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CONTINUITY AND APPROXIMATE DIFFERENTIABILITY OF MULTISUBLINEAR FRACTIONAL MAXIMAL FUNCTIONS

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Abstract. In this note we investigate the continuity and approximate differentiability of the m-sublinear fractional maximal operator

$$\mathfrak{M}_\alpha(\vec{f})(x)=\sup_{r>0}|B(x,r)|^{\alpha/d-m}\prod_{i=1}^{m}\int_{B(x,r)}|f_i(y)|dy,$$

where $m\geqslant 1$, $0\leqslant\alpha<md$ and $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in L^1_{\rm loc}(\mathbb{R}^d)$. More precisely, we prove that $\mathfrak{M}_\alpha$ maps $W^{1,p_1}(\mathbb{R}^d)\times\cdots\times W^{1,p_m}(\mathbb{R}^d)$ into $W^{1,q}(\mathbb{R}^d)$ continuously, provided that $1<p_1,\dots,p_m<\infty$ and $0<\sum_{i=1}^m 1/p_i-\alpha/d=1/q\leqslant 1$. We also show that the multisublinear fractional maximal functions $\mathfrak{M}_\alpha(\vec{f})$ are approximately differentiable a.e. if $\vec{f}=(f_1,f_2,\dots,f_m)$ with each $f_j\in L^1(\mathbb{R}^d)$ being approximately differentiable a.e. As applications, the corresponding results for fractional maximal operators are established.

1. Introduction

Let $d\geqslant 1$ and M denote the centered Hardy-Littlewood maximal operator on $\mathbb{R}^d$, i.e. for $f\in L^1_{\rm loc}(\mathbb{R}^d)$,

$$Mf(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy$$

for any $x\in\mathbb{R}^d$, where $B(x,r)$ is the ball in $\mathbb{R}^d$ centered at x with radius r and $|B(x,r)|$ denotes the volume of $B(x,r)$. One of the cornerstones of harmonic analysis is the celebrated theorem of Hardy-Littlewood-Wiener that asserts that $M:L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)$ is bounded for $1<p\leqslant\infty$. For $p=1$ we have $M:L^1(\mathbb{R}^d)\to L^{1,\infty}(\mathbb{R}^d)$ bounded. During the last several years, a considerable amount of attention has been given to investigate the behavior of differentiability under a maximal operator. The first work in this direction is due to Kinnunen [15] who proved that M is bounded on $W^{1,p}(\mathbb{R}^d)$ for $1<p\leqslant\infty$, where $W^{1,p}(\mathbb{R}^d)$ is the first order Sobolev space, which consists of

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functions $f \in L^p(\mathbb{R}^d)$, whose first weak partial derivatives $D_i f$, $i = 1, 2, \ldots, d$, belong to $L^p(\mathbb{R}^d)$. We endow $W^{1,p}(\mathbb{R}^d)$ with the norm

$$\|f\|_{1,p} = \|f\|_{L^p(\mathbb{R}^d)} + \|\nabla f\|_{L^p(\mathbb{R}^d)},$$

where $\nabla f = (D_1 f, \ldots, D_d f)$ is the weak gradient of f. Later on, this paradigm has been extended to a local version in [16], to a fractional version in [17], to a bilinear version in [11], to a multisublinear fractional version in [25] and to a one-sided version in [24]. Due to the lack of reflexivity of L^1, results for $p = 1$ are subtler. A crucial question on whether the operator $f \mapsto |\nabla Mf|$ is bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, posed by Hajłasz and Onninen in [14], has been restricted to dimension $d = 1$. For example, see [4, 23, 34] for the uncentered Hardy-Littlewood maximal operator, [18] for the centered Hardy-Littlewood maximal operator, [10] for the fractional maximal operators, [26] for the multisublinear fractional maximal operators. Other interesting works related to this topic are [3, 5, 6, 12, 13, 22, 28].

In general, bounded non-sublinear operators need not be continuous (see [7] for a famous example). Continuity of the maximal operator in $L^p(\mathbb{R}^d)$ follows easily from its sublinearity and boundedness. Since we do not have sublinearity for the weak derivatives of the Hardy-Littlewood maximal function, the result of Kinnunen now leads us to another question: is $M : W^{1,p}(\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d)$ continuous for $1 < p < \infty$? This question was posed in [14, Question 3] where it was attributed to T. Iwaniec and has been studied by many authors. In 2007, Luiro first proved that M is continuous on $W^{1,p}(\mathbb{R}^d)$ for $1 < p < \infty$. Subsequently, Carneiro and Moreira [11] extended above result to the bilinear case. Later on, Luiro [30] extended the result of [29] to the local case. Recently, Luiro and Nuutinen [31] established the continuity of a class of discrete maximal operators in Sobolev space $W^{1,p}(\mathbb{R}^d)$ under certain sufficient assumptions.

As well known, the multilinear maximal operator and its fractional version are standard tools in the multilinear Calderón-Zygmund theory and the boundedness of these maximal operators on various function spaces have been extensively studied, for example, see [8, 9, 19, 20, 21, 25] and therein references. In this paper we focus on the regularity of the multisublinear fractional maximal operator. More precisely, let $m \geqslant 1$ and $\vec{f} = (f_1, \ldots, f_m)$ with each $f_j \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. For $0 \leqslant \alpha < md$, we define the multisublinear fractional maximal operator $\mathfrak{M}_\alpha$ by

$$\mathfrak{M}_\alpha(\vec{f})(x) = \sup_{r>0} |B(x,r)|^{\alpha/d - m} \prod_{i=1}^{m} \int_{B(x,r)} |f_i(y)|dy,$$

for any $x \in \mathbb{R}^d$. For $\alpha = 0$, the operator $\mathfrak{M}_\alpha$ recovers the classical multisublinear Hardy-Littlewood maximal operator. Specially, the centered Hardy-Littlewood maximal operator M corresponds to the special case of $\mathfrak{M}_\alpha$ for $m = 1$ and $\alpha = 0$. For $m = 1$ and $0 < \alpha < d$, the operator $\mathfrak{M}_\alpha$ reduces to the classical fractional maximal operator denoted by M_α, which has extensive applications in potential theory and partial differential equations (see [1, 2, 32, 33] for example).

Recently, it was shown in [25] that

$$\mathfrak{M}_\alpha : W^{1,p_1}(\mathbb{R}^d) \times \cdots \times W^{1,p_m}(\mathbb{R}^d) \to W^{1,q}(\mathbb{R}^d)$$

is bounded for $1/q=\sum_{i=1}^m 1/p_i-\alpha/d$ with $1<p_1,\dots,p_m<\infty$ and $1\leqslant q<\infty$. A question that arises naturally is

QUESTION A. *Is the operator $\mathfrak{M}_\alpha$ continuous from $W^{1,p_1}(\mathbb{R}^d)\times\cdots\times W^{1,p_m}(\mathbb{R}^d)$ to $W^{1,q}(\mathbb{R}^d)$, when $1/q=\sum_{i=1}^m 1/p_i-\alpha/d$, $1<p_1,\dots,p_m<\infty$ and $1\leqslant q<\infty$?*

Question A is the main motivation for this work. This problem is solved by our next theorem.

THEOREM 1. *Let $0\leqslant\alpha<md$. Then $\mathfrak{M}_\alpha$ maps $W^{1,p_1}(\mathbb{R}^d)\times\cdots\times W^{1,p_m}(\mathbb{R}^d)$ into $W^{1,q}(\mathbb{R}^d)$ continuously, where $1/q=\sum_{i=1}^m 1/p_i-\alpha/d$, $1<p_1,\dots,p_m<\infty$ and $1\leqslant q<\infty$.*

REMARK 1. Theorem 1 extends the continuity result in [29], which corresponds to the case $m=1$ and $\alpha=0$. In [17], Kinnunen and Saksman observed that $M_\alpha: W^{1,p}(\mathbb{R}^d)\to W^{1,q}(\mathbb{R}^d)$ is bounded if $1/q=1/p-\alpha/d$ with $1<p<\infty$ and $0\leqslant\alpha<d/p$. As an application of Theorem 1, it is known that $M_\alpha: W^{1,p}(\mathbb{R}^d)\to W^{1,q}(\mathbb{R}^d)$ is continuous if $1/q=1/p-\alpha/d$ with $1<p<\infty$ and $0\leqslant\alpha<d/p$.

To the best of my knowledge Hajłasz and Onninen's question remains open in dimension $d\geqslant 2$. Motivated by this challenging problem, Hajłasz and Maly [13] established the approximate differentiability of the Hardy-Littlewood maximal operator, which can be listed as follows.

THEOREM B. ([13]) *Let $f\in L^1(\mathbb{R}^d)$ be approximately differentiable a.e., then Mf is also approximately differentiable a.e.*

For the endpoint regularity of $\mathfrak{M}_\alpha$, it was shown in [26, 27] that if $d\geqslant 2$, $1\leqslant\alpha<m(d-1)+1$ and $\vec{f}=(f_1,\dots,f_m)$ with each $f_i\in W^{1,1}(\mathbb{R}^d)$, then $\mathfrak{M}_\alpha(\vec{f})$ is weakly differentiable and

$$\|\nabla\mathfrak{M}_\alpha(\vec{f})\|_{L^{\frac{d}{m(d-1)-\alpha+1}}(\mathbb{R}^d)}\leqslant C\prod_{j=1}^m\|\nabla f_j\|_{L^1(\mathbb{R}^d)}.$$

A natural and interesting question is that what about the smoothness properties of the maximal function $\mathfrak{M}_\alpha(\vec{f})$ if $0\leqslant\alpha<1$ or $m(d-1)+1\leqslant\alpha<md$ and $\vec{f}=(f_1,\dots,f_m)$ with each $f_i\in W^{1,1}(\mathbb{R}^d)$. Motivated by Theorem B, we shall establish the following result.

THEOREM 2. *Let $0\leqslant\alpha\leqslant md-1$ and $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in L^1(\mathbb{R}^d)$ being approximately differentiable a.e., then $\mathfrak{M}_\alpha(\vec{f})$ is approximately differentiable a.e.*

REMARK 2. Clearly, Theorem 2 extends Theorem B, which corresponds to the case $m=1$ and $\alpha=0$. Since every function $f\in W^{1,1}(\mathbb{R}^d)$ is approximately differentiable a.e., Theorem 2 implies that if $0\leqslant\alpha\leqslant md-1$ and $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in W^{1,1}(\mathbb{R}^d)$, then $\mathfrak{M}_\alpha(\vec{f})$ is approximately differentiable a.e. Moreover, we know from Theorem 2 that the fractional maximal function $M_\alpha f$ is approximately differentiable a.e. if $f\in L^1(\mathbb{R}^d)$ is approximately differentiable a.e. and $0\leqslant\alpha\leqslant d-1$.

The rest of this paper is organized as follows. In Section 2 we prove Theorem 1, the proof of Theorem 2 will be given in Section 3. We remark that the main ideas in our proofs are greatly motivated by [13, 29], but our methods and techniques are more subtle and complex than that of [13, 29]. The main ingredient of our proof of Theorem 1 is to present an explicit formula for the derivative of the multisublinear fractional maximal functions (see Lemma 2). In what follows, we use the following conventions

$$\prod_{i\in\emptyset} a_i = 1 \text{ and } \sum_{i\in\emptyset} a_i = 0.$$

2. Proof of Theorem 1

In this section we follow carefully the proof for the continuity in $W^{1,p}(\mathbb{R}^d)$ of the Hardy-Littlewood maximal operator in [29] and simply adjust the notation to our context.

For $R>0$, we denote by B_R the ball of radius R centered at the origin. For $A\subset\mathbb{R}^d$ and $x\in\mathbb{R}^d$, we define

$$d(x,A) := \inf_{a\in A}|x-a| \text{ and } A_{(\lambda)} := \{x\in\mathbb{R}^d; d(x,A)\leqslant\lambda\} \text{ for } \lambda\geqslant 0.$$

We denote by $\|f\|_{p,A}$ the L^p-norm of $f\chi_A$ for all measurable sets $A\subset\mathbb{R}^d$. Let $\vec{f}=(f_1,\dots,f_m)\in L^{p_1}(\mathbb{R}^d)\times\cdots\times L^{p_m}(\mathbb{R}^d)$ with $1<p_1,\dots,p_m<\infty$, $1\leqslant q<\infty$ and $1/q=\sum_{i=1}^m 1/p_i-\alpha/d$. For a fixed point $x\in\mathbb{R}^d$, we define the set $\mathscr{R}(\vec{f})(x)$ by

$$\mathscr{R}(\vec{f})(x)$$
$$:=\Big\{r\geqslant 0:\ \mathfrak{M}_\alpha(\vec{f})(x)=\limsup_{r_k\to r}|B(x,r_k)|^{\alpha/d-m}\prod_{i=1}^m\int_{B(x,r_k)}|f_i(y)|dy \text{ for some } r_k>0\Big\}.$$

We also define the function $u_{x,\vec{f}}:[0,\infty)\to\mathbb{R}$ by

$$u_{x,\vec{f}}(0)=\begin{cases}\prod_{i=1}^m|f_i(x)|, & \text{if } \alpha=0;\\ 0, & \text{if } 0<\alpha<md,\end{cases}$$

$$u_{x,\vec{f}}(r)=|B(x,r)|^{\alpha/d-m}\prod_{i=1}^m\int_{B(x,r)}|f_i(y)|dy \text{ when } r\in(0,\infty).$$

We notice that the following facts are valid: (i) $u_{x,\vec{f}}$ is continuous on $(0,\infty)$ for all $x\in\mathbb{R}^d$ and at $r=0$ for a.e. $x\in\mathbb{R}^d$; (ii) $\lim_{r\to\infty}u_{x,\vec{f}}(r)=0$ since $u_{x,\vec{f}}(r)\leqslant\prod_{i=1}^m\|f_i\|_{L^{p_i}(\mathbb{R}^d)}|B(x,r)|^{-1/q}$ for any $r>0$ and $x\in\mathbb{R}^d$; (iii) $\mathscr{R}(\vec{f})(x)$ is nonempty and closed for all $x\in\mathbb{R}^d$ and

$$\mathfrak{M}_\alpha(\vec{f})(x)=u_{x,\vec{f}}(r) \text{ if } 0<r\in\mathscr{R}(\vec{f})(x),\ \forall x\in\mathbb{R}^d,$$

$$\mathfrak{M}_\alpha(\vec{f})(x)=u_{x,\vec{f}}(0) \text{ for a.e. } x\in\mathbb{R}^d \text{ such that } 0\in\mathscr{R}(\vec{f})(x).$$

Motivated by the idea in [29], we can get the following

LEMMA 1. *Let $\vec{f}_j = (f_{1,j}, f_{2,j}, \ldots, f_{m,j})$. Suppose that $f_{i,j} \to f_i$ in $L^{p_i}(\mathbb{R}^d)$ when $j \to \infty$ for all $i = 1,2,\ldots,m$, where $1 < p_1,\ldots,p_m < \infty$, $1 \leqslant q < \infty$ and $1/q = \sum_{i=1}^m 1/p_i - \alpha/d$. Then for all $R > 0$ and $\lambda > 0$, we have*

$$\lim_{j\to\infty} |\{x \in B_R; \mathscr{R}(\vec{f}_j)(x) \nsubseteq \mathscr{R}(\vec{f})(x)_{(\lambda)}\}| = 0. \tag{1}$$

Proof. Without loss of generality we may assume that all $f_{i,j} \geqslant 0$ and $f_i \geqslant 0$. By the similar argument as in the proof of [29, Lemma 2.2], we can conclude that the set $\{x \in \mathbb{R}^d;\ \mathscr{R}(\vec{f}_j)(x) \nsubseteq \mathscr{R}(\vec{f})(x)_{(\lambda)}\}$ is measurable for any $j \in \mathbb{Z}$. Let $\lambda > 0$, $R > 0$ and $\varepsilon \in (0,1)$. We can claim that for a.e. $x \in B_R$, there exists $\gamma(x) \in \mathbb{N}\backslash\{0\}$ such that

$$u_{x,\vec{f}}(r) < \mathfrak{M}_\alpha(\vec{f})(x) - \frac{1}{\gamma(x)}, \text{ when } d(r, \mathscr{R}(\vec{f})(x)) > \lambda. \tag{2}$$

Otherwise, for a.e. $x \in B_R$, there exists a bounded sequence of radii $\{r_k\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} u_{x,\vec{f}}(r_k) = \mathfrak{M}_\alpha(\vec{f})(x) \text{ and } d(r_k, \mathscr{R}(\vec{f})(x)) > \lambda.$$

We can choose a subsequence $\{s_k\}_{k=1}^\infty$ of $\{r_k\}_{k=1}^\infty$ such that $s_k \to r$ as $k \to \infty$. It follows that $r \in \mathscr{R}(\vec{f})(x)$ and $d(r, \mathscr{R}(\vec{f})(x)) \geqslant \lambda$, which is a contradiction. Thus, (2) holds. From (2) we can conclude that there exists $\gamma = \gamma(R,\lambda,\varepsilon) \in \mathbb{N}\backslash\{0\}$ and a measurable set E with $|E| < \varepsilon$ such that

$$\begin{aligned} B_R &\subset \{x \in \mathbb{R}^d : u_{x,\vec{f}}(r) < \mathfrak{M}_\alpha(\vec{f})(x) - \gamma^{-1} \text{ if } d(r, \mathscr{R}(\vec{f})(x)) > \lambda\} \cup E \\ &\subset A_{1,j} \cup A_{2,j} \cup A_{3,j} \cup E, \end{aligned} \tag{3}$$

where

$$A_{1,j} := \{x \in \mathbb{R}^d : |\mathfrak{M}_\alpha(\vec{f}_j)(x) - \mathfrak{M}_\alpha(\vec{f})(x)| \geqslant (4\gamma)^{-1}\},$$

$$A_{2,j} := \{x \in \mathbb{R}^d : |u_{x,\vec{f}_j}(r) - u_{x,\vec{f}}(r)| \geqslant (2\gamma)^{-1} \text{ for some } r \text{ such that } d(r, \mathscr{R}(\vec{f})(x)) > \lambda\},$$

$$A_{3,j} := \{x \in \mathbb{R}^d : u_{x,\vec{f}_j}(r) < \mathfrak{M}_\alpha(\vec{f}_j)(x) - (4\gamma)^{-1} \text{ if } d(r, \mathscr{R}(\vec{f})(x)) > \lambda\}.$$

Let $\bar{A}$ be the set of all points x such that x is a Lebesgue point of all f_j. Note that $|\mathbb{R}^d\backslash\bar{A}| = 0$. One can easily check that $A_{3,j} \cap \bar{A} \subset \cup\{x \in \mathbb{R}^d : \mathscr{R}(\vec{f}_j)(x) \subset \mathscr{R}(\vec{f})(x)_{(\lambda)}\}$. This together with (3) yields that

$$\{x \in B_R; \mathscr{R}(\vec{f}_j)(x) \nsubseteq \mathscr{R}(\vec{f})(x)_{(\lambda)}\} \subset A_{1,j} \cup A_{2,j} \cup E \cup (\mathbb{R}^d\backslash\bar{A}).$$

It follows that

$$|\{x \in B_R; \mathscr{R}(\vec{f}_j)(x) \nsubseteq \mathscr{R}(\vec{f})(x)_{(\lambda)}\}| \leqslant |A_{1,j}| + |A_{2,j}| + \varepsilon. \tag{4}$$

Since $f_{i,j} \to f_i$ in $L^{p_i}(\mathbb{R}^d)$ when $j \to \infty$ for all $i = 1,2,\ldots,m$, then for any fixed $\varepsilon \in (0,1)$, there exists $N_0 = N_0(\varepsilon) \in \mathbb{N}$ such that

$$\|f_{i,j} - f_i\|_{L^{p_i}(\mathbb{R}^d)} < \frac{\varepsilon}{\gamma} \text{ and } \|f_{i,j}\|_{L^{p_i}(\mathbb{R}^d)} \leqslant \|f_i\|_{L^{p_i}(\mathbb{R}^d)} + 1 \tag{5}$$

for any $j \geqslant N_0$ and $i = 1,2,\ldots,m$. We can write

$$\begin{aligned}
&|\mathfrak{M}_\alpha(\vec{f}_j)(x) - \mathfrak{M}_\alpha(\vec{f})(x)| \\
&\leqslant \sup_{r>0} |B(x,r)|^{\alpha/d-m} \Big| \prod_{i=1}^m \int_{B(x,r)} f_{i,j}(y)dy - \prod_{i=1}^m \int_{B(x,r)} f_i(y)dy \Big| \\
&\leqslant \sum_{i=1}^m \sup_{r>0} |B(x,r)|^{\alpha/d-m} \prod_{\mu=1}^{i-1} \int_{B(x,r)} f_\mu(y)dy \prod_{\nu=i+1}^m \int_{B(x,r)} f_{\nu,j}(y)dy \\
&\quad \times \int_{B(x,r)} |f_{i,j}(y) - f_i(y)|dy \\
&= \sum_{i=1}^m \mathfrak{M}_\alpha(\vec{F}_j^i)(x)
\end{aligned} \tag{6}$$

for any $x \in \mathbb{R}^d$, where $\vec{F}_j^i = (f_1, \ldots, f_{i-1}, f_{i,j} - f_i, f_{i+1,j}, \ldots, f_{m,j})$. It follows form (5) and (6) that

$$\begin{aligned}
|A_{1,j}| &\leqslant \Big| \Big\{ x \in \mathbb{R}^d : \sum_{i=1}^m \mathfrak{M}_\alpha(\vec{F}_j^i)(x) \geqslant (4\gamma)^{-1} \Big\} \Big| \\
&\leqslant \sum_{i=1}^m |\{x \in \mathbb{R}^d : \mathfrak{M}_\alpha(\vec{F}_j^i)(x) \geqslant (4m\gamma)^{-1}\}| \\
&\leqslant (4m\gamma)^q \sum_{i=1}^m \|\mathfrak{M}_\alpha(\vec{F}_j^i)\|_{L^q(\mathbb{R}^d)}^q \\
&\leqslant C(m,\gamma,q,\alpha,d,p_1,\ldots,p_m)\varepsilon
\end{aligned} \tag{7}$$

for any $j \geqslant N_0$. Similarly, we can get

$$|A_{2,j}| \leqslant C(m,\gamma,q,\alpha,d,p_1,\ldots,p_m)\varepsilon \tag{8}$$

for any $j \geqslant N_0$. (1) follows form (4), (7) and (8). □

Let $e_l = (0,\ldots,0,1,0,\ldots,0)$ be the canonical l-th base vector in $\mathbb{R}^d$ for $l = 1,2,\ldots,d$. For any fixed $i = 1,2,\ldots,m$, $h > 0$ and $f_i \in L^p(\mathbb{R}^d)$ with $p \geqslant 1$, define

$$f_{i,h}^l(x) = \frac{f_{\tau(h)}^{i,l}(x) - f_i(x)}{h} \quad \text{and} \quad f_{\tau(h)}^{i,l}(x) = f_i(x + he_l).$$

It is well known that for $p \geqslant 1$, $f_{\tau(h)}^{i,l} \to f_i$ in $L^p(\mathbb{R}^d)$ when $h \to 0$, and if $f_i \in W^{1,p}(\mathbb{R}^d)$ we have $f_{i,h}^l \to D_l f_i$ in $L^p(\mathbb{R}^d)$ when $h \to 0$. Let A, B be two subsets of $\mathbb{R}^d$, we define the Hausdorff distance of A and B by

$$\pi(A,B) := \inf\{\delta > 0 : A \subset B_{(\delta)} \text{ and } B \subset A_{(\delta)}\}.$$

Applying Lemma 1 and the argument similar to that in the proof of [29, Corollary 2.3], we can get the following result. The details are omitted.

COROLLARY 1. *Let $\vec{f} = (f_1,\ldots,f_m) \in L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d)$ with $1 \leqslant q < \infty$, $1 < p_1,\ldots,p_m < \infty$ and $1/q = \sum_{i=1}^m 1/p_i - \alpha/d$. Then for all $R > 0$, $\lambda > 0$ and $l = 1,2,\ldots,d$, we have*

$$|\{x \in B_R; \pi(\mathscr{R}(\vec{f})(x), \mathscr{R}(\vec{f})(x + he_l)) > \lambda\}| \to 0 \text{ when } h \to 0.$$

We now state several formulas for the derivative of the multisublinear maximal fractional functions, which play a key role in the proof of Theorem 1.

LEMMA 2. *Let $\vec{f}=(f_1,f_2,\dots,f_m)\in W^{1,p_1}(\mathbb{R}^d)\times\cdots\times W^{1,p_m}(\mathbb{R}^d)$ with $1\leqslant q<\infty$, $1<p_1,\dots,p_m<\infty$ and $1/q=\sum_{i=1}^m 1/p_i-\alpha/d$. Then for any $l=1,2,\dots,d$ and a.e. $x\in\mathbb{R}^d$, we have*

$$D_l\mathfrak{M}_\alpha(\vec{f})(x)=\sum_{i=1}^m|B(x,r)|^{\alpha/d-m}\Big(\prod_{1\leqslant j\neq i\leqslant m}\int_{B(x,r)}|f_j(y)|dy\Big)\times\int_{B(x,r)}D_l(|f_i|)(y)dy\ \text{ for all } 0<r\in\mathscr{R}(\vec{f})(x),$$

$$D_l\mathfrak{M}_\alpha(\vec{f})(x)=\begin{cases}\sum_{i=1}^m D_l|f_i|(x)\prod_{1\leqslant j\neq i\leqslant m}|f_j(x)|, & \text{if } \alpha=0 \text{ and } 0\in\mathscr{R}(\vec{f})(x),\\ 0, & \text{if } 0<\alpha<md \text{ and } 0\in\mathscr{R}(\vec{f})(x).\end{cases}$$

Proof. We may assume without loss of generality that all $f_i\geqslant 0$, since $|f_i|\in W^{1,p_i}(\mathbb{R}^d)$ if $f_i\in W^{1,p_i}(\mathbb{R}^d)$. Let $R>0$. Invoking Corollary 1, we can choose a sequence $\{s_k\}_{k=1}^\infty$, $s_k>0$ and $s_k\to 0$ such that $\lim_{k\to\infty}\pi(\mathscr{R}(\vec{f})(x),\mathscr{R}(\vec{f})(x+s_ke_l))=0$ for a.e. $x\in B_R$. Then for any $i=1,2,\dots,m$ and $l=1,2,\dots,d$ we have

$$\|f^{i,l}_{\tau(s_k)}-f_i\|_{L^{p_i}(\mathbb{R}^d)}\to 0 \ \text{ as } k\to\infty,$$

$$\|f^l_{i,s_k}-D_lf_i\|_{L^{p_i}(\mathbb{R}^d)}\to 0 \ \text{ as } k\to\infty,$$

$$\|M(f^{i,l}_{\tau(s_k)}-f_i)\|_{L^{p_i}(\mathbb{R}^d)}\to 0 \ \text{ as } k\to\infty,$$

$$\|M(f^l_{i,s_k}-D_lf_i)\|_{L^{p_i}(\mathbb{R}^d)}\to 0 \ \text{ as } k\to\infty,$$

$$\|(\mathfrak{M}_\alpha(\vec{f}))^l_{s_k}-D_l\mathfrak{M}_\alpha(\vec{f})\|_{L^q(\mathbb{R}^d)}\to 0 \ \text{ as } k\to\infty.$$

Furthermore, there exists a subsequence $\{h_k\}_{k=1}^\infty$ of $\{s_k\}_{k=1}^\infty$ and a measurable set $A_1\subset B_R$ such that $|B_R\backslash A_1|=0$ and

(i) $f^{i,l}_{\tau(h_k)}(x)\to f_i(x)$, $f^l_{i,h_k}(x)\to D_lf_i(x)$, $M(f^{i,l}_{\tau(h_k)}-f_i)(x)\to 0$, $M(f^l_{i,h_k}-D_lf_i)(x)\to 0$ and $(\mathfrak{M}_\alpha(\vec{f}))^l_{h_k}(x)\to D_l\mathfrak{M}_\alpha(\vec{f})(x)$ when $k\to\infty$ for any $x\in A_1$, $i=1,2,\dots,m$ and $l=1,\dots,d$;

(ii) $\lim_{k\to\infty}\pi(\mathscr{R}(\vec{f})(x),\mathscr{R}(\vec{f})(x+h_ke_l))=0$ for any $x\in A_1$.

Let

$$A_2:=\bigcap_{k=1}^\infty\{x\in\mathbb{R}^d:\mathfrak{M}_\alpha(\vec{f})(x+h_ke_l)\geqslant u_{x+h_ke_l,\vec{f}}(0)\},$$

$$A_3:=\{x\in\mathbb{R}^d:\mathfrak{M}_\alpha(\vec{f})(x)=u_{x,\vec{f}}(0) \text{ if } 0\in\mathscr{R}(\vec{f})(x)\},$$

$$A_4:=\bigcap_{k=1}^\infty\{x\in\mathbb{R}^d:\mathfrak{M}_\alpha(\vec{f})(x+h_ke_l)=u_{x+h_ke_l,\vec{f}}(0) \text{ if } 0\in\mathscr{R}(\vec{f})(x+h_ke_l)\}.$$

One can easily check that $|B_R\backslash A_i|=0$ for any $i=2,3,4$. Let $x\in A_1\cap A_2\cap A_3\cap A_4$ be a Lebesgue point of all f_i and $D_l f_i$ and $r\in\mathscr{R}(\vec{f})(x)$, there exists radii $r_k\in\mathscr{R}(\vec{f})(x+h_k e_l)$ such that $\lim_{k\to\infty} r_k=r$. We now complete the rest of proof by considering the following two cases:

Case A ($r>0$). Without loss of generality we may assume that all $r_k>0$. We can write

$$
\begin{aligned}
D_l\mathfrak{M}_\alpha(\vec{f})(x) &= \lim_{k\to\infty}\frac{1}{h_k}(\mathfrak{M}_\alpha(\vec{f})(x+h_k e_l)-\mathfrak{M}_\alpha(\vec{f})(x))\\
&\leqslant \lim_{k\to\infty}\frac{1}{h_k}(u_{x+h_k e_l,\vec{f}}(r_k)-u_{x,\vec{f}}(r_k))\\
&= \lim_{k\to\infty}\frac{1}{h_k}\sum_{j=1}^{m}|B(x,r_k)|^{\alpha/d-m}\int_{B(x,r_k)}(f_j(y+h_k e_l)-f_j(y))dy\\
&\quad\times\prod_{\mu=1}^{j-1}\int_{B(x,r_k)}f_\mu(y+h_k e_l)dy\prod_{\nu=j+1}^{m}\int_{B(x,r_k)}f_\nu(y)dy\\
&= \sum_{j=1}^{m}\lim_{k\to\infty}|B(x,r_k)|^{\alpha/d-m}\int_{B(x,r_k)}f_{j,h_k}^{l}(y)dy\\
&\quad\times\prod_{\mu=1}^{j-1}\int_{B(x,r_k)}f_{\tau(h_k)}^{\mu,l}(y)dy\prod_{\nu=j+1}^{m}\int_{B(x,r_k)}f_\nu(y)dy.
\end{aligned}
\tag{9}
$$

Since $\lim_{k\to\infty}|B(x,r_k)|=|B_r|$, $f_{\tau(h_k)}^{i,l}\chi_{B(x,r_k)}\to f_i\chi_{B(x,r)}$ and $f_{j,h_k}^{l}\chi_{B(x,r_k)}\to D_l f_i\chi_{B(x,r)}$ in $L^1(\mathbb{R}^d)$ as $k\to\infty$. It follows that

$$
D_l\mathfrak{M}_\alpha(\vec{f})(x)\leqslant\sum_{i=1}^{m}|B(x,r)|^{\alpha/d-m}\Big(\prod_{1\leqslant j\neq i\leqslant m}\int_{B(x,r)}f_j(y)dy\Big)\int_{B(x,r)}D_l f_i(y)dy.
$$

On the other hand, we have

$$
\begin{aligned}
D_l\mathfrak{M}_\alpha(\vec{f})(x) &= \lim_{k\to\infty}\frac{1}{h_k}(\mathfrak{M}_\alpha(\vec{f})(x+h_k e_l)-\mathfrak{M}_\alpha(\vec{f})(x))\\
&\geqslant \lim_{k\to\infty}\frac{1}{h_k}(u_{x+h_k e_l,\vec{f}}(r)-u_{x,\vec{f}}(r))\\
&= \lim_{k\to\infty}\frac{1}{h_k}\sum_{j=1}^{m}|B(x,r)|^{\alpha/d-m}\int_{B(x,r)}(f_j(y+h_k e_l)-f_j(y))dy\\
&\quad\times\prod_{\mu=1}^{j-1}\int_{B(x,r)}f_\mu(y+h_k e_l)dy\prod_{\nu=j+1}^{m}\int_{B(x,r)}f_\nu(y)dy\\
&= \sum_{j=1}^{m}|B(x,r)|^{\alpha/d-m}\lim_{k\to\infty}\int_{B(x,r)}f_{j,h_k}^{l}(y)dy\\
&\quad\times\prod_{\mu=1}^{j-1}\int_{B(x,r)}f_{\tau(h_k)}^{\mu,l}(y)dy\prod_{\nu=j+1}^{m}\int_{B(x,r)}f_\nu(y)dy\\
&= \sum_{i=1}^{m}|B(x,r)|^{\alpha/d-m}\int_{B(x,r)}D_l f_i(y)dy\Big(\prod_{1\leqslant j\neq i\leqslant m}\int_{B(x,r)}f_j(y)dy\Big).
\end{aligned}
$$

Case B ($r=0$). When $0<\alpha<md$. Since $\mathfrak{M}_\alpha(\vec{f})(x)=0$ and then $\prod_{i=1}^m f_i(y)=0$ for a.e. $y\in\mathbb{R}^d$. Thus $\mathfrak{M}_\alpha(\vec{f})\equiv 0$ and $D_l\mathfrak{M}_\alpha(\vec{f})(x)=0$. We now consider the case $\alpha=0$. Let us begin with estimating the lower bound of $D_l\mathfrak{M}_\alpha(\vec{f})(x)$. We can write

$$\begin{aligned} D_l\mathfrak{M}_\alpha(\vec{f})(x) &= \lim_{k\to\infty}\frac{1}{h_k}(\mathfrak{M}_\alpha(\vec{f})(x+h_ke_l)-\mathfrak{M}_\alpha(\vec{f})(x)) \\ &\geqslant \lim_{k\to\infty}\frac{1}{h_k}\Big(\prod_{i=1}^m f_i(x+h_ke_l)-\prod_{i=1}^m f_i(x)\Big) \\ &= \lim_{k\to\infty}\frac{1}{h_k}\sum_{j=1}^m (f_j(x+h_ke_l)-f_j(x))\prod_{i=1}^{j-1} f_i(x+h_ke_l)\prod_{i=j+1}^m f_i(x) \\ &= \sum_{i=1}^m D_lf_i(x)\prod_{1\leqslant j\neq i\leqslant m} f_j(x). \end{aligned} \tag{10}$$

Below we estimate the upper bound of $D_l\mathfrak{M}_\alpha(\vec{f})(x)$. If we have $r_k=0$ for infinitely many k, then

$$\begin{aligned} D_l\mathfrak{M}_\alpha(\vec{f})(x) &= \lim_{k\to\infty}\frac{1}{h_k}(\mathfrak{M}_\alpha(\vec{f})(x+h_ke_l)-\mathfrak{M}_\alpha(\vec{f})(x)) \\ &= \lim_{k\to\infty}\frac{1}{h_k}\Big(\prod_{i=1}^m f_i(x+h_ke_l)-\prod_{i=1}^m f_i(x)\Big) \\ &= \lim_{k\to\infty}\frac{1}{h_k}\sum_{j=1}^m (f_j(x+h_ke_l)-f_j(x))\prod_{\mu=1}^{j-1} f_\mu(x+h_ke_l)\prod_{\nu=j+1}^m f_\nu(x) \\ &= \sum_{i=1}^m D_lf_i(x)\prod_{1\leqslant j\neq i\leqslant m} f_j(x). \end{aligned}$$

If there exists $k_0\in\mathbb{N}$ such that $r_k>0$ when $k\geqslant k_0$. We get from (9) that

$$\begin{aligned} D_l\mathfrak{M}_\alpha(\vec{f})(x) \leqslant \sum_{j=1}^m \lim_{k\to\infty} |B(x,r_k)|^{-m}\int_{B(x,r_k)} f_{j,h_k}^l(y)dy \\ \times\prod_{\mu=1}^{j-1}\int_{B(x,r_k)} f_{\tau(h_k)}^{\mu,l}(y)dy\prod_{\nu=j+1}^m\int_{B(x,r_k)} f_\nu(y)dy. \end{aligned} \tag{11}$$

Since

$$\begin{aligned} &\lim_{k\to\infty}\Big|\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)} f_{j,h_k}^l(y)dy-D_lf_j(x)\Big| \\ &= \lim_{k\to\infty}\Big|\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)}(f_{j,h_k}^l(y)-D_lf_j(y))dy\Big| \\ &\leqslant \lim_{k\to\infty} M(f_{j,h_k}^l-D_lf_j)(x)=0. \end{aligned}$$

It follows that

$$\lim_{k\to\infty}\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)} f_{j,h_k}^l(y)dy=D_lf_j(x). \tag{12}$$

By argument similar to those used to derive (11),

$$\lim_{k\to\infty}\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)} f^{\mu,l}_{\tau(h_k)}(y)dy = f_\mu(x). \tag{13}$$

It follows from (11)–(13) that

$$D_l\mathfrak{M}_\alpha(\vec{f})(x) \leqslant \sum_{i=1}^m D_l f_i(x) \prod_{1\leqslant j\neq i\leqslant m} f_j(x),$$

which together with (10) yields

$$D_l\mathfrak{M}_\alpha(\vec{f})(x) = \sum_{i=1}^m D_l f_i(x) \prod_{1\leqslant j\neq i\leqslant m} f_j(x).$$

Since R was arbitrary and $|B_R\backslash(A_1\cap A_2\cap A_3\cap A_4)| = 0$. This proves Lemma 2. □

We are now in the position of proving Theorem 1.

Proof of Theorem 1. Let $p_1,\dots,p_m,q,\alpha$ be given as in Theorem 1 and $\vec{f} = (f_1,\dots,f_m) \in W^{1,p_1}(\mathbb{R}^d)\times\cdots\times W^{1,p_m}(\mathbb{R}^d)$. For $i=1,2,\dots,m$, let $\vec{f}_j = (f_{1,j},\dots,f_{m,j})$ and $f_{i,j}\to f_i$ in $W^{1,p_i}(\mathbb{R}^d)$ when $j\to\infty$. We get from (6) that

$$\begin{aligned}\|\mathfrak{M}_\alpha(\vec{f}_j) - \mathfrak{M}_\alpha(\vec{f})\|_{L^q(\mathbb{R}^d)} &\leqslant \sum_{i=1}^m \|\mathfrak{M}_\alpha(\vec{F}^i_j)\|_{L^q(\mathbb{R}^d)}\\ &\leqslant \sum_{i=1}^m \|f_{i,j}-f_i\|_{L^{p_i}(\mathbb{R}^d)} \prod_{\mu=1}^{i-1}\|f_\mu\|_{L^{p_\mu}(\mathbb{R}^d)} \prod_{\nu=i+1}^m \|f_{\nu,j}\|_{L^{p_\nu}(\mathbb{R}^d)},\end{aligned}$$

where $\vec{F}^i_j$ is given as in (6). It follows that

$$\|\mathfrak{M}_\alpha(\vec{f}_j) - \mathfrak{M}_\alpha(\vec{f})\|_{L^q(\mathbb{R}^d)} \to 0 \text{ when } j\to\infty.$$

It suffices to show that

$$\|D_l\mathfrak{M}_\alpha(\vec{f}_j) - D_l\mathfrak{M}_\alpha(\vec{f})\|_{L^q(\mathbb{R}^d)} \to 0 \text{ when } j\to\infty \tag{14}$$

for any $l=1,2,\dots,d$. We only prove (14) for $l=d$ (since other cases are analogous). We may assume without loss of generality that all $f_i\geqslant 0$ and $f_{i,j}\geqslant 0$.

For any $i=1,2,\dots,m$, let $\vec{f^i} = (f_1,\dots,f_{i-1},D_d f_i,f_{i+1},\dots,f_m)$. Given $\varepsilon>0$, there exists $R>0$ such that $\sum_{i=1}^m \|\mathfrak{M}_\alpha(\vec{f^i})\|_{q,G_1} < \varepsilon$ with $G_1 = \mathbb{R}^d\backslash B_R$. By absolute continuity, there exists $\eta>0$ such that $\sum_{i=1}^m \|\mathfrak{M}_\alpha(\vec{f^i})\|_{q,A} < \varepsilon$ whenever $|A|<\eta$ and A is a measurable subset of B_R. As we already observed, for a.e. $x\in\mathbb{R}^d$ and any $1\leqslant i\leqslant m$, the function $u_{x,\vec{f^i}}$ is uniformly continuous on $[0,\infty)$. Thus, for a.e. $x\in\mathbb{R}^d$, the function $\sum_{i=1}^m u_{x,\vec{f^i}}$ is uniformly continuous on $[0,\infty)$ and we can find $\delta_x>0$ such that

$$\Big|\sum_{i=1}^m u_{x,\vec{f^i}}(r_1) - \sum_{i=1}^m u_{x,\vec{f^i}}(r_2)\Big| < |B_R|^{-1/q}\varepsilon \text{ whenever } |r_1-r_2|<\delta_x.$$

It follows that there exists $\delta > 0$ such that

$$\begin{aligned}&\Big|\Big\{x \in B_R : \Big|\sum_{i=1}^m u_{x,\vec{f}^i}(r_1) - \sum_{i=1}^m u_{x,\vec{f}^i}(r_2)\Big|\\ &\geqslant |B_R|^{-1/q}\varepsilon \text{ for some } r_1, r_2 \text{ with } |r_1 - r_2| < \delta\Big\}\Big|\\ &=: |G_2| < \tfrac{\eta}{2}.\end{aligned}$$

Applying Lemma 1, there exists $j_1 \in \mathbb{N}$ such that

$$|\{x \in B_R; \mathscr{R}(\vec{f}_j)(x) \nsubseteq \mathscr{R}(\vec{f})(x)_{(\delta)}\}| =: |B^j| < \frac{\eta}{2} \text{ when } j \geqslant j_1.$$

Let $\vec{f_{i,j}} = (f_{1,j}, \dots, f_{i-1,j}, D_d f_{i,j}, f_{i+1,j}, \dots, f_{m,j})$ and $r \in \mathscr{R}(\vec{f}_j)(x)$. We consider the following two cases: (i) $r > 0$. We can write

$$\begin{aligned}&|u_{x,\vec{f_{i,j}}}(r) - u_{x,\vec{f}^i}(r)|\\ =&\Big||B(x,r)|^{\alpha/d-m}\Big(\prod_{1\leqslant\mu\neq i\leqslant m}\int_{B(x,r)} f_{\mu,j}(y)dy\Big)\int_{B(x,r)} D_d f_{i,j}(y)dy\\ &-|B(x,r)|^{\alpha/d-m}\Big(\prod_{1\leqslant\mu\neq i\leqslant m}\int_{B(x,r)} f_\mu(y)dy\Big)\int_{B(x,r)} D_d f_i(y)dy\Big|\\ \leqslant&\sum_{\mu=1}^{i-1}\mathfrak{M}_\alpha(\vec{F_{\mu,j}})(x) + \sum_{\nu=i+1}^{m}\mathfrak{M}_\alpha(\vec{G_{\nu,j}})(x) + \mathfrak{M}_\alpha(\vec{H_{i,j}})(x) =: \mathscr{G}_{i,j}(x),\end{aligned}$$

where

$$\vec{F_{\mu,j}} = (f_1, \dots, f_{\mu-1}, f_{\mu,j} - f_\mu, f_{\mu+1,j}, \dots, f_{i-1,j}, D_d f_{i,j}, f_{i+1,j} \dots, f_{m,j}),$$

$$\vec{G_{\nu,j}} = (f_1, \dots, f_{i-1}, D_d f_i, f_{i+1}, \dots, f_{\nu-1}, f_{\nu,j} - f_\nu, f_{\nu+1,j}, \dots, f_{m,j}),$$

$$\vec{H_{i,j}} = (f_1, \dots, f_{i-1}, D_d f_{i,j} - D_d f_i, f_{i+1,j}, \dots, f_{m,j}).$$

(ii) $r = 0$. If $0 < \alpha < md$, then $|u_{x,\vec{f_{i,j}}}(r) - u_{x,\vec{f}^i}(r)| = 0$. If $\alpha = 0$, we have

$$\begin{aligned}&|u_{x,\vec{f_{i,j}}}(r) - u_{x,\vec{f}^i}(r)|\\ \leqslant&\sum_{\mu=1}^{i-1}\Big(\prod_{l_1=1}^{\mu-1} f_{l_1}(x)\Big)(f_{\mu,j}(x) - f_\mu(x))\Big(\prod_{l_2=\mu+1}^{i-1} f_{l_2,j}(x)\Big)|D_d f_{i,j}(x)|\Big(\prod_{l_3=i+1}^{m} f_{l_3,j}(x)\Big)\\ &+\sum_{\nu=i+1}^{m}\Big(\prod_{l_1=1}^{i-1} f_{l_1}(x)\Big)|D_d f_i(x)|\Big(\prod_{l_2=i+1}^{\nu-1} f_{l_2}(x)\Big)|f_{\nu,j}(x) - f_\nu(x)|\Big(\prod_{l_3=\nu+1}^{m} f_{l_3,j}(x)\Big)\\ &+\Big(\prod_{l_1=1}^{i-1} f_{l_1}(x)\Big)|D_d f_{i,j}(x) - D_d f_i(x)|\Big(\prod_{l_2=i+1}^{m} f_{l_2,j}(x)\Big).\end{aligned}$$

From the above, the Lebesgue differential theorem and Lemma 2, we have that for a.e.

$x \in \mathbb{R}^d$,

$$\begin{aligned}
&|D_d\mathfrak{M}_\alpha(\vec{f}_j)(x) - D_d\mathfrak{M}_\alpha(\vec{f})(x)| \\
&= \Big|\sum_{i=1}^m u_{x,\vec{f}_{i,j}}(r_1) - \sum_{i=1}^m u_{x,\vec{f}^i}(r_2)\Big| \\
&\leqslant \Big|\sum_{i=1}^m u_{x,\vec{f}_{i,j}}(r_1) - \sum_{i=1}^m u_{x,\vec{f}^i}(r_1)\Big| + \Big|\sum_{i=1}^m u_{x,\vec{f}^i}(r_1) - \sum_{i=1}^m u_{x,\vec{f}^i}(r_2)\Big| \\
&\leqslant \sum_{i=1}^m \mathscr{G}_{i,j}(x) + \Big|\sum_{i=1}^m u_{x,\vec{f}^i}(r_1) - \sum_{i=1}^m u_{x,\vec{f}^i}(r_2)\Big|
\end{aligned} \tag{15}$$

for any $r_1 \in \mathscr{R}(\vec{f}_j)(x)$ and $r_2 \in \mathscr{R}(\vec{f})(x)$. On can easily check that

$$\lim_{j\to\infty} \|\mathscr{G}_{i,j}\|_{L^q(\mathbb{R}^d)} = 0$$

for any $i = 1,2,\dots,m$. It follows that there exists $j_2 \in \mathbb{N}$ such that $\|\mathscr{G}_{i,j}\|_{L^q(\mathbb{R}^d)} < \varepsilon$ for any $i = 1,2,\dots,m$ and $j \geqslant j_2$.

If $x \notin G_1 \cup G_2 \cup B^j$ we can choose $r_1 \in \mathscr{R}(\vec{f}_j)(x)$ and $r_2 \in \mathscr{R}(\vec{f})(x)$ such that $|r_1 - r_2| < \delta$ and

$$\Big|\sum_{i=1}^m u_{x,\vec{f}^i}(r_1) - \sum_{i=1}^m u_{x,\vec{f}^i}(r_2)\Big| < |B_R|^{-1/q}\varepsilon.$$

On the other hand, we have that for any $i = 1,2,\dots,m$, $r_1 \in \mathscr{R}(\vec{f}_j)(x)$ and $r_2 \in \mathscr{R}(\vec{f})(x)$,

$$\Big|\sum_{i=1}^m u_{x,\vec{f}^i}(r_1) - \sum_{i=1}^m u_{x,\vec{f}^i}(r_2)\Big| \leqslant \sum_{i=1}^m |u_{x,\vec{f}^i}(r_1) - u_{x,\vec{f}^i}(r_2)| \leqslant 2\sum_{i=1}^m \mathfrak{M}_\alpha(\vec{f}^i)(x).$$

Note that $|G_2 \cup B^j| < \eta$ for $j \geqslant j_1$. Thus we get from (15) that

$$\begin{aligned}
&\|D_d\mathfrak{M}_\alpha(\vec{f}_j) - D_d\mathfrak{M}_\alpha(\vec{f})\|_{L^q(\mathbb{R}^d)} \\
&\leqslant \Big\|\sum_{i=1}^m \mathscr{G}_{i,j}\Big\|_{L^q(\mathbb{R}^d)} + \||B_R|^{-1/q}\varepsilon\|_{q,B_R} + 2\Big\|\sum_{i=1}^m \mathfrak{M}_\alpha(\vec{f}^i)\Big\|_{q,G_1\cup G_2\cup B^j} \leqslant C\varepsilon,
\end{aligned}$$

for any $j \geqslant \max\{j_1, j_2\}$, which gives

$$\lim_{j\to\infty} \|D_d\mathfrak{M}_\alpha(\vec{f}_j) - D_d\mathfrak{M}_\alpha(\vec{f})\|_{L^q(\mathbb{R}^d)} = 0.$$

This completes the proof of Theorem 1. □

3. Proof of Theorem 2

This section is devoted to proving Theorem 2. We now recall the definition of approximate differentiability. Let f be a real-valued function defined on a set $E \subset$

$\mathbb{R}^d$. We say that f is approximately differentiable at $x_0 \in E$ if there is a vector $L = (L_1, L_2, \dots, L_d) \in \mathbb{R}^d$ such that for any $\varepsilon > 0$ the set

$$A_\varepsilon = \left\{ x \in \mathbb{R}^d : \frac{|f(x) - f(x_0) - L(x - x_0)|}{|x - x_0|} < \varepsilon \right\}$$

has x_0 as a density point. If this is the case, then x_0 is a density point of E and L is uniquely determined. The vector L is called the approximate differential of f at x_0 and is denoted by $\nabla f(x_0)$. Note that every function $f \in W^{1,1}(\mathbb{R}^d)$ is approximately differentiable a.e. It follows from Theorem A that Mf is approximately differentiable a.e. under the assumption that $f \in W^{1,1}(\mathbb{R}^d)$. However, it is unknown that whether $f \in W^{1,1}(\mathbb{R}^d)$ implies the weak differentiability of Mf when $d \geqslant 2$. For now, the relationship of approximate differentiability and weak differentiability is also not clear.

To prove Theorem 2, we need the following lemma followed from [35], which provides several characterizations of a.e. approximate differentiability of a function.

LEMMA 3. ([35]) *Let $f : E \to \mathbb{R}$ be measurable, $E \subset \mathbb{R}^d$. Then the following conditions are equivalent:*

(i) *f is approximately differentiable a.e.*

(ii) *For any $\varepsilon > 0$, there is a closed set $F \subset E$ and a locally Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$ such that $f = g$ on $x \in F$ and $|E \backslash F| < \varepsilon$.*

(iii) *For any $\varepsilon > 0$, there is a closed set $F \subset E$ and a function $g \in \mathscr{C}^1(\mathbb{R}^d)$ such that $f = g$ on $x \in F$ and $|E \backslash F| < \varepsilon$.*

LEMMA 4. *Let $0 \leqslant \alpha \leqslant md - 1$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in L^1(\mathbb{R}^d)$. For $\varepsilon > 0$, we define the truncated maximal operator $\mathfrak{M}_\alpha^\varepsilon$ by*

$$\mathfrak{M}_\alpha^\varepsilon(\vec{f})(x) = \sup_{r \geqslant \varepsilon} |B(x,r)|^{\alpha/d - m} \prod_{i=1}^m \int_{B(x,r)} |f_i(y)| dy.$$

Then $\mathfrak{M}_\alpha^\varepsilon(\vec{f})$ is Lipschitz continuous for every $\varepsilon > 0$.

Proof. For any $r \geqslant \varepsilon$ and $\delta \geqslant 1$, we have

$$\left(\frac{r}{r + |x - y|}\right)^\delta \geqslant \left(\frac{\varepsilon}{\varepsilon + |x - y|}\right)^\delta \geqslant 1 - \delta \frac{|x - y|/\varepsilon}{1 + |x - y|/\varepsilon} \geqslant 1 - \frac{\delta}{\varepsilon}|x - y|. \tag{16}$$

Fix $x, y \in \mathbb{R}^d$ and $r \geqslant \varepsilon$ we have $B(y,r) \subset B(x, r + |x - y|)$. This together with (16) yields that

$$\begin{aligned} \mathfrak{M}_\alpha^\varepsilon(\vec{f})(x) &\geqslant |B(x, r + |x - y|)|^{\alpha/d - m} \prod_{i=1}^m \int_{B(x, r + |x - y|)} |f_i(y)| dy \\ &\geqslant \left(\frac{r}{r + |x - y|}\right)^{md - \alpha} |B(y,r)|^{\alpha/d - m} \prod_{i=1}^m \int_{B(y,r)} |f_i(y)| dy \\ &\geqslant \left(1 - \frac{md - \alpha}{\varepsilon}|x - y|\right) |B(y,r)|^{\alpha/d - m} \prod_{i=1}^m \int_{B(y,r)} |f_i(y)| dy. \end{aligned}$$

whenever $\alpha \leqslant md-1$. It follows that

$$\mathfrak{M}_\alpha^\varepsilon(\vec{f})(y)-\mathfrak{M}_\alpha^\varepsilon(\vec{f})(x) \leqslant \frac{md-\alpha}{\varepsilon}|x-y|\mathfrak{M}_\alpha^\varepsilon(\vec{f})(y).$$

Similarly,

$$\mathfrak{M}_\alpha^\varepsilon(\vec{f})(x)-\mathfrak{M}_\alpha^\varepsilon(\vec{f})(y) \leqslant \frac{md-\alpha}{\varepsilon}|x-y|\mathfrak{M}_\alpha^\varepsilon(\vec{f})(x).$$

Thus we have

$$\begin{aligned}|\mathfrak{M}_\alpha^\varepsilon(\vec{f})(x)-\mathfrak{M}_\alpha^\varepsilon(\vec{f})(y)| &\leqslant \frac{md-\alpha}{\varepsilon}|x-y|(\mathfrak{M}_\alpha^\varepsilon(\vec{f})(x)+\mathfrak{M}_\alpha^\varepsilon(\vec{f})(y))\\ &\leqslant 2\frac{md-\alpha}{\varepsilon}|x-y|\Big(\frac{1}{\frac{2\pi^{d/2}}{d\Gamma(d/2)}\varepsilon^d}\Big)^{m-\alpha/d}\prod_{j=1}^m\|f_j\|_{L^1(\mathbb{R}^d)}\\ &\leqslant 2\frac{md-\alpha}{\varepsilon^{md-\alpha+1}}\Big(\frac{d\Gamma(d/2)}{2\pi^{d/2}}\Big)^{m-\alpha/d}\prod_{j=1}^m\|f_j\|_{L^1(\mathbb{R}^d)}|x-y|.\end{aligned}$$

Then Lemma 4 is proved. □

We now turn to prove Theorem 2.

Proof of Theorem 2. Let Z_j be the set of all Lebesgue points of f_j and $u_{x,\vec{f}}(r)$ as in Section 2. We set $F=\mathbb{R}^d\backslash(\bigcap_{j=1}^m Z_j)$. Let $x\in\bigcap_{j=1}^m Z_j$ such that $\mathfrak{M}_\alpha(\vec{f})(x)>u_{x,\vec{f}}(0)$. Since $f_j\in L^1(\mathbb{R}^d)$ and $\mathfrak{M}_\alpha(\vec{f})(x)>0$, there exists a sequence of positive bounded numbers $\{r_k\}_{k\geqslant 1}$ such that

$$u_{x,\vec{f}}(r_k)\to\mathfrak{M}_\alpha(\vec{f})(x) \text{ when } k\to\infty.$$

Hence there exists a subsequence $\{s_k\}_{k\geqslant1}\subset\{r_k\}_{k\geqslant1}$ such that $\lim_{k\to\infty}s_k=r>0$. It follows that

$$\mathfrak{M}_\alpha(\vec{f})(x)=u_{x,\vec{f}}(r).$$

This yields that

$$\mathbb{R}^d=F\cup\{x\in\mathbb{R}^d:\mathfrak{M}_\alpha(\vec{f})(x)=u_{x,\vec{f}}(0)\}\cup E,$$

where $E=\bigcup_{k=1}^\infty E_k$ and $E_k=\{x\in\mathbb{R}^d:\mathfrak{M}_\alpha(\vec{f})(x)=\mathfrak{M}_\alpha^{1/k}(\vec{f})(x)\}$. Obviously, $E_k\subset E_{k+1}$. By Lemma 3, we know that $\prod_{j=1}^m|f_j|$ is approximately differentiable a.e. Then $\mathfrak{M}_\alpha(\vec{f})$ is approximately differentiable a.e. in the set $\{x\in\mathbb{R}^d:\mathfrak{M}_\alpha(\vec{f})(x)=u_{x,\vec{f}}(0)\}$. By Lemma 4 we have that $\mathfrak{M}_\alpha^{1/k}(\vec{f})$ is Lipschitz continuous for any $k\geqslant1$. It follows that $\mathfrak{M}_\alpha(\vec{f})\chi_{E_{k+1}\backslash E_k}$ is approximately differentiable a.e. for all $k\geqslant1$. Using Lemma 3 again we have that $\mathfrak{M}_\alpha(\vec{f})\chi_E=\mathfrak{M}_\alpha(\vec{f})\chi_{E_k}+\sum_{k=1}^\infty\mathfrak{M}_\alpha(\vec{f})\chi_{E_{k+1}\backslash E_k}$ is approximately differentiable a.e. Note that $|F|=0$. Therefore, $\mathfrak{M}_\alpha(\vec{f})$ is approximately differentiable a.e. □

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