

A GENERALIZATION OF YOUNG–TYPE INEQUALITIES

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Abstract. In this paper, we prove a simple but useful result and apply it to give a generalization of Young-type inequalities developed by many researchers. Applications to positive definite matrices will be also provided.

1. Introduction

The classical Young's inequality states that if $a, b > 0$ and $0 \leq v \leq 1$, then

$$(1 - v)a + vb \geq a^{1-v}b^v. \quad (1)$$

There are many refinements of Young's inequality and its reverse. Kittaneh and Manasrah [6, 7] improved (1) as follows:

$$(1 - v)a + vb \geq a^{1-v}b^v + r_0(\sqrt{a} - \sqrt{b})^2, \quad (2)$$

$$(1 - v)a + vb \leq a^{1-v}b^v + R_0(\sqrt{a} - \sqrt{b})^2, \quad (3)$$

where $r_0 = \min\{v, 1 - v\}$ and $R_0 = \max\{v, 1 - v\}$ whose notations will be used throughout this paper. Hirzallah and Kittaneh [5] and He and Zou [4], respectively, obtained other refinements:

$$((1 - v)a + vb)^2 \geq (a^{1-v}b^v)^2 + r_0^2(a - b)^2, \quad (4)$$

$$((1 - v)a + vb)^2 \leq (a^{1-v}b^v)^2 + R_0^2(a - b)^2. \quad (5)$$

Recently, Manasrah and Kittaneh refined Young's inequality as

$$((1 - v)a + vb)^m \geq (a^{1-v}b^v)^m + r_0^m(a^{m/2} - b^{m/2})^2 \quad (6)$$

for any positive integer m . We remark that (2) and (4) are the cases $m = 1, 2$ of (6), respectively.

This paper is motivated by the following observation:

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- (2) and (4) hold equality when $v = \frac{1}{2}$, but (6) which gives

$$(a + b)^m \geq a^m + b^m + (2^m - 2)(ab)^{m/2}$$

when $v = \frac{1}{2}$ does not satisfy such a property for $m > 2$.

- Considering the form of (3) and (5), we may expect that

$$((1 - v)a + vb)^m \leq (a^{1-v}b^v)^m + R_0^m(a^{m/2} - b^{m/2})^2$$

holds for any positive integer m . However, as numerical examples show, the above does not hold for $m > 2$.

In this paper, we will present a generalization of (2)–(5), which solves the problems stated above. As applications, we will give some inequalities involving positive definite matrices. In the last section, we will generalize other Young-type inequalities. For readers interested in recent papers closely connected to this paper, we refer to [1, 3].

2. Generalized Young’s inequalities

Our main result is the following theorem which will be proved later in this section.

THEOREM 1. *If $a, b > 0$ and $0 \leq v \leq 1$, then for any positive integer m , we have*

$$((1 - v)a + vb)^m \geq (a^{1-v}b^v)^m + (2r_0)^m \left(\left(\frac{a+b}{2} \right)^m - \sqrt{ab}^m \right), \tag{7}$$

$$((1 - v)a + vb)^m \leq (a^{1-v}b^v)^m + (2R_0)^m \left(\left(\frac{a+b}{2} \right)^m - \sqrt{ab}^m \right), \tag{8}$$

where $r_0 = \min\{v, 1 - v\}$ and $R_0 = \max\{v, 1 - v\}$.

Before we give a proof of the theorem, we observe the following obvious properties of the result:

1. (2)–(5) are the cases $m = 1, 2$ of the theorem.
2. Equality holds in (7) and (8) when $v = \frac{1}{2}$.

We start with a simple, but useful result in studying Young-type inequalities.

PROPOSITION 1. *Let a, b, c, d be positive numbers with $a > c$ and $b > d$, and let $\alpha_m = \left(\frac{a^m - c^m}{b^m - d^m} \right)^{1/m}$ for $m = 1, 2, \dots$. Then,*

1. $\alpha_m \rightarrow \frac{a}{b}$ as $m \rightarrow \infty$.
2. $\{\alpha_m\}$ is decreasing if $ad \geq bc$ and increasing if $ad \leq bc$.

Proof. Since $\alpha_m = \frac{a}{b} \left(\frac{1-(c/a)^m}{1-(d/b)^m} \right)^{1/m}$, $c/a < 1$, and $d/b < 1$, it is obvious that $\alpha_m \rightarrow \frac{a}{b}$ as $m \rightarrow \infty$. Next, we show monotonicity of $\{\alpha_m\}$. Since $\alpha_m = \frac{c}{d} \left(\frac{(a/c)^m - 1}{(b/d)^m - 1} \right)^{1/m}$, we may assume $c = d = 1$ and show that $\alpha_m = \left(\frac{a^m - 1}{b^m - 1} \right)^{1/m}$ is decreasing if $a \geq b > 1$ and increasing if $b \geq a > 1$. Moreover, it suffices only to show the decreasing case, since

$$\left(\frac{a^m - 1}{b^m - 1} \right)^{1/m} = 1 / \left(\frac{b^m - 1}{a^m - 1} \right)^{1/m}.$$

Simple algebra shows that $\alpha_m \geq \alpha_{m+1}$ is equivalent to

$$\frac{(b^{m+1} - 1)^m}{(b^m - 1)^{m+1}} \geq \frac{(a^{m+1} - 1)^m}{(a^m - 1)^{m+1}}. \tag{9}$$

For a fixed positive integer m , let $f(t) = \frac{(t^{m+1} - 1)^m}{(t^m - 1)^{m+1}}$ for $t > 1$. Since

$$f'(t) = \frac{-m(m+1)(t-1)t^{m-1}(t^{m+1} - 1)^{m-1}}{(t^m - 1)^{m+2}} < 0,$$

f is decreasing on $(1, \infty)$. Thus if $a \geq b > 1$, then (9) holds. \square

COROLLARY 1. For $0 < v < 1$ and $a, b > 0$ with $a \neq b$, let $\mu_v = (1 - v)a + vb$ and $\delta_v = a^{1-v}b^v$. For convenience, we denote $\mu_{1/2}$ and $\delta_{1/2}$ by μ and δ , respectively. Then for all positive integers m , if $\frac{\mu_v}{\mu} \geq \frac{\delta_v}{\delta}$, then

$$\frac{\mu_v}{\mu} \leq \left(\frac{\mu_v^m - \delta_v^m}{\mu^m - \delta^m} \right)^{1/m} \leq \frac{\mu_v - \delta_v}{\mu - \delta}$$

and if $\frac{\mu_v}{\mu} \leq \frac{\delta_v}{\delta}$, then

$$\frac{\mu_v - \delta_v}{\mu - \delta} \leq \left(\frac{\mu_v^m - \delta_v^m}{\mu^m - \delta^m} \right)^{1/m} \leq \frac{\mu_v}{\mu}.$$

Equivalently, the following holds for all positive integers m :

$$\min \left\{ \frac{\mu_v}{\mu}, \frac{\mu_v - \delta_v}{\mu - \delta} \right\} \leq \left(\frac{\mu_v^m - \delta_v^m}{\mu^m - \delta^m} \right)^{1/m} \leq \max \left\{ \frac{\mu_v}{\mu}, \frac{\mu_v - \delta_v}{\mu - \delta} \right\}.$$

Proof. Let $\alpha_m = \left(\frac{\mu_v^m - \delta_v^m}{\mu^m - \delta^m} \right)^{1/m}$. Since $\mu_v > \delta_v$ for all $0 < v < 1$, the desired result follows from Proposition 1. \square

Now we prove Theorem 1 as follows.

Proof of Theorem 1. Under the same notation of Corollary 1, the inequalities in Theorem 1 can be written as

$$\begin{aligned} \mu_v^m &\geq \delta_v^m + (2r_0)^m (\mu^m - \delta^m), \\ \mu_v^m &\leq \delta_v^m + (2R_0)^m (\mu^m - \delta^m). \end{aligned}$$

Since they obviously hold when $v = 0$, $v = 1$, or $a = b$, we assume $0 < v < 1$ and $a \neq b$. Then the above inequalities are equivalent to

$$2r_0 \leq \left(\frac{\mu_v^m - \delta_v^m}{\mu^m - \delta^m} \right)^{1/m} \leq 2R_0. \quad (10)$$

Since $\left(\frac{\mu_v^m - \delta_v^m}{\mu^m - \delta^m} \right)^{1/m}$ is between $\frac{\mu_v}{\mu}$ and $\frac{\mu_v - \delta_v}{\mu - \delta}$ by Corollary 1, it suffices to show that $\frac{\mu_v}{\mu}$ and $\frac{\mu_v - \delta_v}{\mu - \delta}$ are between $2r_0$ and $2R_0$. Simple algebra shows that

$$\begin{aligned} 2r_0 &\leq \frac{\mu_v}{\mu} \leq 2R_0 \\ \iff (a+b)r_0 &\leq (1-v)a + vb \leq (a+b)R_0 \\ \iff (1-v-r_0)a + (v-r_0)b &\geq 0, \\ &\text{and } (R_0-1+v)a + (R_0-v)b \geq 0. \end{aligned}$$

Since $r_0 \leq v$ and $1-v \leq R_0$, the above holds for all v . Meanwhile, (10) holds for $m = 1$ by (2) and (3). \square

REMARK 1. By direct computation, we can show that (7) improves (6). Since

$$\begin{aligned} (a^{1-v}b^v)^m + (2r_0)^m \left(\left(\frac{a+b}{2} \right)^m - \sqrt{ab^m} \right) &\geq (a^{1-v}b^v)^m + r_0^m (a^{m/2} - b^{m/2})^2 \\ \iff (a+b)^m - (2\sqrt{ab})^m &\geq a^m + b^m - 2\sqrt{ab^m}, \end{aligned}$$

we will show $f_m(x) \geq 0$ for $x > 0$ and $m = 1, 2, \dots$, where

$$f_m(x) = (1+x^2)^m - 2^m x^m - 1 - x^{2m} + 2x^m.$$

Moreover, since $x^{2m} f_m(x^{-1}) = f_m(x)$, we may assume $x \geq 1$. We use induction on m . Clearly, $f_1 \equiv 0$. Suppose $f_m(x) \geq 0$ for all $x \geq 1$. Then, since

$$(1+x^2)^m \geq 2^m x^m + 1 + x^{2m} - 2x^m,$$

we have

$$\begin{aligned} f'_{m+1}(x) &= 2(m+1) \{x(1+x^2)^m - 2^m x^m - x^{2m+1} + x^m\} \\ &\geq 2(m+1) \{x(2^m x^m + 1 + x^{2m} - 2x^m) - 2^m x^m - x^{2m+1} + x^m\} \\ &= 2(m+1)x \{2^m x^m + 1 - 2x^m - 2^m x^{m-1} + x^{m-1}\}. \end{aligned}$$

Let $g_m(x) = 2^m x^m + 1 - 2x^m - 2^m x^{m-1} + x^{m-1}$. Since $g_m(1) = 0$ and

$$\begin{aligned} g'_m(x) &= x^{m-2} \{m(2^m - 2)x - (m-1)(2^m - 1)\} \\ &\geq x^{m-2} \{m(2^m - 2) - (m-1)(2^m - 1)\} \\ &= x^{m-2} (2^m - m - 1) \\ &\geq 0, \end{aligned}$$

$f'_{m+1}(x) \geq 0$ for $x \geq 1$. Thus $f_{m+1} \geq 0$ follows from $f_{m+1}(1) = 0$.

3. Applications

In this section, we show some applications of Theorem 1 to matrix inequalities. Let M_n be the space of $n \times n$ complex matrices. For $A \in M_n$, $A > 0$ denotes that A is positive definite, *i.e.*, $x^*Ax > 0$ for all nonzero $x \in \mathbb{C}^n$. For $A, B > 0$ and $0 \leq v \leq 1$, the v -arithmetic mean of A and B is defined by $A\nabla_v B = (1-v)A + vB$. Similarly, we define the t -weighted geometric mean of two positive definite matrices $A, B \in M_n$ by $A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ for any real number t . For convenience, we denote $A\nabla_{\frac{1}{2}} B$ by $A\nabla B$.

Using (6), Manasrah and Kittaneh [8] improved the inequalities

$$\begin{aligned} [\operatorname{tr}(A\nabla_v B)]^m &\geq (\operatorname{tr}|A^{1-v}B^v|)^m, \\ [\det(A\nabla_v B)]^m &\geq \det(A^{1-v}B^v)^m, \\ [(1-v)\|AX\| + v\|XB\|]^m &\geq \|A^{1-v}XB^v\|^m \end{aligned}$$

for $A, B, X \in M_n$ with $A, B > 0$ and $0 \leq v \leq 1$, where $|X| = (X^*X)^{1/2}$ and $\|\cdot\|$ stands for any unitarily invariant norm on M_n . Since (7) is stronger than (6), their results can be further improved. We state the following without proof. See [8] for detailed proofs.

THEOREM 2. *Let $A, B \in M_n$ be positive definite. Then for $0 \leq v \leq 1$ and $m = 1, 2, \dots$, we have*

1. $[\operatorname{tr}(A\nabla_v B)]^m \geq (\operatorname{tr}|A^{1-v}B^v|)^m + (2r_0)^m \left((\operatorname{tr}A\nabla B)^m - \sqrt{\operatorname{tr}A\operatorname{tr}B^m} \right),$
2. $[\det(A\nabla_v B)]^m \geq \det(A^{1-v}B^v)^m + (2r_0)^{nm} \left(\left(\frac{\det A^{1/n} + \det B^{1/n}}{2} \right)^{nm} - \sqrt{\det(AB)^m} \right),$
3. $[(1-v)\|AX\| + v\|XB\|]^m \geq \|A^{1-v}XB^v\|^m + (2r_0)^m \left(\left(\frac{\|AX\| + \|XB\|}{2} \right)^m - \sqrt{\|AX\|\|XB\|^m} \right)$

for all $X \in M_n$.

To show matrix (or operator) inequalities corresponding to their scalar versions, we can use the operator monotonicity of continuous functions; that is, if f is a real valued continuous function defined on the spectrum of a self-adjoint operator A , then $f(t) \geq 0$ for every t in the spectrum of A implies $f(A)$ is a positive operator. The following follows from the property and Theorem 1.

THEOREM 3. *Let $A, B \in M_n$ be positive definite. Then for $0 \leq v \leq 1$ and $m = 1, 2, \dots$, we have*

$$A\#_m(A\nabla_v B) \geq A\#_{vm}B + (2r_0)^m (A\#_m(A\nabla B) - A\#_{m/2}B), \tag{11}$$

$$A\#_m(A\nabla_v B) \leq A\#_{vm}B + (2R_0)^m (A\#_m(A\nabla B) - A\#_{m/2}B). \tag{12}$$

Proof. For any positive definite matrix $X \in M_n$, it follows from Theorem 1 with $a = 1$ that

$$((1-v)I + vX)^m \geq X^{vm} + (2r_0)^m \left(\left(\frac{I+X}{2} \right)^m - X^{m/2} \right),$$

where I is the identity matrix in M_n . Letting $X = A^{-1/2}BA^{-1/2}$ and multiplying both sides of the above inequality by $A^{1/2}$ on their left and right hand sides, we get (11). We can also derive (12) in the same way. \square

4. Other generalized inequalities

In this section, we will show more examples which can be generalized by Proposition 1. For convenience of notation, we have used μ_v and δ_v , respectively, for the v -weighted arithmetic and geometric means of a and b in the second section. As shown in [2] and other references, however, we will now use the following well-known notation:

$$\begin{aligned} a\nabla_v b &= (1-v)a + vb, \\ a\#_v b &= a^{1-v}b^v, \\ a!_v b &= ((1-v)a^{-1} + vb^{-1})^{-1} \end{aligned}$$

for $a, b > 0$ and $0 \leq v \leq 1$. When $v = \frac{1}{2}$, we write $a\nabla b$, $a\#b$, and $a!b$ for brevity, respectively.

There are many inequalities which can be generalized by Proposition 1, but we will apply the proposition to the following ones shown in [2] and its references.

THEOREM 4. For $a, b > 0$ and $0 \leq v \leq 1$, we have

1. $a!_v b + 2r_0(a\nabla b - a!b) \leq a\nabla_v b \leq a!_v b + 2R_0(a\nabla b - a!b)$,
2. $K(h, 2)^{r_0} a\#_v b \leq a\nabla_v b \leq K(h, 2)^{R_0} a\#_v b$, where $h = \frac{b}{a}$ and $K(t, 2)$ is the Kantorovich constant defined by $K(t, 2) = \frac{(t+1)^2}{4t}$.

We generalize the first one in the above theorem as follows.

THEOREM 5. For $a, b > 0$, $0 \leq v \leq 1$, and $m = 1, 2, \dots$, we have

$$\begin{aligned} (a\nabla_v b)^m &\geq (a!_v b)^m + (2r_0)^m ((a\nabla b)^m - (a!b)^m), \\ (a\nabla_v b)^m &\leq (a!_v b)^m + (2R_0)^m ((a\nabla b)^m - (a!b)^m). \end{aligned}$$

Proof. Since equality holds in them when either $a = b$ or $v = 0, 1$, we may assume $a \neq b$ and $0 < v < 1$. Then we have

$$2r_0 \leq \frac{a\nabla_v b - a!_v b}{a\nabla b - a!b} \leq 2R_0$$

by the first part of Theorem 4. Thus it suffices to show

$$2r_0 \leq \frac{a\nabla_v b}{a\nabla b} \leq 2R_0$$

by Proposition 1, and the above is already shown in the proof of Theorem 1. \square

Note that the second part of Theorem 4 can be written as

$$\left(\frac{a\nabla b}{a\#b}\right)^{2r_0} a\#_v b \leq a\nabla_v b \leq \left(\frac{a\nabla b}{a\#b}\right)^{2R_0} a\#_v b. \tag{13}$$

or equivalently, using natural logarithm,

$$\begin{aligned} \ln(a\nabla_v b) &\geq \ln(a\#_v b) + 2r_0(\ln(a\nabla b) - \ln(a\#b)), \\ \ln(a\nabla_v b) &\leq \ln(a\#_v b) + 2R_0(\ln(a\nabla b) - \ln(a\#b)). \end{aligned} \tag{14}$$

THEOREM 6. For $a, b > 1$, $0 \leq v \leq 1$, and $m = 1, 2, \dots$, we have

$$\begin{aligned} [\ln(a\nabla_v b)]^m &\geq [\ln(a\#_v b)]^m + (2r_0)^m ([\ln(a\nabla b)]^m - [\ln(a\#b)]^m), \\ [\ln(a\nabla_v b)]^m &\leq [\ln(a\#_v b)]^m + (2R_0)^m ([\ln(a\nabla b)]^m - [\ln(a\#b)]^m). \end{aligned}$$

Proof. Without loss of generality, we may assume $a \neq b$ and $0 < v < 1$. Since

$$2r_0 \leq \frac{\ln(a\nabla_v b) - \ln(a\#_v b)}{\ln(a\nabla b) - \ln(a\#b)} \leq 2R_0$$

by (14), it suffices to show

$$2r_0 \leq \frac{\ln(a\nabla_v b)}{\ln(a\nabla b)} \leq 2R_0$$

by Proposition 1, which is equivalent to

$$(a\nabla b)^{2r_0} \leq a\nabla_v b \leq (a\nabla b)^{2R_0}. \tag{15}$$

Note that (13) can be expressed as

$$(a\nabla b)^{2r_0} a^{1-v-r_0} b^{v-r_0} \leq a\nabla_v b \leq (a\nabla b)^{2R_0} a^{1-v-R_0} b^{v-R_0}. \tag{16}$$

Since $a, b > 1$ and $r_0 \leq v, 1-v \leq R_0$, we have $a^{1-v-r_0} b^{v-r_0} \geq 1$ and $a^{1-v-R_0} b^{v-R_0} \leq 1$. Thus (15) follows from (16). \square

Replacing a, b with a^{-1}, b^{-1} , respectively, in the above theorem, we get the following.

COROLLARY 2. For $a, b \in (0, 1)$, $0 \leq v \leq 1$, and $m = 1, 2, \dots$, we have

$$\begin{aligned} |\ln(a!_v b)|^m &\geq |\ln(a\#_v b)|^m + (2r_0)^m (|\ln(a!b)|^m - |\ln(a\#b)|^m), \\ |\ln(a!_v b)|^m &\leq |\ln(a\#_v b)|^m + (2R_0)^m (|\ln(a!b)|^m - |\ln(a\#b)|^m). \end{aligned}$$

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REFERENCES

- [1] D. CHOI, M. KRNIĆ, AND J. PEČARIĆ, *Improved Jensen-type inequalities via linear interpolation and applications*, J. Math. Inequal. **11** (2017), 301–322.
- [2] D. CHOI AND M. SABABHEH, *Inequalities related to the arithmetic, geometric, and harmonic means*, J. Math. Ineq., vol. 11, no. 1 (2017), 1–16.
- [3] M. G. GHAEMI, N. GHARAKHANLU, AND S. FURUICHI, *On the reverse Young and Heinz inequalities*, J. Math. Inequal. **11** (2017), 641–652.
- [4] C. HE AND L. ZOU, *Some inequalities involving unitarily invariant norms*, Math. Inequal. Appl. **12** (4) (2012) 767–776.
- [5] O. HIRZALLAH AND F. KITTANEH, *Matrix Young inequalities for the Hilbert–Schmidt norm*, Linear Algebra Appl. **308** (2000) 77–84.
- [6] F. KITTANEH AND Y. MANASRAH, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl. **361** (2010), pp. 262–269.
- [7] F. KITTANEH AND Y. MANASRAH, *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear Algebra. **59** (2011), pp. 1031–1037.
- [8] Y. MANASRAH AND F. KITTANEH, *A generalization of two refined Young inequalities*, Positivity (2015) 19: 757–768.

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