqslant 1/p > 2$ and (2.24)–(2.27) lead to the conclusion that $f_1'(x)/g_1'(x)$ is strictly decreasing on $(0,\infty)$. Therefore, the function $x \to R_p(a,x)$ is strictly decreasing on $(0,\infty)$ follows from Lemma 2.2, (2.22), (2.23) and the monotonicity of the function $f_1'(x)/g_1'(x)$ on the interval $(0,\infty)$.

Case 5 $p \in (0,1/2)$ and $a \leqslant \min\{1/p,2\}$. Then we clearly see that $a \leqslant 2 < 1/p$ and (2.24)–(2.27) lead to the conclusion that $f_1'(x)/g_1'(x)$ is strictly increasing on $(0,\infty)$. Therefore, the function $x \to R_p(a,x)$ is strictly increasing on $(0,\infty)$ follows from Lemma 2.2, (2.22), (2.23) and the monotonicity of the function $f_1'(x)/g_1'(x)$ on the interval $(0,\infty)$.

Case 6 $p \in (0,1/2)$ and $\min\{1/p,2\} < a < \max\{1/p,2\}$. Then we clearly see that

$$2 < a < 1/p, \tag{2.31}$$

and (2.20), (2.21) and (2.29) lead to the conclusion that

$$h(\infty) = +\infty, \quad h_1(\infty) = +\infty \tag{2.32}$$

and $h_1(u)$ is strictly decreasing on $(1,u_0)$ and strictly increasing on (u_0,∞).

It follows from (2.24), (2.26), (2.32) and the piecewise monotonicity of the function $h_1(u)$ on the interval $(1,\infty)$ that there exists $u_2 \in (1,\infty)$ such that $h(u)$ is strictly decreasing on $(1,u_2)$ and strictly increasing on (u_2,∞). Then (2.24), (2.25), (2.32) and the monotonicity of the function $x \to u = u(x)$ lead to the conclusion that there exists $x_2 \in (0,\infty)$ such that $f_1'(x)/g_1'(x)$ is strictly increasing on $(0,x_2)$ and strictly decreasing on (x_2,∞).

Therefore, part (3) follows from Lemma 2.1(2), Lemma 2.5(2), (2.22), (2.23) and (2.31) together with the piecewise monotonicity of $f_1'(x)/g_1'(x)$ on $(0,\infty)$. □

REMARK 2.7. Let $R_p(a,x)$ be defined by (2.19). Then from (2.15), (2.16), (2.22) and (2.23) we clearly see that

$$R_p(a,0^+) = \frac{a^{1/p}}{\Gamma\left(1+\frac{1}{p}\right)}, \tag{2.33}$$

$$R_p(a,\infty) = \lim_{x\to\infty} \frac{f_1'(x)}{g_1'(x)} = 1 + \lim_{u\to 1^-} \frac{(pa-1)u + u^{1-p} - pa}{u^p - 1} = a. \tag{2.34}$$

REMARK 2.8. Let $p \in (0,1/2)\cup(1/2,1)$, $a,x \in (0,\infty)$ and $R_p(a,x)$ be defined by (2.19). Then from Lemma 2.6(3) and (4) we know that the equation

$$\frac{dR_p(a,x)}{dx} = 0$$

has a unique solution $x = \mu_0$ on the interval $(0,\infty)$ if $\min\{1/p,2\} < a < \max\{1/p,2\}$.

From Lemma 2.6 and Remarks 2.7 and 2.8 we get Corollary 2.9 immediately.

COROLLARY 2.9. *Let $p \in (0,1/2)\cup(1/2,1)$, $a,x \in (0,\infty)$, $I_p(x)$ be defined by (1.2) and μ_0 be defined by Remark 2.8. Then the following statements are true:*

(1) *If* $a \geqslant \max\{1/p, 2\}$*, then the double inequality*

$$\frac{\Gamma\left(1+\frac{1}{p}\right)}{a^{1/p}}\left[(x^p+a)^{1/p}-x\right] < I_p(x) < \frac{1}{a}\left[(x^p+a)^{1/p}-x\right] \tag{2.35}$$

holds for all $x \in (0,\infty)$*, and inequality (2.35) is reversed if* $a \leqslant \min\{1/p, 2\}$*;*

(2) *If* $p \in (0,1/2)$ *and* $2 = \min\{1/p,2\} < a < \max\{1/p,2\} = 1/p$*, then the double inequality*

$$\frac{1}{R_p(a,\mu_0)}\left[(x^p+a)^{1/p}-x\right] \leqslant I_p(x) < \max\left\{\frac{\Gamma\left(1+\frac{1}{p}\right)}{a^{1/p}}, \frac{1}{a},\right\}\left[(x^p+a)^{1/p}-x\right]$$

takes place for all $x \in (0,\infty)$*;*

(3) *If* $p \in (1/2,1)$ *and* $1/p = \min\{1/p,2\} < a < \max\{1/p,2\} = 2$*, then the double inequality*

$$\min\left\{\frac{\Gamma\left(1+\frac{1}{p}\right)}{a^{1/p}}, \frac{1}{a},\right\}\left[(x^p+a)^{1/p}-x\right] < I_p(x) \leqslant \frac{1}{R_p(a,\mu_0)}\left[(x^p+a)^{1/p}-x\right]$$

is valid for all $x \in (0,\infty)$*.*

REMARK 2.10. Let $a,x > 0$ and $R_p(a,x)$ be defined by (2.19). Then from (1.3) we clearly see that

$$R_{1/2}(a,x) = a + \frac{a(a-2)}{2(1+\sqrt{x})}, \quad R_{1/2}(2,x) = 2,$$

the identities (2.33) and (2.34) are also valid for $p = 1/2$, $R_{1/2}(a,x)$ is strictly decreasing from $(0,\infty)$ onto $(a, a^2/2)$ if $a > 2$ and strictly increasing from $(0,\infty)$ onto $(a^2/2, a)$ if $a < 2$, inequality (2.35) holds for $p = 1/2$ and all $x > 0$ if $a > 2$ and the reversed inequality of (2.35) takes place for $p = 1/2$ and all $x > 0$ if $a < 2$.

3. Main results

THEOREM 3.1. *Let* $p \in (0,1)$*,* $a,b > 0$*,* $x > 0$*,* $I_p(x)$ *be defined by (1.2) and* λ_0 *be defined by (1.5). Then the following statements are true:*

(1) *If* $p \in (0,1/2)$*, then the double inequality*

$$\frac{1}{a}\left[(x^p+a)^{1/p}-x\right] < I_p(x) < \frac{1}{b}\left[(x^p+b)^{1/p}-x\right] \tag{3.1}$$

holds for all $x > 0$ *if and only if* $a \leqslant 2$ *and* $b \geqslant \lambda_0$*;*

(2) *If* $p \in (1/2,1)$*, then inequality (3.1) takes place for all* $x > 0$ *if and only if* $a \leqslant \lambda_0$ *and* $b \geqslant 2$*;*

(3) *If* $p=1/2$*, then inequality (3.1) is valid for all* $x>0$ *if and only if* $a<2$ *and* $b>2$*;*

(4) *If* $p=1/2$ *and* $a=2$*, then the identity*

$$I_p(x)=\frac{1}{a}\left[(x^p+a)^{1/p}-x\right] \tag{3.2}$$

holds for all $x>0$.

Proof. (1) For $p\in(0,1/2)$, we first prove that the inequality

$$I_p(x)>\frac{1}{a}\left[(x^p+a)^{1/p}-x\right] \tag{3.3}$$

holds for all $x>0$ if and only if $a\leqslant 2$.

If $p\in(0,1/2)$ and $a\leqslant 2$, then $a\leqslant\min\{1/p,2\}=2<1/p$ and Corollary 2.9(1) leads to the conclusion that inequality (3.3) holds for all $x>0$.

If $p\in(0,1/2)$ and inequality (3.3) holds for all $x>0$, then we use the proof by contradiction to prove that $a\leqslant 2$. We divide the proof into two cases.

Case 1 $p\in(0,1/2)$ and $a\geqslant 1/p$. Then $a\geqslant\max\{1/p,2\}$ and Corollary 2.9(1) leads to the conclusion that $I_p(x)<\left[(x^p+a)^{1/p}-x\right]/a$ for all $x>0$, which contradicts with inequality (3.3).

Case 2 $p\in(0,1/2)$ and $2<a<1/p$. Then $\min\{1/p,2\}<a<\max\{1/p,2\}$. Let $R_p(a,x)$ be defined by (2.19), then from Lemma 2.6(3) and (2.34) we clearly see that there exists $x_0\in(0,\infty)$ such that $I_p(x)<\left[(x^p+a)^{1/p}-x\right]/a$ for all $x\in(x_0,\infty)$, which also contradicts with inequality (3.3).

Next, we prove that the inequality

$$I_p(x)<\frac{1}{b}\left[(x^p+b)^{1/p}-x\right] \tag{3.4}$$

holds for all $p\in(0,1/2)$ and $x>0$ if and only if $b\geqslant\lambda_0$.

It follows from (1.5) and Lemma 2.3 together with Corollary 2.9(2) that $2=\min\{1/p,2\}<\lambda_0<\max\{1/p,2\}=1/p$, $1/\lambda_0=\Gamma(1+1/p)/\lambda_0^{1/p}$ and

$$\begin{aligned} I_p(x)&<\max\left\{\frac{\Gamma\left(1+\frac{1}{p}\right)}{\lambda_0^{1/p}},\frac{1}{\lambda_0}\right\}\left[(x^p+\lambda_0)^{1/p}-x\right] \\ &=\frac{1}{\lambda_0}\left[(x^p+\lambda_0)^{1/p}-x\right] \end{aligned} \tag{3.5}$$

for all $x>0$.

Therefore, inequality (3.4) holds for $p\in(0,1/2)$, $b\geqslant\lambda_0$ and all $x>0$ follows from Lemma 2.4 and (3.5).

If inequality (3.4) holds for $p\in(0,1/2)$ and all $x>0$. Then from (2.19) and (2.33) we get

$$\frac{R_p(b,0^+)}{b}=\frac{b^{1/p-1}}{\Gamma\left(1+\frac{1}{p}\right)}\geqslant 1,\quad b\geqslant\lambda_0.$$

(2) For $p \in (1/2,1)$, we first prove that the inequality

$$I_p(x) > \frac{1}{a}\left[(x^p+a)^{1/p} - x\right] \tag{3.6}$$

holds for all $x > 0$ if and only if $a \leqslant \lambda_0$.

If $p \in (1/2,1)$, then from (1.5), Lemma 2.3 and Corollary 2.9(3) we clearly see that $1/p = \min\{1/p,2\} < \lambda_0 < \max\{1/p,2\} = 2$, $1/\lambda_0 = \Gamma(1+1/p)/\lambda_0^{1/p}$ and

$$\begin{aligned} I_p(x) &> \min\left\{\frac{\Gamma\left(1+\frac{1}{p}\right)}{\lambda_0^{1/p}}, \frac{1}{\lambda_0}\right\}\left[(x^p+\lambda_0)^{1/p} - x\right] \\ &= \frac{1}{\lambda_0}\left[(x^p+\lambda_0)^{1/p} - x\right] \end{aligned} \tag{3.7}$$

for all $x > 0$. Therefore, inequality (3.6) holds for $p \in (1/2,1)$, $a \leqslant \lambda_0$ and all $x > 0$ follows from Lemma 2.4 and (3.7).

If $p \in (1/2,1)$ and inequality (3.6) holds all $x > 0$, then equations (2.19) and (2.33) lead to the conclusion that

$$\frac{R_p(a,0^+)}{a} = \frac{a^{1/p-1}}{\Gamma\left(1+\frac{1}{p}\right)} \leqslant 1, \quad a \leqslant \lambda_0.$$

Next, we prove that the second inequality of (3.1) holds for all $p \in (1/2,1)$ and $x > 0$ if and only if $b \geqslant 2$.

If $p \in (1/2,1)$ and $b \geqslant 2$. Then $b \geqslant \max\{1/p,2\} = 2 > 1/p$ and the second inequality of (3.1) holds for all $x > 0$ follows from Corollary 2.9(1).

We use the proof by contradiction to prove that $b \geqslant 2$ if $p \in (1/2,1)$ and the second inequality of (3.1) holds for all $x > 0$. We divide the proof into two cases.

Case 1 $p \in (1/2,1)$ and $b \leqslant \min\{1/p,2\} = 1/p$. Then Corollary 2.9(1) leads to the conclusion that $I_p(x) > \left[(x^p+b)^{1/p} - x\right]/b$ for all $x > 0$, which contradicts with the second inequality of (3.1).

Case 2 $p \in (1/2,1)$ and $1/p = \min\{1/p,2\} < b < \max\{1/p,2\} = 2$. Then from Lemma 2.6(4) and (2.34) we know that there exists $x^* \in (0,\infty)$ such that the function $x \to R_p(b,x)$ is strictly decreasing on $(0,x^*)$ and strictly increasing on (x^*,∞) and $I_p(x) > \left[(x^p+b)^{1/p} - x\right]/b$ for all $x \in (x^*,\infty)$, which also contradicts with the second inequality of (3.1).

(3) If $p = 1/2$, then inequality (3.1) holds for $a < 2$, $b > 2$ and all $x > 0$ follows easily from Remark 2.10.

If $p = 1/2$ and inequality (3.1) holds for all $x > 0$. Then from (1.3) and (3.1) we get

$$2\sqrt{x} + a < 2(\sqrt{x}+1) < 2\sqrt{x} + b$$

and

$$a < 2, \quad b > 2.$$

(4) If $p=1/2$ and $a=2$, then from (1.3) we clearly see that

$$I_p(x)=\frac{1}{a}\left[(x^p+a)^{1/p}-x\right]=2(\sqrt{x}+1). \quad \square$$

It is well known that the incomplete gamma function $\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt$ has many important applications in the fields of special functions [16, 32, 35], statistics [15], integral equations [10], semiconductors [20] and transcendence [21]. For more information about the incomplete gamma function, we refer the interested readers to see [5, 19, 33, 34, 38, 40, 41, 42, 43, 44, 45]. As applications of Theorem 3.1, we next present new bounds for the incomplete gamma function to end this article.

Let $p\in(0,1)$, $a,x>0$, $q=1/p\in(1,\infty)$ and $u=x^p>0$. Then from (1.1) and (1.2) one has

$$I_p(x)=qe^u\Gamma(q,u), \quad (x^p+a)^{1/p}-x=(u+a)^q-u^q,$$

and Corollary 2.9, Remark 2.10 and Theorem 3.1 lead to Corollaries 3.2 and 3.3 immediately.

COROLLARY 3.2. *Let $q>1$, $a>0$ and $u>0$. Then the following statements are true:*

(1) *If $(q,a)\in\{(q,a)|a\geqslant q>2\}\cup\{(q,a)|a\geqslant 2>q>1\}\cup\{(q,a)|a>q=2\}$, then the double inequality*

$$\frac{\Gamma(1+q)\left[(u+a)^q-u^q\right]}{qa^q}<e^u\Gamma(q,u)<\frac{(u+a)^q-u^q}{qa} \tag{3.8}$$

holds for all $u>0$, and inequality (3.8) is reversed if $(q,a)\in\{(q,a)|q>2\geqslant a\}\cup\{(q,a)|a\leqslant q,1<q<2\}\cup\{(q,a)|a<q=2\}$;

(2) *If $q>a>2$, then the double inequality*

$$\frac{e^{u_0}\Gamma(q,u_0)}{(u_0+a)^q-u_0^q}\left[(u+a)^q-u^q\right]\leqslant e^u\Gamma(q,u)<\max\left\{\frac{\Gamma(1+q)}{a^q},\frac{1}{a}\right\}\frac{(u+a)^q-u^q}{q}$$

takes place for all $u>0$, where u_0 is the unique solution of the equation

$$\frac{d}{du}\left[\frac{(u+a)^q-u^q}{e^u\Gamma(q,u)}\right]=0$$

on the interval $(0,\infty)$;

(2) *If $1<q<a<2$, then the double inequality*

$$\min\left\{\frac{\Gamma(1+q)}{a^q},\frac{1}{a}\right\}\frac{(u+a)^q-u^q}{q}<e^u\Gamma(q,u)\leqslant\frac{e^{u_0}\Gamma(q,u_0)}{(u_0+a)^q-u_0^q}\left[(u+a)^q-u^q\right]$$

is valid for all $u>0$.

COROLLARY 3.3. *Let $q > 1$ and $a,b,u > 0$ and λ_0 be defined by (1.5). Then the following statements are true:*

(1) *If $q > 2$, then the double inequality*

$$\frac{(u+a)^q - u^q}{qa} < e^u\Gamma(q,u) < \frac{(u+b)^q - u^q}{qb} \tag{3.9}$$

holds for all $u > 0$ if and only if $a \leqslant 2$ and $b \geqslant \lambda_0$;

(2) *If $1 < q < 2$, then inequality (3.9) takes place for all $u > 0$ if and only if $a \leqslant \lambda_0$ and $b \geqslant 2$;*

(3) *If $q = 2$, then inequality (3.9) is valid for all $u > 0$ if and only if $a < 2$ and $b > 2$;*

(4) *If $a = q = 2$, then the identity*

$$e^u\Gamma(q,u) = \frac{(u+a)^q - u^q}{qa}$$

holds for all $u > 0$.

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