

## OPTIMALITY OF THE REARRANGEMENT INEQUALITY WITH APPLICATIONS TO LORENTZ-TYPE SEQUENCE SPACES

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*Abstract.* We characterize the sequences  $(w_i)_{i=1}^\infty$  of non-negative numbers for which

$$\sum_{i=1}^{\infty} a_i w_i \quad \text{is of the same order as} \quad \sup_n \sum_{i=1}^n a_i w_{1+n-i}$$

when  $(a_i)_{i=1}^\infty$  runs over all non-increasing sequences of non-negative numbers. As a by-product of our work we settle a problem raised in [1] and prove that Garling sequence spaces have no symmetric basis.

### 1. Introduction

The rearrangement inequality states that, for  $n \in \mathbb{N}$ , if  $(a_i)_{i=1}^n$  and  $(b_i)_{i=1}^n$  are a pair of non-increasing  $n$ -tuples of non-negative scalars then we have

$$\sum_{i=1}^n a_i b_{1+n-i} \leq \sum_{i=1}^n a_i b_{\sigma(i)} \leq \sum_{i=1}^n a_i b_i$$

for every permutation  $\sigma$  of the set  $\{1, \dots, n\}$  (see Theorem 368 of [3]). Consequently, if  $(a_i)_{i=1}^\infty$  and  $(w_i)_{i=1}^\infty$  are non-increasing sequences of non-negative scalars,

$$\sup_{n \in \mathbb{N}} \sum_{i=1}^n a_i w_{1+n-i} \leq \sum_{i=1}^{\infty} a_i w_i.$$

In this note we wonder about which are the non-increasing sequences  $(w_i)_{i=1}^\infty$  of non-negative scalars that verify a reverse inequality, i.e., in which cases there is a constant  $C < \infty$  such that

$$\sum_{i=1}^{\infty} a_i w_i \leq C \sup_{n \in \mathbb{N}} \sum_{i=1}^n a_i w_{1+n-i} \tag{1}$$

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for every sequence  $(a_i)_{i=1}^\infty$  of non-negative scalars. For the time being some simple answers can be given. Indeed, on the one hand, if  $w_\infty := \inf_i w_i > 0$  then

$$\sum_{i=1}^\infty a_i w_i \leq w_1 \sum_{i=1}^\infty a_i = w_1 \sup_n \sum_{i=1}^n a_i \leq \frac{w_1}{w_\infty} \sup_n \sum_{i=1}^n a_i w_{1+n-i}.$$

On the other hand, if we consider  $W := \sum_{i=1}^\infty w_i < \infty$  and let  $w_1 > 0$  (the case  $w_1 = 0$  is trivial) then

$$\sum_{i=1}^\infty a_i w_i \leq a_1 \sum_{i=1}^\infty w_i = \frac{W}{w_1} a_1 w_1 \leq \frac{W}{w_1} \sup_n \sum_{i=1}^n a_i w_{1+n-i}.$$

In fact, as we will show below, these two cases are the only ones for which (1) holds. This will be our main result as far as inequalities is concerned:

**THEOREM 1. (Main Theorem)** *Let  $(w_i)_{i=1}^\infty$  be a non-increasing sequence consisting of non-negative scalars. The following are equivalent:*

(i) *There is a constant  $C < \infty$  such that*

$$\sum_{i=1}^\infty a_i w_i \leq C \sup_{n \in \mathbb{N}} \sum_{i=1}^n a_i w_{1+n-i}$$

*for every sequence  $(a_i)_{i=1}^\infty$  of non-negative scalars.*

(ii) *Either  $\sum_{i=1}^\infty w_i < \infty$  or  $\inf_{i \in \mathbb{N}} w_i > 0$ .*

Section 2 is devoted to proving Theorem 1. In Section 3 we use Theorem 1 to give some functional analytic properties of a recently introduced class of Lorentz-type spaces, called Garling sequence spaces. In particular, Theorem 1 is applied to show that Garling sequence spaces have no symmetric basis, answering thus a problem that was recently posed in [1].

Throughout this note we use standard terminology and notation in Banach space theory. As is customary, we denote by  $\ell_q$ ,  $1 \leq q \leq \infty$ , the Banach space consisting of all  $q$ -summable sequences (bounded sequences in the case  $q = \infty$ ) and by  $c_0$  the subspace of  $\ell_\infty$  consisting of all sequences converging to zero. For background on bases in Banach spaces we refer the reader to [2].

## 2. Proof of the Main Theorem

*Proof of Theorem 1.* As explained in the Introduction, we must only prove that (i) implies (ii).

Assume that (ii) does not hold, that is,  $\mathbf{w} = (w_i)_{i=1}^\infty \in c_0 \setminus \ell_1$ . Let us denote by  $\mathcal{D}$  the set of (nonzero) non-increasing sequences of non-negative integers. For  $f = (a_i)_{i=1}^\infty \in \mathcal{D}$  and  $n \in \mathbb{N}$  we put

$$A(f, \mathbf{w}) = \sum_{i=1}^\infty a_i w_i, \text{ and}$$

$$B(f, \mathbf{w}) = \sup_{n \in \mathbb{N}} B(f, \mathbf{w}, n),$$

where, for  $n \in \mathbb{N}$ ,

$$B(f, \mathbf{w}, n) = \sum_{i=1}^n a_i w_{1+n-i}.$$

With this notation we must prove that

$$S(\mathbf{w}) := \sup_{f \in \mathcal{D}} \frac{A(f, \mathbf{w})}{B(f, \mathbf{w})} = \infty.$$

We will use the convention that  $\sum_{i=1}^0 c_i = 0$  for all sequences of scalars  $(c_i)_{i=1}^\infty$ .

For  $n \in \mathbb{N}$  put  $W(n) = \sum_{i=1}^n w_i$ . Since  $\mathbf{w} \notin \ell_1$  we have

$$\lim_n W(n) = \infty.$$

Moreover, since  $\mathbf{w} \in c_0$ ,

$$\lim_{n \in \mathbb{N}} (W(s+n) - W(n)) = 0$$

for any non-negative integer  $s$ . We use these properties to recursively construct an increasing sequence  $(d_k)_{k=0}^\infty$  of non-negative integers with  $d_0 = 0$  verifying

- (i)  $W(\sum_{j=1}^{k-1} d_j) \leq 2^{-1} W(d_k)$ , and
- (ii)  $W(d_{k-1} + d_k) - W(d_k) \leq 2^{1-k} W(d_{k-1})$

for  $k = 1, 2, \dots$ .

For every integer  $k \geq 0$  put  $n_k = \sum_{j=1}^k d_j$ . For each  $r \in \mathbb{N}$  we define a sequence  $f^{(r)} = (a_{i,r})_{i=1}^\infty$  by

$$a_{i,r} = \begin{cases} 1/W(d_k) & \text{if, for some } 1 \leq k \leq r, n_{k-1} < i \leq n_k \\ 0 & \text{if } i > n_r. \end{cases}$$

It is clear that  $f^{(r)} \in \mathcal{D}$  for all  $r \in \mathbb{N}$ . Taking into account the inequality in (i) we obtain

$$\begin{aligned} A(f^{(r)}, \mathbf{w}) &= \sum_{k=1}^r \frac{1}{W(d_k)} \sum_{i=1+n_{k-1}}^{n_k} w_i = \sum_{k=1}^r \frac{W(n_k) - W(n_{k-1})}{W(d_k)} \\ &\geq \sum_{k=1}^r \frac{W(d_k) - 2^{-1} W(d_k)}{W(d_k)} = \sum_{k=1}^r \frac{1}{2} = \frac{r}{2}. \end{aligned}$$

Let  $n \in \mathbb{N}$ . In case that  $n > n_r$  we have

$$B(f^{(r)}, \mathbf{w}, n) = \sum_{i=1}^{n_r} a_{i,r} w_{1+n-i} \leq \sum_{i=1}^{n_r} a_{i,r} w_{1+n_r-i} = B(f^{(r)}, \mathbf{w}, n_r).$$

In case that  $n \leq n_r$ , pick  $1 \leq k \leq r$  with  $n_{k-1} < n \leq n_k$ . We have

$$\begin{aligned} B(f^{(r)}, \mathbf{w}, n) &= \frac{W(n - n_{k-1})}{W(d_k)} + \sum_{j=1}^{k-1} \frac{W(n - n_{j-1}) - W(n - n_j)}{W(d_j)} \\ &\leq \frac{W(n_k - n_{k-1})}{W(d_k)} + \sum_{j=1}^{k-1} \frac{W(n - n_{j-1}) - W(n - n_j)}{W(d_j)} \\ &= 1 + \sum_{j=1}^{k-1} \frac{W(n - n_{j-1}) - W(n - n_j)}{W(d_j)}. \end{aligned}$$

If  $k = 1$  we get  $B(f^{(r)}, \mathbf{w}, n) \leq 1$ . Assume that  $k \geq 2$ . Taking into account inequality (ii) and that, since  $\mathbf{w}$  is non-increasing, the sequence  $(W(n+t) - W(n+s))_{n=1}^\infty$  is non-increasing for any  $s \leq t$ , we obtain

$$\begin{aligned} B(f^{(r)}, \mathbf{w}, n) &\leq 1 + \frac{W(n - n_{k-2}) - W(n - n_{k-1})}{W(d_{k-1})} + \sum_{j=1}^{k-2} \frac{W(n - n_{j-1}) - W(n - n_j)}{W(d_j)} \\ &\leq 1 + \frac{W(n_{k-1} - n_{k-2}) - W(n_{k-1} - n_{k-1})}{W(d_{k-1})} \\ &\quad + \sum_{j=1}^{k-2} \frac{W(n_{j+1} - n_{j-1}) - W(n_{j+1} - n_j)}{W(d_j)} \\ &= 2 + \sum_{j=1}^{k-2} \frac{W(d_{j+1} + d_j) - W(d_{j+1})}{W(d_j)} \leq 2 + \sum_{j=1}^{k-2} 2^{-j} = 3 - 2^{2-k}. \end{aligned}$$

Therefore  $B(f^{(r)}, \mathbf{w}) \leq 3$ . Thus

$$S(\mathbf{w}) \geq \sup_{r \in \mathbb{N}} \frac{A(f^{(r)}, \mathbf{w})}{B(f^{(r)}, \mathbf{w})} \geq \sup_{r \in \mathbb{N}} \frac{r}{6} = \infty,$$

and the proof is over.  $\square$

### 3. Applications to Garling sequence spaces

Let  $1 \leq p < \infty$  and let  $\mathbf{w} = (w_n)_{n=1}^\infty$  be a non-increasing sequence of positive scalars. Given a sequence of (real or complex) scalars  $f = (b_k)_{k=1}^\infty$  we put

$$\|f\|_{g(\mathbf{w}, p)} = \sup_{\phi \in \mathcal{O}} \left( \sum_{i=1}^\infty |b_{\phi(i)}|^p w_i \right)^{1/p}$$

where  $\mathcal{O}$  denotes the set of increasing functions from  $\mathbb{N}$  to  $\mathbb{N}$ . The Garling sequence space  $g(\mathbf{w}, p)$  is the Banach space consisting of all sequences  $f$  with  $\|f\|_{g(\mathbf{w}, p)} < \infty$ .

Notice that if in (3) we replace “ $\phi \in \mathcal{O}$ ” with “ $\phi$  is a permutation of  $\mathbb{N}$ ” we obtain the norm that defines the weighted Lorentz sequence space

$$d(\mathbf{w}, p) := \left\{ (b_k)_{k=1}^\infty \in c_0 : \left( \sum_{i=1}^\infty (b_i^*)^p w_i \right)^{1/p} < \infty \right\},$$

where  $(b_i^*)_{i=1}^\infty$  denotes the decreasing rearrangement of  $(b_k)_{k=1}^\infty$ . So, the Garling sequence space  $g(\mathbf{w}, p)$  can be regarded as a variation of the weighted Lorentz sequence space  $d(\mathbf{w}, p)$ .

Imposing the further conditions  $\mathbf{w} \in c_0$  and  $\mathbf{w} \notin \ell_1$  will prevent us, respectively, from having  $g(\mathbf{w}, p) = \ell_p$  or  $g(\mathbf{w}, p) = \ell_\infty$ . We will assume as well that  $\mathbf{w}$  is normalized, i.e.,  $w_1 = 1$ . Thus, we put

$$\mathscr{W} := \{(w_i)_{i=1}^\infty \in c_0 \setminus \ell_1 : 1 = w_1 \geq w_2 \geq \dots \geq w_i \geq w_{i+1} \geq \dots > 0\}$$

and we restrict our attention to weights  $\mathbf{w} \in \mathscr{W}$ .

For  $n \in \mathbb{N}$ , we will denote  $\mathbf{e}_n = (\delta_{i,n})_{i=1}^\infty$ , where  $\delta_{i,n} = 1$  if  $n = i$  and  $\delta_{i,n} = 0$  otherwise. We have (see Theorem 3.1 of [1]) that the canonical sequence  $(\mathbf{e}_n)_{n=1}^\infty$  is a Schauder basis for  $g(\mathbf{w}, p)$ . A question posed and partially solved in [1] is to determine the weights  $\mathbf{w} \in \mathscr{W}$  and the indices  $p \in [1, \infty)$  for which  $(\mathbf{e}_n)_{n=1}^\infty$  is a symmetric basis of  $g(\mathbf{w}, p)$ . Here we provide a complete intrinsic solution to this problem, in the sense that our approach is entirely based on Theorem 1.

LEMMA 2. *The canonical sequence  $(\mathbf{e}_n)_{n=1}^\infty$  is not a symmetric basis for  $g(\mathbf{w}, p)$  for any  $\mathbf{w} \in \mathscr{W}$  and any  $1 \leq p < \infty$ .*

*Proof.* Assume that  $(\mathbf{e}_n)_{n=1}^\infty$  is a symmetric basis for  $g(\mathbf{w}, p)$ . Then, there is a constant  $C$  so that

$$\|g\|_{g(\mathbf{w}, p)} \leq C \|f\|_{g(\mathbf{w}, p)}$$

whenever the sequence  $g$  is a permutation of the sequence  $f$ .

Given  $r \in \mathbb{N}$  and  $\phi \in \mathcal{O}$  let  $n(r, \phi)$  be the largest integer  $n$  such that  $\phi(n) \leq r$ . We have  $\phi(i) \leq i + r - n(r, \phi)$  for  $1 \leq i \leq n(r, \phi)$ . Given a non-increasing sequence  $(a_i)_{i=1}^\infty$  of non-negative numbers we have

$$\begin{aligned} \sum_{i=1}^\infty a_i w_i &= \sup_r \sum_{i=1}^r a_i w_i \leq \sup_{r \in \mathbb{N}} \left\| \sum_{i=1}^r a_i^{1/p} \mathbf{e}_i \right\|_{g(\mathbf{w}, p)}^p \\ &\leq C \sup_{r \in \mathbb{N}} \left\| \sum_{i=1}^r a_{1+r-i}^{1/p} \mathbf{e}_i \right\|_{g(\mathbf{w}, p)}^p = C \sup_{r \in \mathbb{N}, \phi \in \mathcal{O}} \sum_{i=1}^{n(r, \phi)} a_{1+r-\phi(i)} w_i \\ &\leq C \sup_{r \in \mathbb{N}, \phi \in \mathcal{O}} \sum_{i=1}^{n(r, \phi)} a_{1+n(r, \phi)-i} w_i = C \sup_{n \in \mathbb{N}} \sum_{i=1}^n a_{1+n-i} w_i \\ &= C \sup_{n \in \mathbb{N}} \sum_{i=1}^n a_i w_{1+n-i}. \end{aligned}$$

Theorem 1 yields the absurdity  $\mathbf{w} \in \ell_1$  or  $\mathbf{w} \notin c_0$ .  $\square$

Now we are ready to establish the advertised structural properties of Garling sequence spaces.

**THEOREM 3.** *Let  $\mathbf{w} \in \mathscr{W}$  and  $1 \leq p < \infty$ .*

- (i) *There is no symmetric basis for  $g(\mathbf{w}, p)$ .*
- (ii)  *$d(\mathbf{w}, p) \subsetneq g(\mathbf{w}, p)$ .*
- (iii) *No subspace of  $d(\mathbf{w}, p)$  is isomorphic to  $g(\mathbf{w}, p)$ .*
- (iv) *Let  $I_{d,g}: d(\mathbf{w}, p) \rightarrow g(\mathbf{w}, p)$  be the natural inclusion map, and let  $T: g(\mathbf{w}, p) \rightarrow d(\mathbf{w}, p)$  be a bounded linear operator. Then (despite the fact that  $I_{d,g}$  is not a strictly singular operator)  $T \circ I_{d,g}$  does not preserve a copy of  $d(\mathbf{w}, p)$ , i.e., if  $X$  is a subspace of  $d(\mathbf{w}, p)$  isomorphic to  $d(\mathbf{w}, p)$  then  $T \circ I_{d,g}|_X$  is not an isomorphism.*

*Proof.* It follows using Lemma 2 in combination with Theorem 5.1 of [1].  $\square$

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