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A NOTE ON HÖLDER'S INEQUALITY FOR MATRIX–VALUED MEASURES

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Abstract. Following [1], we prove a version of Holder's inequality for matrix-valued measures. As corollaries, an integral version of moment type inequalities in [3] and Minkowski inequality are derived.

1. Introduction and preliminary notation

We present a useful generalization of Hölder's inequality to matrix-valued probability measures. Compared to the scalar case, the inequality holds only for a very restricted set of couples (p,q), where $q=(1-1/p)^{-1}$ is the Hölder conjugate, but only if the random objects integrated are matrix-valued.

Before stating the main result, we introduce some notation and concepts. We refer to Farenick and Zhou (2007) for more details.

For $n\in\mathbb{N}$, let H^n denote the vector space of $n\times n$ Hermitian matrices over the field $\mathbb{C}$. The space H^n is a partially ordered set, and we say $A\leqslant B$ if and only if $\langle Av,v\rangle\leqslant\langle Bv,v\rangle$ for all $v\in\mathbb{C}^n$, and $\langle\cdot,\cdot\rangle$ the inner product in $\mathbb{C}^n$.

We denote by $\|v\|=(\langle v,v\rangle)^{1/2}$ the (Euclidean) norm in $\mathbb{C}^n$ and by $\|A\|$ the operator norm induced on H^n, namely:

$$\|A\|=\max_{\|v\|=1}\{\|Av\|\},\qquad \forall A\in H^n.$$

By the spectral theorem, $\|A\|=\max\{|\lambda_1|,\ldots,|\lambda_n|\}$ where λ_j are the eigenvalues of A. We denote by $\lambda(A)$ the spectrum of A, i.e. the set of all (real) eigenvalues of A.

Positive definiteness of A is equivalent to $\min\{\lambda(A)\}>0$. If $\min\{\lambda(A)\}\geqslant 0$ then A is said to be positive semi-definite.

Let $J\subset\mathbb{R}$ be an interval and $\lambda(A)\in J$. Let $\varphi: J\mapsto\mathbb{R}$ be a continuous function. Then the operator $\varphi(A)$, $A\in H^n$ is defined and has spectrum $\{\varphi(\lambda),\lambda\in\lambda(A)\}$.

We say that $\varphi: J\mapsto\mathbb{R}$ is an operator-convex function if for all n, $A,B\in H^n$ such that $\lambda(A)\cup\lambda(B)\subset J$ and for all $t\in[0,1]$,

$$\varphi(tA+(1-t)B)\leqslant t\varphi(A)+(1-t)\varphi(B).$$

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DEFINITION 1. Given a measurable space $(X,\mathscr{S})$, a function $f: X \mapsto H^n$ is measurable if and only if for all $v \in \mathbb{C}^n$, the function $\langle fv, v\rangle : X \mapsto \mathbb{R}$ is measurable, namely, for all $E \in \mathscr{B}(\mathbb{R})$, $\{x :\ \langle f(x)v, v\rangle \in E\} \in \mathscr{S}$.

DEFINITION 2. Let $(X,\mathscr{S},\mu)$ be a probability space and $f : X \mapsto H^n$. Then f is integrable if for every $v \in \mathbb{C}^n$, the function $\langle fv, v\rangle$ is integrable and the integral is denoted by

$$\int_X \langle f(x)v, v\rangle d\mu(x).$$

There exists a unique matrix $A \in H^n$, such that

$$\langle Av, u\rangle = \int_X \langle f(x)v, u\rangle d\mu(x) \qquad \text{for all } u, v \in \mathbb{C}^n.$$

The matrix A is the Bochner integral and is denoted by $\int f d\mu$. Two important properties of this integral are linearity and monotonicity:

$$\int (f+g)d\mu = \int f d\mu + \int g d\mu$$

$$\int f d\mu \leqslant \int g d\mu \text{ iff } f(x) \leqslant g(x) \text{ (in } H^n\text{) for all } x \in X.$$

DEFINITION 3. If $(X,\mathscr{S})$ is a measurable space, a matrix-valued probability measure is a function $\nu : \mathscr{S} \mapsto H^n$ such that $\nu(\emptyset) = \mathbf{0}$, $\nu(E)$ is positive semi-definite, ν is countably additive and $\nu(X) = I_n$ (the identity matrix).

Note that the function $\mu_X(E) = \frac{1}{k} trace(\nu(E))$, $E \in \mathscr{S}$ is a scalar-valued measure and is absolutely continuous w.r.t ν.

We say that $f : X \mapsto H^n$ is a nonnegative (positive) function if $f(x)$ is nonnegative (positive) definite for all $x \in X$.

2. Hölder's inequality

THEOREM 1. (Hölder's inequality for matrix functions) Let $1 \leqslant p \leqslant 2$, and $q = 1/(1-p^{-1})$ its Hölder conjugate.

(i) (Scalar measures) Let $(X,\mathscr{S},\mu)$ be a probability space. Let $f : X \mapsto H^n$, $g : X \mapsto H^n$ be two positive $\mu-$measurable functions and let $J \subset \mathbb{R}$ be a closed subset, such that $\lambda(f(x)) \cup \lambda(g(y)) \in J$ for all $x, y \in X$. Let c be the matrix-valued function satisfying $g(x) = c(x)c(X)^*$ for all x.

$$\int_X cfc^* d\mu \leqslant \left(\int_X g^q d\mu\right)^{1/2q} \left(\int_X f^p d\mu\right)^{1/p} \left(\int_X g^q d\mu\right)^{1/2q}. \tag{1}$$

(ii) (matrix-valued measures) Let ν a matrix-valued probability measure defined on $(X,\mathscr{S})$. Let $f : X \mapsto H^n$, $g : X \mapsto H^n$ be two positive $\nu-$measurable functions and let $J \subset \mathbb{R}$ be a closed subset, such that $\lambda(f(x)) \cup \lambda(g(y)) \in J$ for all $x, y \in X$. Let c be the matrix-valued function satisfying $g(x) = c(x)c(X)^*$ for all x. Then,

$$\int_X cfc^* d\nu \leqslant \left(\int_X g^q d\nu\right)^{1/2q} \left(\int_X f^p d\nu\right)^{1/p} \left(\int_X g^q d\nu\right)^{1/2q}. \tag{2}$$

Proof. (i) The function $\varphi_p : \mathbb{R} \mapsto \mathbb{R},\ \varphi_p(z) = z^p$ is operator convex for all $1 \leqslant p \leqslant 2$ or $-1 \leqslant p \leqslant 0$.

Let

$$\nu = \left(\int g^q d\mu\right)^{-1/2} c^q \mu\,(c^*)^q \left(\int g^q d\mu\right)^{-1/2}$$

be a matrix-valued probability measure. Then ν is absolutely continuous w.r.t. μ, with Radon-Nicodỳm derivative equal to the p.s.d. matrix $\frac{d\nu}{d\mu}$ such that

$$\int_E \frac{d\nu}{d\mu} d\mu = \int_E d\nu, \quad \forall E \in \mathscr{S}.$$

Note that, for any integrable function h, the integral $\int_X h d\nu$ can be written as an integral of the scalar measure μ:

$$\begin{aligned}\int_X h d\nu &= \int_X \left(\frac{d\nu}{d\mu}\right)^{1/2} h \left(\frac{d\nu}{d\mu}\right)^{1/2} d\mu \\ &= \left(\int g^q d\mu\right)^{-1/2} \int c^q h (c^*)^q d\mu \left(\int g^q d\mu\right)^{-1/2}.\end{aligned}$$

Since f is $\nu-$measurable, we can apply Theorem 4.2 of Farenick and Zhou (2007) to the operator-convex function φ_p and to the nonnegative function $h = c^{1-q} f (c^*)^{1-q}$:

$$\varphi_p\left(\int h d\nu\right) \leqslant \int \varphi_p(h)\, d\nu$$

which also implies, since $1/p \in [1/2, 1]$ and $z^{1/p}$ is operator monotone:

$$\int c^{1-q} f (c^*)^{1-q} d\nu \leqslant \left[\int \varphi_p\left(c^{1-q} f (c^*)^{1-q}\right) d\nu\right]^{1/p}$$

Then,

$$\begin{aligned}\left(\int g^q d\mu\right)^{-1/2} \int_X cfc^* d\mu \left(\int g^q d\mu\right)^{-1/2} &= \int_X c^{1-q} f (c^*)^{1-q} d\nu \\ \leqslant \left[\int_X \left(c^{1-q} f (c^*)^{1-q}\right)^p d\nu\right]^{1/p} &= \left(\int_X c^{-q} f^p (c^*)^{-q} d\nu\right)^{1/p} \\ = \left(\int g^q d\mu\right)^{-1/2p} \left(\int_X f^p d\mu\right)^{1/p} \left(\int g^q d\mu\right)^{-1/2p}.\end{aligned}$$

By noting that $(\int g^q d\mu)^{(p-1)/2p} = (\int g^q d\mu)^{1/2q}$ the result follows.

(ii) Let the scalar measure μ be defined as $\mu(E) = \frac{1}{k} trace(\nu(E))$. Then ν is absolutely continuous w.r. to μ, with Radon-Nicodŷm derivative $h = d\nu/d\mu$ and f, g and c are $\mu-$measurable and $\mu-$integrable. Equation (2) then writes:

$$\int_X h^{1/2} c f c^* h^{1/2} d\mu \leqslant \left(\int_X h^{1/2} g^q h^{1/2} d\mu\right)^{1/2q} \left(\int_X h^{1/2} f^p h^{1/2} d\mu\right)^{1/p} \times \left(\int_X h^{1/2} g^q h^{1/2} d\mu\right)^{1/2q}.$$

Since μ is a scalar probability measure, (1) holds:

$$\int_X c f c^* d\mu \leqslant \left(\int_X g^q d\mu\right)^{1/2q} \left(\int_X f^p d\mu\right)^{1/p} \left(\int_X g^q d\mu\right)^{1/2q}.$$

Define the matrix-valued measure $\tilde{\nu}$ by

$$\int_E d\tilde{\nu} = \left(\int h^{1/2} g^q h^{1/2} d\mu\right)^{-1/2} \int_E h^{1/2} c^q \, d\mu \, (c^*)^q h^{1/2} \left(\int h^{1/2} g^q h^{1/2} d\mu\right)^{-1/2},$$

such that

$$\left(\int h^{1/2} g^q h^{1/2} d\mu\right)^{-1/2} \int_E h^{1/2} c f c^* h^{1/2} d\mu \left(\int h^{1/2} g^q h^{1/2} d\mu\right)^{-1/2} = \int c^{1-q} f (c^*)^{1-q} d\tilde{\nu}.$$

Then, by repeating the same steps as in (i), we get (2). □

REMARK 1. If f and g are commuting functions, inequalities (1) and (2) simplify to:

$$\int_X f g d\mu \leqslant \left(\int_X f^p d\mu\right)^{1/p} \left(\int_X g^q d\mu\right)^{1/q}$$

that is equivalent to the scalar Hölder's inequality, if μ is a probability measure and

$$\int_X f g d\nu \leqslant \left(\int_X g^q d\nu\right)^{1/2q} \left(\int_X f^p d\nu\right)^{1/p} \left(\int_X g^q d\nu\right)^{1/2q}$$

if ν is a matrix-valued probability.

By taking alternatively $g = I$, $f = f^r$ and $p = s/r$, or $f = f^s$ and $p = r/s$ one obtains an integral version of Theorem 2.3 in [3], as a corollary of Hölder's inequality and (for negative values of p) of Jensen's inequality.

COROLLARY 1. Let $\nu\{f>0\}=I$. If $s \geqslant r$ and $(s,r) \notin (-1,1)^2$, or $1/2 \leqslant r \leqslant 1 \leqslant s$ or $-1/2 \geqslant s \geqslant -1 \geqslant r$ then

$$\left(\int f^r d\nu\right)^{1/r} \leqslant \left(\int f^s d\nu\right)^{1/s}.$$

Theorem 1 can be used to prove a Minkowski inequality for matrix-valued measures and random elements.

THEOREM 2. Let $1 \leqslant p \leqslant 2$ and $(X,\mathscr{S},\mu)$ be a probability space. If $f : X \mapsto H^n$, $g : X \mapsto H^n$ are two real positive $\mu-$measurable functions and $J \subset \mathbb{R}$ be a closed subset, such that $\lambda(f(x)) \cup \lambda(g(y)) \in J$ for all $x,y \in X$, then,

$$\left(\int_X (f+g)^p d\mu\right)^{1/p} \leqslant \left(\int_X f^p d\mu\right)^{1/p} + \left(\int_X g^p d\mu\right)^{1/p}. \tag{3}$$

Let ν a matrix-valued probability measure defined on $(X,\mathscr{S})$. If f,g are $\nu-$measurable,

$$\left(\int_X (f+g)^p d\nu\right)^{1/p} \leqslant \left(\int_X f^p d\nu\right)^{1/p} + \left(\int_X g^p d\nu\right)^{1/p}. \tag{4}$$

Proof. We consider the more general case (4). Since f and g are Hermitian and nonnegative, we can write $(f+g)^{p-1} = hh^*$. Thus, from Theorem 1:

$$\begin{aligned}\int (f+g)^p d\nu &\leqslant \int h(f+g)h^* d\nu \\ &\leqslant \left(\int (f+g)^{q(p-1)} d\nu\right)^{1/2q} \int (f+g) d\nu \left(\int (f+g)^{q(p-1)} d\nu\right)^{1/2q} \\ &\leqslant \left(\int (f+g)^p d\nu\right)^{1/2q} \left(\left(\int f^p d\nu\right)^{1/p} + \left(\int g^p d\nu\right)^{1/p}\right) \\ &\quad \times \left(\int (f+g)^p d\nu\right)^{1/2q}\end{aligned} \tag{5}$$

where we have exploited the linearity of the Bochner integral and applied (1) to f and to g in the particular case $r=1$, $s=2$.

Now by pre and post-multiplying both terms of (5) by $(\int (f+g)^p)^{-1/2q}$, we get:

$$\left(\int (f+g)^p\right)^{1-1/q} \leqslant \left(\int f^p d\nu\right)^{1/p} + \left(\int g^p d\nu\right)^{1/p}$$

which completes the proof once noting that $1-1/q = 1/p$. $\square$

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