

MAXIMAL AND SINGULAR INTEGRAL OPERATORS AND THEIR COMMUTATORS ON GENERALIZED WEIGHTED MORREY SPACES WITH VARIABLE EXPONENT

VAGIF S. GULIYEV, JAVANSHIR J. HASANOV AND XAYYAM A. BADALOV

(Communicated by J. Soria)

Abstract. We consider the generalized weighted Morrey spaces $\mathcal{M}_\omega^{p(\cdot),\varphi}(\Omega)$ with variable exponent $p(x)$ and a general function $\varphi(x,r)$ defining the Morrey-type norm. In case of unbounded sets $\Omega \subset \mathbb{R}^n$ we prove the boundedness of the Hardy-Littlewood maximal operator and Calderón-Zygmund singular operators with standard kernel, in such spaces. We also prove the boundedness of the commutators of maximal operator and Calderón-Zygmund singular operators in the generalized weighted Morrey spaces with variable exponent

1. Introduction

The classical Morrey spaces were originally introduced by Morrey in [53] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [22, 23, 25, 53]. Mizuhara [54] and Nakai [57] introduced generalized Morrey spaces. Later, Guliyev [25] defined the generalized Morrey spaces $M^{p,\varphi}$ with normalized norm. Recently, Komori and Shirai [48] considered the weighted Morrey spaces $L_w^{p,\kappa}$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Guliyev [26] gave a concept of generalized weighted Morrey space $M_w^{p,\varphi}$ which could be viewed as extension of both generalized Morrey space $M^{p,\varphi}$ and weighted Morrey space $L_w^{p,\kappa}$. In [26] the boundedness of the classical operators and its commutators in spaces $M_w^{p,\varphi}$ also was studied, see also [33, 42].

As it is known, last two decades there is an increasing interest to the study of variable exponent spaces and operators with variable parameters in such spaces, we refer for instance to the surveying papers [20, 44, 47, 60], on the progress in this field, including topics of Harmonic Analysis and Operator Theory, see also references therein. For mapping properties of maximal functions and singular integrals on Lebesgue spaces with variable exponent we refer to [13, 14, 15, 17, 18, 19, 46, 50].

Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$, were introduced and studied in [3] and [55] in the Euclidean setting and in [45] in the setting of metric measure spaces,

Mathematics subject classification (2010): 42B20, 42B25, 42B35.

Keywords and phrases: Maximal operator, singular integral operators, generalized weighted Morrey space with variable exponent, BMO space.

in case of bounded sets. The boundedness of the maximal operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)$ under the log-condition on $p(\cdot)$, $\lambda(\cdot)$ was proved in [3]. In [56] the maximal operator was considered in a somewhat more general space, but under more restrictive conditions on $p(x)$. P. Hästö in [35] used his new “local-to-global” approach to extend the result of [3] on the maximal operator to the case of the whole space \mathbb{R}^n . The boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot), \lambda(\cdot)}$ in the general setting of metric measure spaces was proved in [45].

Generalized Morrey spaces of such a kind in the case of constant p were studied in [6], [51], [54], [57]. In the case of bounded sets the boundedness of the maximal operator, singular integral operators and the potential operator in generalized variable exponent Morrey type spaces was proved in [29], [30], [31] and in the case of unbounded sets in [32], see also [39, 40, 58, 62]. Also, in the case of bounded sets the boundedness of these operators in generalized variable exponent weighted Morrey spaces for the power weights was proved in [37] and [38].

In the case of constant p and λ , the results on the boundedness of potential operators and classical Calderón-Zygmund singular operators go back to [1] and [59], respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [16]; for further results in the case of constant p and λ (see, for instance, [5]–[7]).

We introduce the generalized variable exponent weighted Morrey spaces $\mathcal{M}_\omega^{p(\cdot), \varphi}(\Omega)$ over an open set $\Omega \subseteq \mathbb{R}^n$. Within the frameworks of the spaces $\mathcal{M}_\omega^{p(\cdot), \varphi}(\Omega)$, over unbounded sets $\Omega \subseteq \mathbb{R}^n$ we consider the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} |B(x, r)|^{-1} \int_{\tilde{B}(x, r)} |f(y)| dy$$

and Calderón-Zygmund type singular operator

$$Tf(x) = \int_{\Omega} K(x, y) f(y) dy,$$

where $K(x, y)$ is a “standard singular kernel”, that is, a continuous function defined on $\{(x, y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x, y)| \leq C|x - y|^{-n} \text{ for all } x \neq y,$$

$$|K(x, y) - K(x, z)| \leq C \frac{|y - z|^\sigma}{|x - y|^{n+\sigma}}, \quad \sigma > 0, \text{ if } |x - y| > 2|y - z|,$$

$$|K(x, y) - K(\xi, y)| \leq C \frac{|x - \xi|^\sigma}{|x - y|^{n+\sigma}}, \quad \sigma > 0, \text{ if } |x - y| > 2|x - \xi|.$$

Let

$$T^*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$$

be the maximal singular operator, where $T_\varepsilon f(x)$ is the usual truncation

$$T_\varepsilon f(x) = \int_{\{y \in \Omega : |x - y| \geq \varepsilon\}} K(x, y) f(y) dy.$$

We find the condition on the Morrey function $\varphi(x, r)$ for the boundedness of the maximal operator M and the singular integral operators T in generalized weighted Morrey space $\mathcal{M}_\omega^{p(\cdot), \varphi}(\Omega)$ with variable $p(x)$ under the log-condition on $p(\cdot)$.

The paper is organized as follows. In Section 2 we provide necessary preliminaries on variable exponent weighted Lebesgue and generalized weighted Morrey spaces. In Section 3 we deal with the maximal operator and its commutator. In Section 4 we treat Calderón-Zygmund singular operators and its commutators.

The main results are given in Theorems 3.4, 3.5, 3.8, 3.9, 4.2, 4.3, 4.5, 4.6. We emphasize that the results we obtain for generalized weighted Morrey spaces are new even in the case when $p(x)$ is constant, because we do not impose any monotonicity type condition on $\varphi(x, r)$.

We use the following notation: \mathbb{R}^n is the n -dimensional Euclidean space, $\Omega \subset \mathbb{R}^n$ is an open set, $\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$, $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$, $\tilde{B}(x, r) = B(x, r) \cap \Omega$, by c, C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line. By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries on variable exponent weighted Lebesgue and generalized weighted Morrey spaces

We refer to the book [18] for variable exponent Lebesgue spaces but give some basic definitions and facts. Let $p(\cdot)$ be a measurable function on Ω with values in $(1, \infty)$. An open set Ω which may be unbounded throughout the whole paper. We mainly suppose that

$$1 < p_- \leq p(x) \leq p_+ < \infty, \tag{2.1}$$

where $p_- := \operatorname{ess\,inf}_{x \in \Omega} p(x)$, $p_+ := \operatorname{ess\,sup}_{x \in \Omega} p(x)$. By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \eta > 0 : I_{p(\cdot)} \left(\frac{f}{\eta} \right) \leq 1 \right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent.

The space $L^{p(\cdot)}(\Omega)$ coincides with the space

$$\left\{ f(x) : \left| \int_\Omega f(y)g(y)dy \right| < \infty \text{ for all } g \in L^{p'(\cdot)}(\Omega) \right\} \tag{2.2}$$

up to the equivalence of the norms

$$\|f\|_{L^{p(\cdot)}(\Omega)} \approx \sup_{\|g\|_{L^{p'(\cdot)}} \leq 1} \left| \int_{\Omega} f(y)g(y)dy \right| \quad (2.3)$$

see [41, Proposition 2.2], see also [49, Theorem 2.3], or [61, Theorem 3.5].

For the basics on variable exponent Lebesgue spaces we refer to [63], [49].

$\mathcal{P}(\Omega)$ is the set of bounded measurable functions $p : \Omega \rightarrow [1, \infty)$;

$\mathcal{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}(\Omega)$ satisfying the local log-condition

$$|p(x) - p(y)| \leq \frac{A}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2}, \quad x, y \in \Omega, \quad (2.4)$$

where $A = A(p) > 0$ does not depend on x, y ;

$\mathcal{A}^{log}(\Omega)$ is the set of bounded exponents $p : \Omega \rightarrow \mathbb{R}^n$ satisfying the condition (2.4);

$\mathbb{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}^{log}(\Omega)$ with $1 < p_- \leq p_+ < \infty$;

for Ω which may be unbounded, by $\mathcal{P}_{\infty}(\Omega)$, $\mathcal{P}_{\infty}^{log}(\Omega)$, $\mathbb{P}_{\infty}^{log}(\Omega)$, $\mathcal{A}_{\infty}^{log}(\Omega)$ we denote the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded)

$$|p(x) - p(\infty)| \leq \frac{A_{\infty}}{\ln(2+|x|)}, \quad x \in \mathbb{R}^n. \quad (2.5)$$

where $p_{\infty} = \lim_{x \rightarrow \infty} p(x) > 1$.

We will also make use of the estimate provided by the following lemma (see [18], Corollary 4.5.9).

$$\|\mathcal{X}_{\tilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \leq Cr^{\theta_p(x,r)}, \quad x \in \Omega, \quad p \in \mathbb{P}_{\infty}^{log}(\Omega), \quad (2.6)$$

where $\theta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & r \leq 1, \\ \frac{n}{p(\infty)}, & r \geq 1. \end{cases}$

By ω we always denote a weight, i.e. a positive, locally integrable function with domain Ω . The weighted Lebesgue space $L_{\omega}^{p(\cdot)}(\Omega)$ is defined as the set of all measurable functions for which

$$\|f\|_{L_{\omega}^{p(\cdot)}(\Omega)} = \|f\omega\|_{L^{p(\cdot)}(\Omega)}.$$

Let us define the class $A_{p(\cdot)}(\Omega)$ (see [20], [50]) to consist of those weights ω for which

$$\sup_B |B|^{-1} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))} \|\omega^{-1}\|_{L^{p'(\cdot)}(\tilde{B}(x,r))} < \infty.$$

THEOREM 2.1. ([36, Theorem 1.1]) *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}_{\infty}^{log}(\Omega)$. Then $M : L_{\omega}^{p(\cdot)}(\Omega) \rightarrow L_{\omega}^{p(\cdot)}(\Omega)$ if and only if $\omega \in A_{p(\cdot)}(\Omega)$.*

Singular operators within the framework of the spaces with variable exponents were studied in [19]. From Theorem 4.8 and Remark 4.6 of [19] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω , but valid for an arbitrary open set Ω under the corresponding condition in $p(x)$ at infinity.

THEOREM 2.2. ([19, Theorem 4.8]) *Let $\Omega \subset \mathbb{R}^n$ be a unbounded open set and $p \in \mathbb{P}^{log}(\Omega)$. Then the singular integral operator T is bounded in $L^{p(\cdot)}(\Omega)$.*

Let $\lambda(x)$ be a measurable function on Ω with values in $[0, n]$. The variable Morrey space $\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)$ and variable weighted Morrey space $\mathcal{L}_\omega^{p(\cdot), \lambda(\cdot)}(\Omega)$ are defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{\mathcal{L}^{p(\cdot), \lambda(\cdot)}(\Omega)} = \sup_{x \in \Omega, t > 0} t^{-\frac{\lambda(x)}{p(x)}} \|f \chi_{\tilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)},$$

$$\|f\|_{\mathcal{L}_\omega^{p(\cdot), \lambda(\cdot)}(\Omega)} = \sup_{x \in \Omega, t > 0} t^{-\frac{\lambda(x)}{p(x)}} \|f \chi_{\tilde{B}(x,t)}\|_{L_\omega^{p(\cdot)}(\Omega)},$$

respectively.

Let M^\sharp be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r > 0} |B(x, r)|^{-1} \int_{\tilde{B}(x,r)} |f(y) - f_{\tilde{B}(x,r)}| dy,$$

where $f_{\tilde{B}(x,t)}(x) = |\tilde{B}(x,t)|^{-1} \int_{\tilde{B}(x,t)} f(y) dy$.

DEFINITION 2.1. We define the $BMO(\Omega)$ space as the set of all locally integrable functions f with finite norm

$$\|f\|_{BMO} = \sup_{x \in \Omega} M^\sharp f(x) = \sup_{x \in \Omega, r > 0} |B(x, r)|^{-1} \int_{\tilde{B}(x,r)} |f(y) - f_{\tilde{B}(x,r)}| dy.$$

DEFINITION 2.2. We define the $BMO_{p(\cdot), \omega}(\Omega)$ space as the set of all locally integrable functions f with finite norm

$$\|f\|_{BMO_{p(\cdot), \omega}} = \sup_{x \in \Omega, r > 0} \frac{\|(f(\cdot) - f_{\tilde{B}(x,r)}) \chi_{\tilde{B}(x,r)}\|_{L_\omega^{p(\cdot)}(\Omega)}}{\|\chi_{\tilde{B}(x,r)}\|_{L_\omega^{p(\cdot)}(\Omega)}}.$$

THEOREM 2.3. ([41, Theorem 4.4]) *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{log}(\Omega)$ and ω be a Lebesgue measurable function. If $\omega \in A_{p(\cdot)}(\Omega)$, then the norms $\|\cdot\|_{BMO_{p(\cdot), \omega}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

Everywhere in the sequel the functions $\varphi(x, r)$, $\varphi_1(x, r)$ and $\varphi_2(x, r)$ used in the body of the paper, are non-negative measurable functions on $\Omega \times (0, \infty)$. We find it convenient to define the generalized weighted Morrey spaces in the form as follows.

DEFINITION 2.3. Let $1 \leq p(x) < \infty$, $x \in \Omega$. The variable exponent generalized Morrey space $\mathcal{M}^{p(\cdot), \varphi}(\Omega)$ and variable exponent generalized weighted Morrey space $\mathcal{M}_\omega^{p(\cdot), \varphi(\cdot)}(\Omega)$ are defined as the set of integrable functions f on Ω with the finite norms

$$\|f\|_{\mathcal{M}^{p(\cdot), \varphi}} = \sup_{x \in \Omega, r > 0} \frac{1}{\varphi(x, r) r^{\theta_p(x,r)}} \|f\|_{L^{p(\cdot)}(\tilde{B}(x,r))},$$

$$\|f\|_{\mathcal{M}_\omega^{p(\cdot),\varphi}} = \sup_{x \in \Omega, r > 0} \frac{1}{\varphi(x, r) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, r))}} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, r))},$$

respectively.

According to this definition, we recover the space $\mathcal{L}_\omega^{p(\cdot), \lambda(\cdot)}(\Omega)$ under the choice $\varphi(x, r) = r^{\theta_p(x, r) - \frac{\lambda(x)}{p(x)}}$:

$$\mathcal{L}_\omega^{p(\cdot), \lambda(\cdot)}(\Omega) = \mathcal{M}_\omega^{p(\cdot), \varphi(\cdot)}(\Omega) \Big|_{\varphi(x, r) = r^{\theta_p(x, r) - \frac{\lambda(x)}{p(x)}}}.$$

3. The maximal operator and its commutators in $\mathcal{M}_\omega^{p(\cdot), \varphi}(\Omega)$

Let $L_v^\infty(\mathbb{R}_+)$ be the weighted L^∞ -space with the norm

$$\|g\|_{L_v^\infty(\mathbb{R}_+)} = \operatorname{ess\,sup}_{t > 0} v(t)g(t).$$

In the sequel $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+; \uparrow)$ stand for the set of Lebesgue-measurable functions on \mathbb{R}_+ , and its subspaces of nonnegative and nonnegative non-decreasing functions, respectively. We also denote

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(\mathbb{R}_+; \uparrow) : \lim_{t \rightarrow 0^+} \varphi(t) = 0 \right\}.$$

Let u be a continuous and non-negative function on \mathbb{R}_+ . We define the supremal operator \bar{S}_u by

$$(\bar{S}_u g)(t) := \|u g\|_{L_1(0, t)}, \quad t \in (0, \infty).$$

The following theorem was proved in [5].

THEOREM 3.1. *Suppose that v_1 and v_2 are nonnegative measurable functions such that $0 < \|v_1\|_{L_\infty(0, t)} < \infty$ for every $t > 0$. Let u be a continuous nonnegative function on \mathbb{R} . Then the operator \bar{S}_u is bounded from $L_{v_1}^\infty(\mathbb{R}_+)$ to $L_{v_2}^\infty(\mathbb{R}_+)$ on the cone \mathbb{A} if and only if*

$$\left\| v_2 \bar{S}_u \left(\|v_1\|_{L_\infty(0, \cdot)}^{-1} \right) \right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

We will use the following results on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_0^t g(s)w(s)ds, \quad H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [27, 28].

THEOREM 3.2. *Let v_1 , v_2 and w be weights on $(0, \infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \leq C \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s) ds}{\operatorname{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

THEOREM 3.3. *Let v_1 , v_2 and w be weights on $(0, \infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w g(t) \leq C \sup_{t>0} v_1(t) g(t) \tag{3.1}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_0^t \frac{w(s) ds}{\sup_{0<\tau<s} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (3.1).

The following weighted local estimates are valid.

THEOREM 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{\log}(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$. Then*

$$\|Mf\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq C \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \sup_{r \geq t} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,r))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))}^{-1}, \tag{3.2}$$

for every $f \in L_\omega^{p(\cdot)}(\Omega)$, where C does not depend on $f, x \in \Omega$ and t .

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y) \chi_{\tilde{B}(x,2t)}(y), \quad f_2(y) = f(y) \chi_{\Omega \setminus \tilde{B}(x,2t)}(y), \quad t > 0, \tag{3.3}$$

and have

$$\|Mf\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq \|Mf_1\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} + \|Mf_2\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))}.$$

By Theorem 2.1 we obtain

$$\|Mf_1\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq \|Mf_1\|_{L_\omega^{p(\cdot)}(\Omega)} \leq C \|f_1\|_{L_\omega^{p(\cdot)}(\Omega)} = C \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))}, \tag{3.4}$$

where C does not depend on f . From (3.4) we obtain

$$\|Mf_1\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq C \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \sup_{r \geq t} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,r))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))}^{-1}. \tag{3.5}$$

Let z be an arbitrary point in $B(x, r)$. If $B(z, r) \cap \overset{\circ}{B}(x, 2t) \neq \emptyset$, then $r > t$. Indeed, if $y \in B(z, r) \cap \overset{\circ}{B}(x, 2t)$, then $r > |y - z| \geq |x - z| - |x - y| > 2t - t = t$.

On the other hand, $B(z, r) \cap \overset{\circ}{B}(x, 2t) \subset B(x, 2r)$. Indeed, for $y \in B(z, r) \cap \overset{\circ}{B}(x, 2t)$, then we get $|x - y| \leq |y - z| + |x - z| < t + r < 2r$.

$$\begin{aligned}
Mf_2(z) &= \sup_{r>0} |B(z, r)|^{-1} \int_{\tilde{B}(z, r)} |f_2(y)| dy \\
&\leq C \sup_{r \geq 2t} |B(x, 2r)|^{-1} \int_{\overset{\circ}{B}(x, 2t) \cap \tilde{B}(z, r)} |f(y)| dy \\
&\leq C \sup_{r \geq t} |B(x, r)|^{-1} \int_{\tilde{B}(x, r)} |f(y)| dy \\
&\leq C \sup_{r \geq t} |B(x, r)|^{-1} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, r))} \|\omega^{-1}\|_{L^{p'(\cdot)}(\tilde{B}(x, r))} \\
&\leq C \sup_{r \geq t} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, r))} \|\omega\|_{L^{p'(\cdot)}(\tilde{B}(x, r))}^{-1}.
\end{aligned}$$

Thus, the function $Mf_2(z)$, with fixed x and t , is dominated by the expression not depending on z . Then

$$\|Mf_2\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, t))} \leq C \|\omega\|_{L^{p'(\cdot)}(\tilde{B}(x, t))} \sup_{r \geq t} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, r))} \|\omega\|_{L^{p'(\cdot)}(\tilde{B}(x, r))}^{-1}. \quad (3.6)$$

We then obtain (3.2) from (3.5) and (3.6). \square

THEOREM 3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{\log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$ and the function $\varphi_1(x, r)$ and $\varphi_2(x, r)$ satisfy the condition*

$$\sup_{t>r} \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x, s) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, s))}}{\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \leq C \varphi_2(x, r), \quad (3.7)$$

where C does not depend on $x \in \Omega$ and r . Then the operator M is bounded from the space $\mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)$ to the space $\mathcal{M}_\omega^{p(\cdot), \varphi_2}(\Omega)$.

Proof. Let $f \in \mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)$. By Theorems 3.1 and 3.4 we obtain

$$\begin{aligned}
&\|Mf\|_{\mathcal{M}_\omega^{p(\cdot), \varphi_2}(\Omega)} \\
&\leq C \sup_{x \in \Omega, t > 0} \frac{1}{\varphi_2(x, t)} \sup_{r \geq t} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, r))} \|\omega\|_{L^{p'(\cdot)}(\tilde{B}(x, r))}^{-1} \\
&\leq C \sup_{x \in \Omega, t > 0} \frac{1}{\varphi_1(x, t) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, t))} t^{-n} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))} \int_t^\infty s^{n-1} ds \\
&= C \|f\|_{\mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)}
\end{aligned}$$

by (3.7), which completes the proof. \square

In the case $\omega = 1$ we get

COROLLARY 3.1. ([32, Theorem 3.5]) *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{\log}(\Omega)$ and the functions $\varphi_1(x, r)$ and $\varphi_2(x, r)$ satisfy the condition*

$$\sup_{t>r} \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x, s) s^{\theta_p(x, s)}}{t^{\theta_p(x, t)}} \leq C \varphi_2(x, r), \tag{3.8}$$

where C does not depend on $x \in \Omega$ and r . Then the operator M is bounded from the space $\mathcal{M}^{p(\cdot), \varphi_1}(\Omega)$ to the space $\mathcal{M}^{p(\cdot), \varphi_2}(\Omega)$.

The commutator generated by M and a suitable function b is formally defined by

$$[M, b]f = M(bf) - bM(f).$$

Given a measurable function b the maximal commutator is defined by

$$M_b(f)(x) := \sup_{r>0} |B(x, r)|^{-1} \int_{B(x, r)} |b(x) - b(y)| |f(y)| dy$$

for all $x \in \mathbb{R}^n$.

This operator plays an important role in the study of commutators of singular integral operators with BMO symbols (see, for instance [24], [52]). The maximal operator M_b has been studied intensively and there exist plenty of results about it. Pu Zhang and Jianglong Wu [64] proved the following statement.

THEOREM 3.6. [64, Theorem 3.1] *Let $b \in L_1^{loc}(\mathbb{R}^n)$ and $p \in \mathbb{P}_\infty^{\log}(\mathbb{R}^n)$, then the operator M_b is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself if and only if $b \in BMO(\mathbb{R}^n)$.*

Operators M_b and $[M, b]$ essentially differ from each other. For example, M_b is a positive and sublinear operator, but $[M, b]$ is neither positive nor sublinear. However, if b satisfies some additional conditions, then operator M_b controls $[M, b]$.

LEMMA 3.1. ([2, Lemma 3.1]) *Let b be any non-negative locally integrable function. Then*

$$|[M, b]f(x)| \leq M_b(f)(x), \quad x \in \mathbb{R}^n$$

holds for all $f \in L_1^{loc}(\mathbb{R}^n)$.

THEOREM 3.7. ([2, Theorem 1.13]) *Let $b \in BMO(\mathbb{R}^n)$. Suppose that X is a Banach space of measurable functions defined on \mathbb{R}^n . Assume that M is bounded on X . Then the operator M_b is bounded on X , and the inequality*

$$\|M_b f\|_X \leq C \|b\|_* \|f\|_X$$

holds with constant C independent of f .

COROLLARY 3.2. *Let $b \in BMO(\Omega)$, $p \in \mathbb{P}_\infty^{\log}(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$, then the operator M_b is bounded on $L_\omega^{p(\cdot)}(\mathbb{R}^n)$.*

Before proving the main theorems, we need the following lemma.

LEMMA 3.2. ([34, Lemma 2]) *Let $b \in BMO(\Omega)$. Then there is a constant $C > 0$ such that*

$$\left| b_{\tilde{B}(x,r)} - b_{\tilde{B}(x,t)} \right| \leq C \|b\|_* \ln \frac{t}{r} \text{ for } 0 < 2r < t,$$

where C is independent of b , x , r , and t .

The following weighted local estimates are valid.

THEOREM 3.8. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{\log}(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$, $b \in BMO(\Omega)$. Then*

$$\|M_b f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq C \|b\|_* \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \sup_{r \geq t} \left(1 + \ln \frac{r}{t} \right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,r))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))}^{-1} \quad (3.9)$$

for every $f \in L_\omega^{p(\cdot)}(\Omega)$, where C does not depend on $f, x \in \Omega$ and t .

Proof. We represent function f as in (3.3) and have

$$\|M_b f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq \|M_b f_1\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} + \|M_b f_2\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))}.$$

By Corollary 3.2 we obtain

$$\|M_b f_1\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq \|M_b f_1\|_{L_\omega^{p(\cdot)}(\Omega)} \leq C \|b\|_* \|f_1\|_{L_\omega^{p(\cdot)}(\Omega)} = C \|b\|_* \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))}, \quad (3.10)$$

where C does not depend on f . From (3.10) we obtain (see also (3.4))

$$\|M_b f_1\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq C \|b\|_* \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \sup_{r \geq t} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,r))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))}^{-1}, \quad (3.11)$$

easily obtained from the fact that $\|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))}$ is non-decreasing in t , so that $\|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))}$ on the right-hand side of (3.10) is dominated by the right-hand side of (3.11).

For $z \in \tilde{B}(x,t)$ we get

$$\begin{aligned} M_b f_2(z) &= \sup_{r>0} |B(z,r)|^{-1} \int_{\tilde{B}(z,r)} |b(z) - b(y)| |f_2(y)| dy \\ &\leq C \sup_{r \geq 2t} |B(x,2r)|^{-1} \int_{\tilde{B}(x,2r) \cap \tilde{B}(z,r)} |b(z) - b(y)| |f(y)| dy \\ &\leq C \sup_{r \geq t} |B(x,r)|^{-1} \int_{\tilde{B}(x,r)} |b(z) - b(y)| |f(y)| dy \\ &\leq \sup_{r \geq t} |B(x,r)|^{-1} \int_{\tilde{B}(x,r)} |b(y) - b_{\tilde{B}(x,r)}| |f(y)| dy \\ &\quad + \sup_{r \geq t} |B(x,r)|^{-1} \int_{\tilde{B}(x,r)} |b(z) - b_{\tilde{B}(x,r)}| |f(y)| dy = I_1 + I_2. \end{aligned}$$

By Hölder inequality and Theorem 2.3 we obtain

$$\begin{aligned}
 I_1 &= \sup_{r \geq t} |B(x, r)|^{-1} \int_{\tilde{B}(x, r)} |b(y) - b_{\tilde{B}(x, r)}| |f(y)| dy \\
 &\leq C \sup_{r \geq t} |B(x, r)|^{-1} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, r))} \|b(\cdot) - b_{\tilde{B}(x, r)}\|_{L_{\omega^{-1}}^{p'(\cdot)}(\tilde{B}(x, r))} \\
 &\leq C \|b\|_* \sup_{r \geq t} |B(x, r)|^{-1} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, r))} \|\omega^{-1}\|_{L^{p'(\cdot)}(\tilde{B}(x, r))} \\
 &\leq C \|b\|_* \sup_{r \geq t} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, r))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, r))}^{-1}.
 \end{aligned}$$

To estimate I_2 , by Lemma 3.2 we get

$$\begin{aligned}
 I_2 &= \sup_{r \geq t} |B(x, r)|^{-1} |b(z) - b_{\tilde{B}(x, r)}| \int_{\tilde{B}(x, r)} |f(y)| dy \\
 &\leq \sup_{r \geq t} |B(x, r)|^{-1} |b(z) - b_{\tilde{B}(x, t)}| \int_{\tilde{B}(x, r)} |f(y)| dy \\
 &\quad + \sup_{r \geq t} |B(x, r)|^{-1} |b_{\tilde{B}(x, t)} - b_{\tilde{B}(x, r)}| \int_{\tilde{B}(x, r)} |f(y)| dy \\
 &\leq CM_b \chi_{B(x, t)}(z) \sup_{r \geq t} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, r))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, r))}^{-1} \\
 &\quad + C \|b\|_* \sup_{r \geq t} \ln \frac{r}{t} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, r))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, r))}^{-1}.
 \end{aligned}$$

Then Corollary 3.2 we have

$$\begin{aligned}
 \|M_b f_2\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, t))} &\leq \|I_1\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, t))} + \|I_2\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, t))} \\
 &\leq C \|b\|_* \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))} \sup_{r \geq t} \left(1 + \ln \frac{r}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, r))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, r))}^{-1} \\
 &\quad + C \|M_b \chi_{B(x, t)}\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, t))} \sup_{r \geq t} \left(1 + \ln \frac{r}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, r))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, r))}^{-1} \\
 &\leq C \|b\|_* \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))} \sup_{r \geq t} \left(1 + \ln \frac{r}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, r))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, r))}^{-1}.
 \end{aligned} \tag{3.12}$$

Then from (3.11) and (3.12) we obtain (3.9). \square

THEOREM 3.9. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{\log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$, $b \in BMO(\Omega)$ and the function $\varphi_1(x, r)$ and $\varphi_2(x, r)$ satisfy the condition*

$$\sup_{t > r} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x, s) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, s))}}{\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \leq C \varphi_2(x, r), \tag{3.13}$$

where C does not depend on $x \in \Omega$ and t . Then the operator M_b is bounded from the space $\mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)$ to the space $\mathcal{M}_\omega^{p(\cdot), \varphi_2}(\Omega)$.

Proof. Let $f \in \mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)$. By (3.13), Theorems 3.2 and 3.8 we obtain

$$\begin{aligned} & \|M_b f\|_{\mathcal{M}_\omega^{p(\cdot), \varphi_2}(\Omega)} \\ & \leq C \|b\|_* \sup_{x \in \Omega, t > 0} \frac{\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))}}{\varphi_2(x,t) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))}} \sup_{t > r} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))}^{-1} \\ & \leq C \|b\|_* \sup_{x \in \Omega, t > 0} \frac{1}{\varphi_1(x,t) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))}} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} = C \|b\|_* \|f\|_{\mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)} \end{aligned}$$

which completes the proof. \square

4. Singular integral operators and its commutators in $\mathcal{M}_\omega^{p(\cdot), \varphi}(\Omega)$

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderón [8, 9] studied a kind of commutators, appearing in Cauchy integral problems of Lipschitz curve. Let K be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [10] states that the commutator operator $[b, K]f = K(bf) - bKf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [11], [12], [21], [22], [23]).

The following statement holds (see [19, Lemma 3.5]):

PROPOSITION A. *Let $\Omega \subset \mathbb{R}^n$ be unbounded and $p \in \mathbb{P}_\infty^{log}(\Omega)$. Then for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$ there holds*

$$\left| \int_\Omega f(y)g(y)dy \right| \leq C \int_\Omega M^\sharp f(y)Mg(y)dy$$

with a constant $C > 0$ not depending on f .

LEMMA 4.3. *Let $\Omega \subset \mathbb{R}^n$ be unbounded and $p \in \mathbb{P}_\infty^{log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$. Then*

$$\|f\omega\|_{L^{p(\cdot)}(\Omega)} \leq C \|\omega M^\sharp f\|_{L^{p(\cdot)}(\Omega)}$$

with a constant $C > 0$ not depending on f .

Proof. By (2.3) we have

$$\|f\omega\|_{L^{p(\cdot)}(\Omega)} \leq C \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)} \leq 1} \left| \int_\Omega f(y)g(y)\omega(y)dy \right|.$$

According to Proposition A,

$$\|f\omega\|_{L^{p(\cdot)}(\Omega)} \leq C \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)} \leq 1} \int_\Omega M^\sharp f(y)M(g\omega)(y)dy.$$

By the Hölder inequality and Theorem 2.1, we derive

$$\begin{aligned} \|f\omega\|_{L^{p(\cdot)}(\Omega)} &\leq C \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)} \leq 1} \|\omega M^\sharp f\|_{L^{p(\cdot)}(\Omega)} \|\omega^{-1} M(g\omega)\|_{L^{p'(\cdot)}(\Omega)} \\ &\leq C \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)} \leq 1} \|\omega M^\sharp f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)} \leq C \|\omega M^\sharp f\|_{L^{p(\cdot)}(\Omega)}. \quad \square \end{aligned}$$

PROPOSITION B. ([4, Theorem 2.1]) *Let T be a Calderón-Zygmund operator. Then for arbitrary s , $0 < s < 1$, there exists a constant $C_s > 0$ such that*

$$[(|Tf|^s)^\sharp]^\frac{1}{s}(x) \leq C_s Mf(x)$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{\log}(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$. Then the operators T and T^* are bounded in the space $L_\omega^{p(\cdot)}(\Omega)$.*

Proof. In [46, Theorem 4.2.] was proved that infinitely differentiable functions is dense in $L_\omega^{p(\cdot)}(\Omega)$ with any positive, locally integrable function ω . Then by the Proposition B, Lemma 4.3 and Theorem 2.1, we derive the operator T is bounded in the space $L_\omega^{p(\cdot)}(\Omega)$.

The boundedness of the operator T^* on $L_\omega^{p(\cdot)}(\Omega)$ follows from the known estimate

$$T^*f(x) \lesssim M(Tf)(x) + Mf(x),$$

from Theorem 2.1 and the boundedness of the operator T on $L_\omega^{p(\cdot)}(\Omega)$. \square

The following weighted local estimates are valid.

THEOREM 4.2. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{\log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$ and $f \in L_\omega^{p(\cdot)}(\Omega)$. Then*

$$\|Tf\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq C \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}, \quad (4.1)$$

where C does not depend on f , $x \in \Omega$ and t .

Proof. We represent function f as in (3.3) and have

$$\|Tf\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq \|Tf_1\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} + \|Tf_2\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))}.$$

By the Theorem 4.1 we obtain

$$\|Tf_1\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq \|Tf_1\|_{L_\omega^{p(\cdot)}(\Omega)} \leq C \|f_1\|_{L_\omega^{p(\cdot)}(\Omega)},$$

so that

$$\|Tf_1\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq C \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))}.$$

Taking into account the inequality

$$\|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq C \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s},$$

we get

$$\|Tf_1\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq C \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}. \quad (4.2)$$

To estimate $\|Tf_2\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))}$, we observe that

$$|Tf_2(z)| \leq C \int_{\Omega \setminus B(x,2t)} \frac{|f(y)| dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \leq t$, $|z-y| \geq 2t$ imply $\frac{1}{2}|z-y| \leq |x-y| \leq \frac{3}{2}|z-y|$, and therefore

$$|Tf_2(z)| \leq C \int_{\Omega \setminus \tilde{B}(x,2t)} |x-y|^{-n} |f(y)| dy,$$

To estimate Tf_2 , we first prove the following auxiliary inequality

$$\begin{aligned} & \int_{\Omega \setminus \tilde{B}(x,t)} |x-y|^{-n} |f(y)| dy \\ & \leq Ct^{-n} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^\infty s^{n-1} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}. \end{aligned} \quad (4.3)$$

To this end, we choose $\delta > 0$ and proceed as follows

$$\begin{aligned} \int_{\Omega \setminus \tilde{B}(x,t)} |x-y|^{-n} |f(y)| dy & \leq \delta \int_{\Omega \setminus \tilde{B}(x,t)} |x-y|^{-n+\delta} |f(y)| dy \int_{|x-y|}^\infty s^{-\delta-1} ds \\ & \leq Ct^{-n} \int_t^\infty \frac{ds}{s} \int_{\{y \in \Omega: 2t \leq |x-y| \leq s\}} |f(y)| dy \\ & \leq Ct^{-n} \int_t^\infty \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s} \\ & \leq Ct^{-n} \int_t^\infty s^{n-1} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} ds. \end{aligned} \quad (4.4)$$

Hence by inequality (4.4), we get

$$\begin{aligned} \|Tf_2\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} & \leq C \|\chi_{\tilde{B}(x,t)}\|_{L_\omega^{p(\cdot)}(\Omega)} \int_t^\infty \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s} \\ & = C \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}. \end{aligned} \quad (4.5)$$

From (4.2) and (4.5) we arrive at (4.1). \square

THEOREM 4.3. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_{\infty}^{log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$ and $\varphi_1(x, t)$ and $\varphi_2(x, r)$ fulfill condition*

$$\int_t^{\infty} \frac{\operatorname{ess\,inf}_{s < r < \infty} \varphi_1(x, r) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, r))} ds}{\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, s))}} \frac{ds}{s} \leq C \varphi_2(x, t), \quad (4.6)$$

where C does not depend on $x \in \Omega$ and t . Then the singular integral operators T and T^* are bounded from the space $\mathcal{M}_{\omega}^{p(\cdot), \varphi_1}(\Omega)$ to the space $\mathcal{M}_{\omega}^{p(\cdot), \varphi_2}(\Omega)$.

Proof. Let $f \in \mathcal{M}_{\omega}^{p(\cdot), \varphi_1}(\Omega)$. As usual, when estimating the norm

$$\|Tf\|_{\mathcal{M}_{\omega}^{p(\cdot), \varphi_2}(\Omega)} = \sup_{x \in \Omega, t > 0} \frac{1}{\varphi_2(x, t) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \|Tf \chi_{\tilde{B}(x, t)}\|_{L_{\omega}^{p(\cdot)}(\Omega)}. \quad (4.7)$$

We estimate $\|Tf \chi_{\tilde{B}(x, t)}\|_{L_{\omega}^{p(\cdot)}(\Omega)}$ in (4.7) by means of Theorem 4.2 and obtain

$$\begin{aligned} \|Tf\|_{\mathcal{M}_{\omega}^{p(\cdot), \varphi_2}(\Omega)} &\leq C \sup_{x \in \Omega, t > 0} \frac{\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))}}{\varphi_2(x, t) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \int_t^{\infty} s^{-1} \|f\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x, s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, s))}^{-1} ds \\ &\leq C \sup_{x \in \Omega, t > 0} \frac{1}{\varphi_1(x, t) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \|f\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x, t))} = C \|f\|_{\mathcal{M}_{\omega}^{p(\cdot), \varphi_1}(\Omega)}. \end{aligned}$$

It remains to make use of condition (4.6). \square

LEMMA 4.4. ([21, Lemma 1]) *Let $1 < s < \infty$, $b \in BMO(\mathbb{R}^n)$, then there exists $C > 0$ such that for all $x \in \mathbb{R}^n$, the following inequality holds*

$$M^{\sharp}([b, T]f)(x) \leq C \|b\|_* \left((M|Tf|^s)^{\frac{1}{s}}(x) + (M|f|^s)^{\frac{1}{s}}(x) \right).$$

THEOREM 4.4. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $b \in BMO(\Omega)$, $p \in \mathbb{P}_{\infty}^{log}(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$. Then the commutator operator $[b, T]$ is bounded on the space $L_{\omega}^{p(\cdot)}(\Omega)$.*

Proof. By Lemma 4.4, Lemma 4.3, Theorem 2.1 and Theorem 4.1, we derive the operator $[b, T]$ is bounded in the space $L_{\omega}^{p(\cdot)}(\Omega)$. \square

In the case $\omega \equiv 1$, we have the following corollary, which proved in [43].

COROLLARY 4.3. ([43, Theorem 1.1]) *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $b \in BMO(\Omega)$ and $p \in \mathbb{P}_{\infty}^{log}(\Omega)$. Then the operator $[b, T]$ is bounded on the space $L^{p(\cdot)}(\Omega)$.*

The following weighted local estimates are valid.

THEOREM 4.5. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{\log}(\Omega)$, $b \in BMO(\Omega)$ and $\omega \in A_{p(\cdot)}(\Omega)$. Then*

$$\begin{aligned} & \| [b, T]f \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \\ & \leq C \|b\|_* \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s} \end{aligned} \quad (4.8)$$

for every $f \in L_\omega^{p(\cdot)}(\Omega)$, where C does not depend on $f, x \in \Omega$ and t .

Proof. We represent function f as in (3.3) and have

$$\| [b, T]f \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \leq \| [b, T]f_1 \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} + \| [b, T]f_2 \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))}.$$

By Theorem 4.4 we obtain

$$\begin{aligned} \| [b, T]f_1 \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} & \leq \| [b, T]f_1 \|_{L_\omega^{p(\cdot)}(\Omega)} \leq C \|b\|_* \|f_1\|_{L_\omega^{p(\cdot)}(\Omega)} \\ & = C \|b\|_* \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))}, \end{aligned} \quad (4.9)$$

where C does not depend on f . From (4.9) we obtain

$$\begin{aligned} & \| [b, T]f_1 \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))} \\ & \leq C \|b\|_* \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s} \end{aligned} \quad (4.10)$$

easily obtained from the fact that $\|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))}$ is non-decreasing in t , so that $\|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,2t))}$ on the right-hand side of (4.9) is dominated by the right-hand side of (4.10). To estimate $\| [b, T]f_2 \|_{L_\omega^{p(\cdot)}(\tilde{B}(x,t))}$, we observe that

$$| [b, T]f_2(z) | \leq C \int_{\Omega \setminus B(x,2t)} |b(z) - b(y)| \frac{|f(y)| dy}{|y - z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x - z| \leq t$, $|z - y| \geq 2t$ imply $\frac{1}{2}|z - y| \leq |x - y| \leq \frac{3}{2}|z - y|$, and therefore

$$| [b, T]f_2(z) | \leq C \int_{\Omega \setminus \tilde{B}(x,2t)} |x - y|^{-n} |b(z) - b(y)| |f(y)| dy.$$

To estimate $[b, T]f_2$, we first prove the following auxiliary inequality

$$\begin{aligned} & \int_{\Omega \setminus \tilde{B}(x,t)} |x - y|^{-n} |b(z) - b(y)| |f(y)| dy \\ & \leq C \|b\|_* \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x,s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,s))}^{-1} \frac{ds}{s}. \end{aligned} \quad (4.11)$$

To estimate $[b, T]f_2(z)$, we observe that for $z \in \tilde{B}(x, t)$ we have

$$\begin{aligned} & \int_{\Omega \setminus \tilde{B}(x, t)} |x-y|^{-n} |b(z) - b(y)| |f(y)| dy \\ & \leq \int_{\Omega \setminus \tilde{B}(x, t)} |x-y|^{-n} |b(y) - b_{\tilde{B}(x, t)}| |f(y)| dy \\ & \quad + \int_{\Omega \setminus \tilde{B}(x, t)} |x-y|^{-n} |b(z) - b_{\tilde{B}(x, t)}| |f(y)| dy = J_1 + J_2. \end{aligned}$$

To this end, we choose $\delta > 0$, by Theorem 2.3 and Lemma 3.2 we obtain

$$\begin{aligned} J_1 &= \int_{\Omega \setminus \tilde{B}(x, t)} |x-y|^{-n} |b(y) - b_{\tilde{B}(x, t)}| |f(y)| dy \\ &\leq \delta \int_{\Omega \setminus \tilde{B}(x, t)} |x-y|^{-n+\delta} |b(y) - b_{\tilde{B}(x, t)}| |f(y)| dy \int_{|x-y|}^{\infty} s^{-\delta-1} ds \\ &\leq C \int_t^{\infty} s^{-n-1} \int_{\{y \in \Omega: 2t \leq |x-y| \leq s\}} |b(y) - b_{\tilde{B}(x, t)}| |f(y)| dy ds \\ &\leq C \int_t^{\infty} s^{-n-1} \|b(\cdot) - b_{\tilde{B}(x, s)}\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x, s))} \|f\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x, s))} ds \\ &\quad + C \int_t^{\infty} s^{-n-1} |b_{\tilde{B}(x, t)} - b_{\tilde{B}(x, s)}| \int_{\tilde{B}(x, s)} |f(y)| dy ds \\ &\leq C \|b\|_* \int_t^{\infty} s^{-n-1} \|\omega^{-1}\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \|f\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x, s))} ds \\ &\quad + C \|b\|_* \int_t^{\infty} s^{-n-1} \ln \frac{s}{t} \|\omega^{-1}\|_{L^{p(\cdot)}(\tilde{B}(x, s))} \|f\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x, s))} ds \\ &\leq C \|b\|_* \int_t^{\infty} \left(1 + \ln \frac{s}{t}\right) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, s))}^{-1} \|f\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x, s))} \frac{ds}{s}. \end{aligned}$$

To estimate J_2 , by (4.3), we have

$$\begin{aligned} J_2 &= |b(z) - b_{\tilde{B}(x, t)}| \int_{\Omega \setminus \tilde{B}(x, t)} |x-y|^{-n} |f(y)| dy \\ &\leq C |B(x, t)|^{-1} \int_{\tilde{B}(x, t)} |b(z) - b(y)| dy \int_t^{\infty} \|f\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x, s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, s))}^{-1} \frac{ds}{s} \\ &\leq CM_b \chi_{B(x, t)}(z) \int_t^{\infty} \|f\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x, s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, s))}^{-1} \frac{ds}{s}, \end{aligned}$$

where C does not depend on x, t .

Hence by inequality (4.11), we get

$$\begin{aligned} & \| [b, T]f_2 \|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x, t))} \\ & \lesssim \|\chi_{\tilde{B}(x, t)}\|_{L_{\omega}^{p(\cdot)}(\Omega)} \|b\|_* \int_t^{\infty} \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x, s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, s))}^{-1} \frac{ds}{s} \\ & = \|b\|_* \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))} \int_t^{\infty} \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x, s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, s))}^{-1} \frac{ds}{s}. \end{aligned} \quad (4.12)$$

From (4.10) and (4.12) we arrive at (4.8). \square

THEOREM 4.6. *Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}_\infty^{\log}(\Omega)$, $\omega \in A_{p(\cdot)}(\Omega)$, $b \in BMO(\Omega)$ and the functions $\varphi_1(x, r)$ and $\varphi_2(x, r)$ satisfy the condition (3.13). Then the operator $[b, T]$ is bounded from the space $\mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)$ to the space $\mathcal{M}_\omega^{p(\cdot), \varphi_2}(\Omega)$.*

Proof. Let $f \in \mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)$. We have

$$\|[b, T]f\|_{\mathcal{M}_\omega^{p(\cdot), \varphi_2}(\Omega)} = \sup_{x \in \Omega, t > 0} \frac{1}{\varphi_2(x, t) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \|[b, T]f\|_{L^{p(\cdot)}(\tilde{B}(x, t))}.$$

By (3.13), Theorems 3.2 and 4.5 we obtain

$$\begin{aligned} & \|[b, T]f\|_{\mathcal{M}_\omega^{p(\cdot), \varphi_2}(\Omega)} \\ & \leq C \|b\|_* \sup_{x \in \Omega, t > 0} \frac{\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))}}{\varphi_2(x, t) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, s))} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, s))}^{-1} \frac{ds}{s} \\ & \leq C \|b\|_* \sup_{x \in \Omega, t > 0} \frac{1}{\varphi_1(x, t) \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x, t))}} \|f\|_{L_\omega^{p(\cdot)}(\tilde{B}(x, t))} = C \|b\|_* \|f\|_{\mathcal{M}_\omega^{p(\cdot), \varphi_1}(\Omega)} \end{aligned}$$

which completes the proof. \square

Acknowledgement. The research of V. S. Guliyev is partially supported by the Ministry of Education and Science of the Russian Federation (the Agreement number: 02.a03.21.0008).

REFERENCES

- [1] D. R. ADAMS, *A note on Riesz potentials*, Duke Math. **42** (1975), 765–778.
- [2] M. AGCAYAZI, A. GOGATISHVILI, K. KOCA, AND R. MUSTAFAYEV, *A note on maximal commutators and commutators of maximal functions*, J. Math. Soc. Japan. **67**, 2 (2015), 581–593.
- [3] A. ALMEIDA, J. J. HASANOV, S. G. SAMKO, *Maximal and potential operators in variable exponent Morrey spaces*, Georgian Math. J. **15**, 2 (2008), 195–208.
- [4] J. ALVAREZ AND C. PÉREZ, *Estimates with A_∞ weights for various singular integral operators*, Boll. Un. Mat. Ital. A (7) **8**, 1 (1994), 123–133.
- [5] V. BURENKOV, A. GOGATISHVILI, V. S. GULIYEV, R. MUSTAFAYEV, *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Complex Var. Elliptic Equ. **55**, 8–10 (2010), 739–758.
- [6] V. I. BURENKOV, H. V. GULIYEV, *Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces*, Studia Math. **163**, 2 (2004), 157–176.
- [7] V. I. BURENKOV, V. S. GULIYEV, A. SERBETCI AND T. V. TARARYKOVA, *Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces*, Dokl. Akad. Nauk **422**, 1 (2008), 11–14.
- [8] A. P. CALDERÓN, *Commutators of singular integral operators*, Proc. Natl. Acad. Sci. USA **53** (1965), 1092–1099.
- [9] A. P. CALDERÓN, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Natl. Acad. Sci. USA **74**, 4 (1977), 1324–1327.
- [10] R. COIFMAN, R. ROCHBERG, G. WEISS, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103**, 2 (1976), 611–635.
- [11] F. CHIARENZA, M. FRASCA, P. LONGO, *$W^{2,p}$ -solvability of Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. **336** (1993), 841–853.
- [12] Y. CHEN, *Regularity of solutions to elliptic equations with VMO coefficients*, Acta Math. Sin. (Engl. Ser.) **20** (2004), 1103–1118.

- [13] D. CRUZ-URIBE, A. FIORENZA, AND C. J. NEUGEBAUER, *The maximal function on variable L_p spaces*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238.
- [14] D. CRUZ-URIBE, A. FIORENZA, J. M. MARTELL, C. PEREZ, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 239–264.
- [15] D. CRUZ-URIBE, A. FIORENZA, *Variable Lebesgue spaces: Foundations and harmonic analysis*, Birkhauser/Springer, 2013. MR 3026953.
- [16] F. CHIARENZA, M. FRASCA, *Morrey spaces and Hardy–Littlewood maximal function*, Rend. Math. **7** (1987), 273–279.
- [17] L. DIENING, *Maximal functions on generalized Lebesgue spaces $L^{p(\cdot)}$* , Math. Inequal. Appl. **7**, 2 (2004), 245–253.
- [18] L. DIENING, P. HARJULEHTO, HÄSTÖ, AND M. RUŽIČKA, *Lebesgue and Sobolev spaces with variable exponents*, Springer-Verlag, Lecture Notes in Mathematics, vol. 2017, Berlin, 2011.
- [19] L. DIENING AND M. RŮŽIČKA, *Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics*, J. Reine Angew. Math. **563** (2003), 197–220.
- [20] L. DIENING, P. HASTO AND A. NEKVINDA, *Open problems in variable exponent Lebesgue and Sobolev spaces*, “Function Spaces, Differential Operators and Nonlinear Analysis”, Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28 – June 2, 2004, Math. Inst. Acad. Sci. Czech Republic, Praha, 2005, 38–58.
- [21] G. DI FAZIO AND M. A. RAGUSA, *Commutators and Morrey spaces*, Boll. Un. Mat. Ital. A (7) **5**, 3 (1991), 323–332.
- [22] G. DI FAZIO AND M. A. RAGUSA, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. **112** (1993), 241–256.
- [23] D. FAN, S. LU AND D. YANG, *Boundedness of operators in Morrey spaces on homogeneous spaces and its applications*, Acta Math. Sinica (N. S.) **14** (1998), 625–634.
- [24] J. GARCIA-CUERVA, E. HARBOURE, C. SEGOVIA, J. L. TORREA, *Weighted norm inequalities for commutators of strongly singular integrals*, Indiana Univ. Math. J. **40**, 4 (1991), 1397–1420.
- [25] V. S. GULIYEV, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. Art. ID 503948, (2009), 20 pp.
- [26] V. S. GULIYEV, *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J. **3**, 3 (2012), 33–61.
- [27] V. S. GULIYEV, *Local generalized Morrey spaces and singular integrals with rough kernel*, Azerb. J. Math. **3**, 2 (2013), 79–94.
- [28] V. S. GULIYEV, *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci. (N. Y.) **193**, 2 (2013), 211–227.
- [29] V. S. GULIYEV, J. J. HASANOV, S. G. SAMKO, *Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces*, Math. Scand. **107**, (2010), 285–304.
- [30] V. S. GULIYEV, J. J. HASANOV, S. G. SAMKO, *Boundedness of the maximal, potential and singular integral operators in the generalized variable exponent Morrey type spaces $\mathcal{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$* , J. Math. Sci. (N. Y.) **170**, 4 (2010), 423–443.
- [31] V. S. GULIEV, J. J. HASANOV, S. G. SAMKO, *Maximal, potential and singular operators in the local “complementary” variable exponent Morrey type spaces*, J. Math. Anal. Appl. **401**, 1 (2013), 72–84.
- [32] V. S. GULIEV, S. G. SAMKO, *Maximal, potential and singular operators in the generalized variable exponent Morrey spaces on unbounded sets*, J. Math. Sci. (N. Y.) **193**, 2 (2013), 228–248.
- [33] V. S. GULIYEV, T. KARAMAN, R. CH. MUSTAFAYEV AND A. SERBETCI, *Commutators of sublinear operators generated by Calderón-Zygmund operator on generalized weighted Morrey spaces*, Czechoslovak Math. J. **64** (139), 2 (2014), 365–386.
- [34] S. JANSON, *Mean oscillation and commutators of singular integral operators*, Ark. Mat. **16** (1978), 263–270.
- [35] P. HÄSTÖ, *Local-to-global results in variable exponent spaces*, Math. Res. Lett. **16**, 2 (2009), 263–278.
- [36] P. HÄSTÖ, L. DIENING, *Muckenhoupt weights in variable exponent spaces*, preprint, <http://www.helsinki.fi/pharjule/varsob/publications.shtml>.
- [37] J. J. HASANOV, *Hardy-Littlewood-Stein-Weiss inequality in the variable exponent Morrey spaces*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **39** (2013), 47–62.
- [38] J. J. HASANOV, *Hardy-Littlewood-Stein-Weiss Inequality in the generalized Morrey spaces with variable exponent*, Caspian Journal of Applied Mathematics, Ecology and Economics **2**, 1 (2014), 38–59.

- [39] K.-P. HO, *Vector-valued singular integral operators on Morrey type spaces and variable Triebel-Lizorkin-Morrey spaces*, Ann. Acad. Sci. Fenn. Math. **37** (2012), 375–406.
- [40] K.-P. HO, *Vector-valued operators with singular kernel and Triebel-Lizorkin-block spaces with variable exponents*, Kyoto J. Math. **56** (2016), 97–124.
- [41] KWOK-PUN HO, *Singular integral operators, John-Nirenberg inequalities and Tribel-Lizorkin type spaces on weighted Lebesgue spaces with variable exponents*, Revista De La Union Matematica Argentina **57**, 1 (2016), 85–101.
- [42] T. KARAMAN, V. S. GULIYEV AND A. SERBETCI, *Boundedness of sublinear operators generated by Calderón-Zygmund operators on generalized weighted Morrey spaces*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N. S.) **60**, 1 (2014), 227–244.
- [43] A. KARLOVICH AND A. LERNER, *Commutators of singular integrals on generalized L^p spaces with variable exponent*, Publ. Mat. **49**, 1 (2005), 111–125.
- [44] V. KOKILASHVILI, *On a progress in the theory of integral operators in weighted Banach function spaces*, “Function Spaces, Differential Operators and Nonlinear Analysis”, Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28 – June 2, 2004, Math. Inst. Acad. Sci. Czech Republic, Praha, 2005, 152–175.
- [45] V. KOKILASHVILI AND A. MESKHI, *Boundedness of maximal and singular operators in Morrey spaces with variable exponent*, Arm. J. Math. (Electronic) **1**, (1) (2008), 18–28.
- [46] V. KOKILASHVILI AND S. SAMKO, *Singular integrals in weighted Lebesgue spaces with variable exponent*, Georgian Math. J. **10**, (1) (2003), 145–156.
- [47] V. KOKILASHVILI AND S. SAMKO, *Weighted boundedness of the maximal, singular and potential operators in variable exponent spaces*, Analytic Methods of Analysis and Differential Equations, Cambridge Scientific Publishers, Eds.: A. A. Kilbas and S. V. Rogosin, 139–164, 2008.
- [48] Y. KOMORI AND S. SHIRAI, *Weighted Morrey spaces and a singular integral operator*, Math. Nachr. **282**, 2 (2009), 219–231.
- [49] O. KOVACIK AND J. RAKOSNIK, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$* , Czechoslovak Math. J. **41** (116), 4 (1991), 592–618.
- [50] T. S. KOPALIANI, *Infimal convolution and Muckenhoupt $A_{p(\cdot)}$ condition in variable L^p spaces*, Arch. Math. **89**, 2 (2007), 185–192.
- [51] K. KURATA, S. NISHIGAKI AND S. SUGANO, *Boundedness of integral operators on generalized Morrey spaces and its application to Schrödinger operators*, Proc. AMS **128**, 4 (1999), 1125–1134.
- [52] D. LI, G. HU, X. SHI, *Weighted norm inequalities for the maximal commutators of singular integral operators*, J. Math. Anal. Appl. **319**, 2 (2006), 509–521.
- [53] C. B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.
- [54] T. MIZUHARA, *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer-Verlag, Tokyo (1991), 183–189.
- [55] Y. MIZUTA AND T. SHIMOMURA, *Sobolev embeddings for Riesz potentials of functions in Morrey spaces of variable exponent*, J. Math. Japan **60** (2008), 583–602.
- [56] Y. MIZUTA AND T. SHIMOMURA, *Weighted Morrey spaces of variable exponent and Riesz potentials*, Math. Nachr. **288**, 8–9 (2015), 984–1002.
- [57] E. NAKAI, *Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166** (1994), 95–103.
- [58] E. NAKAI, *Generalized fractional integrals on generalized Morrey spaces*, Math. Nachr. **287** (2014), 339–351.
- [59] J. PEETRE, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces*, J. Funct. Anal. **4** (1969), 71–87.
- [60] S. SAMKO, *On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators*, Integral Transform. Spec. Funct. **16**, 5–6 (2005), 461–482.
- [61] S. G. SAMKO, *Differentiation and integration of variable order and the spaces $L^{p(x)}$* , Proceed. of Intern. Conference “Operator Theory and Complex and Hypercomplex Analysis”, 12–17 December 1994, Mexico City, Mexico, Contemp. Math. **212** (1998), 203–219.
- [62] Y. SAWANO, D. HAKIM, H. GUNAWAN, *Non-smooth atomic decomposition for generalized Orlicz-Morrey spaces*, Math. Nachr. **288** (2015), 1741–1775.

- [63] I. I. SHARAPUDINOV, *The topology of the space $\mathcal{L}^{p(t)}([0, 1])$* , *Mat. Zametki* **26**, 3–4 (1979), 613–632.
- [64] PU ZHANG, JIANGLONG WU, *Commutators of the fractional maximal function on variable exponent Lebesgue spaces*, *Czechoslovak Math. J.* **64** (139), 1 (2014), 183–197.

(Received March 1, 2017)

Vagif S. Guliyev

Ahi Evran University

Department of Mathematics

40100 Kirsehir, Turkey

and

S. M. Nikol'skii Institute of Mathematics at RUDN University

6 Miklukho-Maklay St, Moscow, 117198

and

Institute of Mathematics and Mechanics

Az 1141, B. Vahabzadeh str. 9, Baku, Azerbaijan

e-mail: vagif@guliyev.com

Javanshir J. Hasanov

Azerbaijan State Oil and Industry University

Azadlig av. 20, AZ 1601, Baku, Azerbaijan

and

Institute of Mathematics and Mechanics

Az 1141, B. Vahabzadeh str. 9, Baku, Azerbaijan

e-mail: hasanovjavanshir@yahoo.com.tr,

hasanov.javanshir@math.ab.az

Xayyam A. Badalov

Institute of Mathematics and Mechanics

Az 1141, B. Vahabzadeh str. 9, Baku, Azerbaijan

e-mail: xayyambadalov@gmail.com