

COVERING UNIT SPHERES AND BALLS OF NORMED SPACES BY SMALLER BALLS

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Abstract. We amend a widely used result given by Doyle, Lagarias, and Randall concerning the side length of equilateral Minkowski m -gons inscribed in the unit circle of a Minkowski plane. Based on this, we obtain the smallest positive number γ such that the unit circle S_X of a Minkowski plane X can be covered by m translates of γB_X , where B_X is the unit ball of X . Moreover, we improve a recent estimation of the smallest positive number γ such that the unit ball B_X of a Minkowski space X can be covered by m translates of γB_X .

1. Introduction

Let $n \geq 2$ be a positive integer. We denote by $[n]$ the set $\{m \in \mathbb{Z}^+ : 1 \leq m \leq n\}$ and by $X = (\mathbb{R}^n, \|\cdot\|)$ an n -dimensional (normed or) Minkowski space whose unit ball and unit sphere are denoted by B_X and S_X , respectively. Clearly, B_X is a convex body (i.e., a compact convex set having interior points) symmetric with respect to the origin of \mathbb{R}^n . For each $m \in \mathbb{Z}^+$, put

$$\Gamma_m(X) = \inf \left\{ \gamma > 0 : \exists \{x_i : i \in [m]\} \subseteq X \text{ s.t. } B_X \subseteq \bigcup_{i \in [m]} (x_i + \gamma B_X) \right\},$$

$$\gamma_m(X) = \inf \left\{ \gamma > 0 : \exists \{x_i : i \in [m]\} \subseteq X \text{ s.t. } S_X \subseteq \bigcup_{i \in [m]} (x_i + \gamma B_X) \right\}.$$

It is clear that

$$\gamma_m(X) \leq \Gamma_m(X) \leq 1, \forall m \in \mathbb{Z}^+,$$

and that “inf”s in the definition of $\Gamma_m(X)$ and $\gamma_m(X)$ can be replaced by “min”s.

As shown in [11], [16], [8], and [12], $\Gamma_m(X)$ and $\gamma_m(X)$ are both closely related to the special case of the famous Hadwiger’s covering problem when the convex body under consideration is centrally symmetric. We refer to [4], [13], [2], and [3] for more

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information and more references concerning this open problem. Recently several progresses have been made in this direction by using tools and methods from Banach space theory, see [15], [8], and [12].

In Section 2, we amend a result given by Doyle, Lagarias, and Randall in [5] saying that the side length of equilateral Minkowski m -gons inscribed in the unit circle of a *Minkowski plane* (i.e., a two-dimensional Minkowski space) having a given point x_0 as one of its vertices is uniquely determined by x_0 , which is generally not true (see Example 1). Based on this, in Section 3 we compute the precise value of $\gamma_m(X)$ and show that $\gamma_3(X) = \Gamma_3(X)$ when X is a Minkowski plane. Inspired by the proof idea of Theorem 19 in [12], we provide an estimation of $\Gamma_m(X)$ for Minkowski spaces which is better than the one given in [8] in the last section. As in [8] and [12], we are following the philosophy provided in [17]: studying classical problems from Discrete and Convex Geometry by introducing and studying proper functionals defined on the space of convex bodies.

For each pair of points $u, v \in S_X$ satisfying $u \neq -v$, we denote by

$$\text{arc}(u, v) := \left\{ \frac{\alpha u + \beta v}{\|\alpha u + \beta v\|} : \alpha, \beta \geq 0, \alpha u + \beta v \neq o \right\}$$

the *minor arc connecting u and v* , see Figure 1.

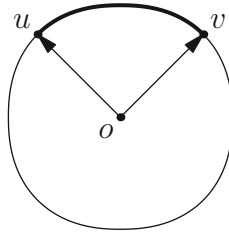


Figure 1: *The definition of a minor arc*

For each bounded subset A of X , put

$$r(A) = \inf \{ \gamma \geq 0 : \exists x \in X \text{ s.t. } A \subseteq (x + \gamma B_X) \}.$$

It is not difficult to verify that, when A is compact, then “inf” in the definition of $r(A)$ can be replaced by “min”. By the triangle inequality, one can easily verify that

$$r(A) \geq \frac{1}{2} \delta(A),$$

where $\delta(A)$ is the *diameter* of A .

2. Side lengths of equilateral Minkowski m -gons

We shall use the following Monotonicity Lemma.

LEMMA 1. (Monotonicity Lemma, cf. Proposition 31 in [14]) *Let X be a Minkowski plane, and $a, b, c \neq o$ be three points with $a \neq c$ such that the ray $[o, b)$ lies between $[o, a)$ and $[o, c)$. If $\|b\| = \|c\|$, then $\|a - b\| \leq \|a - c\|$, with equality if and only if either*

1. $b=c$;
2. or o and b are on opposite sides of $\langle a, c \rangle$, and

$$\left[\frac{c-a}{\|c-a\|}, \frac{b}{\|b\|} \right] \subseteq S_X;$$

3. or o and b are on the same side of $\langle a, c \rangle$, and

$$\left[\frac{c-a}{\|c-a\|}, \frac{-c}{\|-c\|} \right] \subseteq S_X.$$

LEMMA 2. *Let X be a Minkowski plane, $u, v \in S_X$ be two points satisfying $u \neq -v$. Then*

$$r(\text{arc}(u, v)) = \frac{1}{2} \|u - v\|.$$

Proof. Clearly,

$$r(\text{arc}(u, v)) \geq \frac{1}{2} \delta(\text{arc}(u, v)) \geq \frac{1}{2} \|u - v\|.$$

Hence we only need to prove that

$$r(\text{arc}(u, v)) \leq \frac{1}{2} \|u - v\|. \quad (1)$$

Put

$$w = \frac{u+v}{\|u+v\|} \quad \text{and} \quad p = \frac{1}{2}(u+v).$$

Let q be an arbitrary point in $\text{arc}(u, v)$. If $q = u$ or $q = v$, then

$$\|p - q\| = \frac{1}{2} \|u - v\|.$$

If $q = w$, then

$$\|p - q\| = 1 - \|p\| \leq 1 - (\|u\| - \|u - p\|) = \frac{1}{2} \|u - v\|.$$

In the following we may assume, without loss of generality, that

$$q \in \text{arc}(u, w) \setminus \{u, w\}.$$

Then the ray $[o, q)$ lies between $[o, u)$ and $[o, p)$, and $\|u\| = \|q\|$. From Lemma 1 it follows that

$$\|q - p\| \leq \|u - p\| = \frac{1}{2} \|u - v\|.$$

Therefore, (1) holds. \square

LEMMA 3. *Let X be a Minkowski plane, $x \in X$, and $\gamma \in (0, 1)$. If*

$$B_X \cap (x + \gamma B_X) \neq \emptyset,$$

then there exist two points $u, v \in S_X$ such that

$$S_X \cap (x + \gamma B_X) = \text{arc}(u, v) \quad \text{and} \quad \frac{x}{\|x\|} \in \text{arc}(u, v).$$

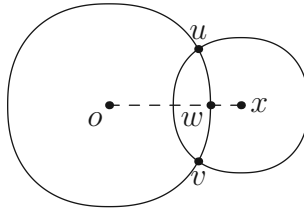


Figure 2: *The intersection of the unit circle and a smaller disc*

Proof. Put $w = x/\|x\|$, see Figure 2. The hypothesis shows that

$$1 - \gamma \leq \|x\| \leq 1 + \gamma.$$

Therefore,

$$\|x - w\| = \left\| x - \frac{x}{\|x\|} \right\| = |\|x\| - 1| \leq \gamma,$$

which shows that

$$w \in S_X \cap (x + \gamma B_X).$$

Since

$$\|x - (-w)\| = \|x + w\| = \left\| x + \frac{x}{\|x\|} \right\| = 1 + \|x\| > 1 > \gamma,$$

there exist two points p and q in S_X lying on different open half-planes bounded by the line $\langle -x, x \rangle$ such that $\min \{\|p - x\|, \|q - x\|\} \geq 1 > \gamma$. Put

$$\lambda_1 = \max \left\{ \lambda \in [0, 1] : \left\| x - \frac{\lambda p + (1 - \lambda)w}{\|\lambda p + (1 - \lambda)w\|} \right\| \leq \gamma \right\},$$

$$\lambda_2 = \max \left\{ \lambda \in [0, 1] : \left\| x - \frac{\lambda q + (1 - \lambda)w}{\|\lambda q + (1 - \lambda)w\|} \right\| \leq \gamma \right\},$$

$$u = \frac{\lambda_1 p + (1 - \lambda_1)w}{\|\lambda_1 p + (1 - \lambda_1)w\|}, \quad \text{and} \quad v = \frac{\lambda_2 q + (1 - \lambda_2)w}{\|\lambda_2 q + (1 - \lambda_2)w\|}.$$

By applying Lemma 1, we have

$$\text{arc}(u, w) \cup \text{arc}(v, w) = S_X \cap (x + \gamma B_X).$$

Since $\delta(\text{arc}(u, w) \cup \text{arc}(v, w)) \leq 2\gamma < 2$, we have

$$\text{arc}(u, w) \cup \text{arc}(v, w) = \text{arc}(u, v). \quad \square$$

LEMMA 4. *Let X be a Minkowski plane, $u, v \in S_X$ be two distinct points such that $[u, v] \cap \text{int} B_X \neq \emptyset$, H^+ be an open halfplane bounded by $\langle u, v \rangle$ not containing o . If s and t are two distinct points in $S_X \cap H^+$, then*

$$\|s - t\| \leq \|u - v\|; \quad (2)$$

equality holds if and only if $\|s - t\| = 2$.

Proof. The case when $u = -v$ is clear. In the following we assume that $u \neq -v$.

The inequality (2) follows directly from the Monotonicity Lemma (Lemma 1). We only need to consider the case when equality holds. Clearly, if $\|s - t\| = 2$, then

$$2 = \|s - t\| \leq \|u - v\| \leq 2,$$

which implies that $\|s - t\| = \|u - v\|$.

Conversely, suppose that $\|s - t\| = \|u - v\|$.

Case I: $[s, t]$ is parallel to $[u, v]$. Clearly, the parallelogram P having $s, t, -s, -t$ as vertices is contained in B_X . Then

$$\langle u, v \rangle \cap P \subseteq \langle u, v \rangle \cap B_X = [u, v].$$

Since $\langle u, v \rangle \cap P$ is a segment parallel to $[s, t]$ whose length is $\|s - t\|$, we have

$$\langle u, v \rangle \cap P = [u, v].$$

Then the line $\langle -s, t \rangle$ contains three distinct points from S_X , which shows that $[-s, t] \subseteq S_X$. Similarly, $[s, -t] \subseteq S_X$. Hence $\|u - v\| = \|s - t\| = 2$.

Case II: $[s, t]$ is not parallel to $[u, v]$. We may assume that the distance from s to the line $\langle u, v \rangle$ is strictly smaller than the distance from t to $\langle u, v \rangle$. Let t' be the point of intersection of S_X and the line passing through s and parallel to $\langle u, v \rangle$. Then Lemma 1 shows that

$$\|s - t\| \leq \|s - t'\| \leq \|u - v\|.$$

It follows that

$$\|s - t\| = \|s - t'\| = \|u - v\|.$$

By Case I, $\|s - t'\| = 2$. Hence $\|s - t\| = 2$. \square

The following example shows that Lemma 2.4 in [5] is not true.

EXAMPLE 1. Let $X = (\mathbb{R}^2, \|\cdot\|)$, where the norm is defined by $\|(\alpha, \beta)\| = \max\{|\alpha|, |\beta|\}$. Put $a = (1, 1)$, $b = (-1, 1)$, $c = (1, -1)$, $b' = (0, 1)$, $c' = (1, 0)$. Then both the triangles abc and $ab'c'$ are equilateral, but their side lengths are different, see Figure 3.

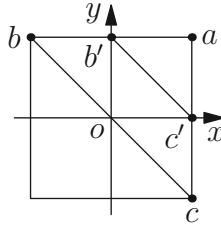


Figure 3: Two equilateral triangles inscribed in S_X

PROPOSITION 5. Let X be a Minkowski plane. If there exist a point $a \in S_X$ and two equilateral triangles T and T' inscribed in S_X having a as one vertex whose side lengths are different, then S_X is a parallelogram.

Proof. Let $\{a, b, c\}$ and $\{a, b', c'\}$ be the set of vertices of T and T' , respectively.

First we show that at least one of T and T' does not contain o . Otherwise both T and T' contain o . Without loss of generality, we may assume that the side length of T is strictly less than the side length of T' , which is at most 2. Then it is clear that T contains o in its interior. Also, $-a \notin [b, c]$, since otherwise we would have

$$\|a - b\| = \|a - c\| = \|-a - a\| = 2,$$

which implies that the triangles T and T' have the same side length.

In this situation we have

$$S_X = \text{arc}(a, b) \cup \text{arc}(a, c) \cup \text{arc}(b, c).$$

Lemma 1 and the assumption that $\|a - c'\| = \|a - b'\| > \|a - b\|$ show that

$$b', c' \in \text{arc}(b, c) \setminus \{b, c\}.$$

Then Lemma 4 shows that $\|b' - c'\| < \|b - c\|$, a contradiction.

Thus at least one of T and T' does not contain o . Assume without loss of generality that $o \notin T$ and that a and o lie on different sides of $\langle b, c \rangle$. Therefore, $\{o, a, b, c\}$ is the set of vertices of a convex quadrilateral having $[o, a]$ and $[b, c]$ as diagonals. We have

$$2(\|o - a\| + \|b - c\|) = \|o - b\| + \|a - c\| + \|o - c\| + \|a - b\|.$$

By Corollary 8 in [14], S_X is a parallelogram. \square

The following theorem amends a false statement in Lemma 2.4 in [5] as we have claimed.

THEOREM 6. *Let X be a Minkowski plane. For any integer $m \geq 3$ and each $x \in S_X$, there exists a convex m -gon inscribed in B_X and having x as a vertex which is equilateral with respect to $\|\cdot\|$ (called an equilateral Minkowski m -gon having x as a vertex). If X is strictly convex, this m -gon is unique. Moreover, if S_X is not a parallelogram or if $m \geq 4$, the side length of the m -gon is uniquely determined by x .*

Proof. Note that Lemma 2.4 in [5] correctly proved the existence of such convex m -gons and the uniqueness of the equilateral Minkowski m -gon having $x \in S_X$ as a vertex when X is strictly convex. Proposition 5 shows that when S_X is not a parallelogram, for each $x \in S_X$, the length of equilateral triangles inscribed in S_X having x as a vertex is uniquely determined. In the rest we show that, when $m \geq 4$, for each Minkowski plane X and each $x \in S_X$ the length of an equilateral Minkowski m -gon having x as a vertex is uniquely determined.

For two distinct points $x, y \in S_X$, we denote by $\overrightarrow{\text{arc}}(x, y)$ the directed arc that connects x counter-clockwisely with y . Hence

$$\overrightarrow{\text{arc}}(x, y) \neq \overrightarrow{\text{arc}}(y, x) \quad \text{and} \quad S_X = \overrightarrow{\text{arc}}(x, y) \cup \overrightarrow{\text{arc}}(y, x).$$

First we show that if $m \geq 4$ and if $\{a_1, \dots, a_m\}$ is the set of vertices of an equilateral Minkowski m -gon P inscribed in S_X , which are ordered counter-clockwisely on S_X , then

$$\overrightarrow{\text{arc}}(a_i, a_{i+1}) = \text{arc}(a_i, a_{i+1}), \quad \forall i \text{ with } 1 \leq i \leq m,$$

where a_{m+1} is set to be a_1 . Take $\overrightarrow{\text{arc}}(a_1, a_2)$ for example. If $\overrightarrow{\text{arc}}(a_1, a_2) \neq \text{arc}(a_1, a_2)$, then either $a_2 = -a_1$ or P is contained in the closed halfplane bounded by $\langle a_1, a_2 \rangle$ not containing o . If $a_2 = -a_1$, then the side length of P is 2 and $\overrightarrow{\text{arc}}(a_2, a_1)$ is a semicircle whose length is not smaller than

$$\|a_m - a_1\| + \|a_m - a_{m-1}\| + \|a_{m-1} - a_{m-2}\| = 6.$$

This shows that the circumference of S_X is at least 12. Since the circumference of S_X ranges from 6 to 8 (see, e.g., [14, p. 130]), this is impossible. Now suppose that P is contained in the closed halfplane bounded by $\langle a_1, a_2 \rangle$ not containing o . In this case, $[a_3, a_4]$ is contained in the open halfplane bounded by $\langle a_1, a_2 \rangle$ not containing o . Since $\|a_3 - a_4\| = \|a_1 - a_2\|$, Lemma 4 shows that the side length of P is 2. Again this would show that the circumference of S_X is at least 12.

Note that these arguments also show that o is in the interior of P .

Suppose that there exist two equilateral Minkowski m -gons P and P' having a_1, \dots, a_m and b_1, \dots, b_m as vertices, respectively, where

1. $a_1 = b_1 = x$,
2. both a_1, \dots, a_m and b_1, \dots, b_m are in counter-clockwise order,
3. $\|a_2 - a_1\| > \|b_2 - b_1\|$.

Since P and P' both contain o in their interiors, a_2 and b_2 are both in $\overrightarrow{\text{arc}}(a_1, -a_1)$. Lemma 1 shows that b_2 is in the relative interior of $\overrightarrow{\text{arc}}(a_1, a_2)$. Now suppose that we have proved that b_i is in the relative interior of $\overrightarrow{\text{arc}}(a_1, a_i)$ for some integer $2 \leq i < m$. We distinguish two cases.

Case I: $-b_i \in \overrightarrow{\text{arc}}(-a_1, a_1)$. If $a_{i+1} \in \overrightarrow{\text{arc}}(-b_i, a_1)$, then

$$\overrightarrow{\text{arc}}(a_1, a_{i+1}) = \overrightarrow{\text{arc}}(a_1, b_i) \cup \overrightarrow{\text{arc}}(b_i, -b_i) \cup \overrightarrow{\text{arc}}(-b_i, a_{i+1}),$$

and $b_{i+1} \in \overrightarrow{\text{arc}}(b_i, -b_i) \setminus \{-b_i\}$. Thus b_{i+1} is in the relative interior of $\overrightarrow{\text{arc}}(a_1, a_{i+1})$.

If $a_{i+1} \in \overrightarrow{\text{arc}}(b_i, -b_i)$, then a_i is in the relative interior of $\text{arc}(b_i, a_{i+1})$. If $b_{i+1} \in \text{arc}(a_{i+1}, -b_i)$, then Lemma 1 shows that

$$\|b_{i+1} - b_i\| \geq \|a_{i+1} - a_i\|,$$

a contradiction. Thus b_{i+1} is in the relative interior of $\overrightarrow{\text{arc}}(b_i, a_{i+1})$ which is contained in the relative interior of $\overrightarrow{\text{arc}}(a_1, a_{i+1})$.

Case II: $-b_i \in \overrightarrow{\text{arc}}(a_1, -a_1) \setminus \{-a_1\}$. In this case we have

$$\text{arc}(a_i, a_{i+1}) \subseteq \text{arc}(b_i, a_1) = \overrightarrow{\text{arc}}(b_i, a_1).$$

Again, Lemma 1 shows that b_{i+1} is in the relative interior of $\text{arc}(b_i, a_{i+1})$ which is contained in the relative interior of $\overrightarrow{\text{arc}}(a_1, a_{i+1})$.

By induction, we know that b_m is in the relative interior of $\overrightarrow{\text{arc}}(a_1, a_m)$. By Lemma 1 again, we have

$$\|b_m - a_1\| = \|b_m - b_1\| \geq \|a_m - a_1\|,$$

a contradiction. \square

For each integer $m \geq 3$, each Minkowski plane X , and each $x \in S_X$, denote by $\alpha_m(x, X)$ the maximal side lengths of equilateral Minkowski m -gons inscribed in S_X having x as a vertex. Theorem 6 shows that if B_X is not a parallelogram or if $m \geq 4$, the side length of each equilateral Minkowski m -gon inscribed in S_X having x as a vertex is $\alpha_m(x, X)$. It is not difficult to verify that, for fixed x , $\alpha_m(x, X)$ is non-increasing with respect to m , see [5, p. 178]. Put

$$S(m, X) = \inf \{ \alpha_m(x, X) : x \in S_X \}.$$

PROPOSITION 7. *Let X be a Minkowski plane. Then $S(3, X) = 2$ if and only if B_X is a parallelogram.*

Proof. It is not difficult to verify that if B_X is a parallelogram then $S(3, X) = 2$.

Conversely, suppose that $S(3, X) = 2$. Then there exists an equilateral triangle T having vertices $a, b, c \in S_X$ whose side length is 2.

First suppose that one pair of points from a, b, c , say a and b , are linearly dependent. Then $b = -a$. Then we have

$$\|c + a\| = \|c - a\| = 2.$$

It follows that $[a, c] \subseteq S_X$ and $[b, c] \subseteq S_X$, which implies that B_X is a parallelogram having $a, b, c, -c$ as vertices.

Now we assume that a, b, c are pairwise linearly independent. Since

$$2 = \|a - b\| = \|a\| + \|b\|,$$

we have $[a, -b] \subseteq S_X$. Suppose that $[s, t]$ is the longest segment contained in S_X and containing $[a, -b]$. Let u be the midpoint of $[s, t]$. By the hypothesis, there exist two points $v, w \in S_X$ such that

$$\|u - v\| = \|u - w\| = \|v - w\| = 2.$$

It follows that $[-u, v] \subseteq S_X$. However, $[-s, -t]$ is the unique segment containing $-u$, which implies that $v \in [-s, -t]$. In a similar way, we can show that $w \in [-s, -t]$. Since

$$2 = \|v - w\| \leq \|s - t\| \leq 2,$$

$[s, t]$ is a segment contained in S_X whose length is 2. Hence, B_X is a parallelogram having $s, t, -s, -t$ as vertices. \square

3. $\gamma_m(X)$ of Minkowski planes

For the discussion in the sequel, we shall use the following equivalent representations of the so called *non-square constants* $J(X)$ and $S(X)$, which were provided in [10] (see also [7]):

$$J(X) := \sup\{\|x + y\| : x, y \in S_X, \|x + y\| = \|x - y\|\}$$

and

$$S(X) := \inf\{\|x + y\| : x, y \in S_X, \|x + y\| = \|x - y\|\}.$$

It is always true that (see, e.g., [6] and [7])

$$1 \leq S(X) \leq \sqrt{2} \leq J(X) \leq 2.$$

REMARK 8. One can easily verify that, for each Minkowski plane X , $S(4, X)$ is the Schäffer constant $S(X)$. Therefore

$$1 \leq S(4, X) \leq \sqrt{2}.$$

THEOREM 9. Let X be a Minkowski plane and $m \geq 3$. Then

$$\gamma_m(X) = \frac{1}{2}S(m, X).$$

Proof. The case when $m = 3$ and S_X is a parallelogram is clear. In the following we always assume that $m > 3$ or S_X is not a parallelogram. In this case the side length of equilateral Minkowski m -gons inscribed in S_X having $x \in S_X$ as a vertex is uniquely determined by m and x , and each equilateral Minkowski m -gon inscribed in S_X contains o in its interior. By Lemma 2, we only need to show that $\gamma_m(X) \geq (1/2)S(m, X)$. Suppose the contrary, namely that

$$\gamma := \gamma_m(X) < \frac{1}{2}S(m, X) \leq 1.$$

Then there exists a set $\{p_i : i \in [m]\}$ such that

$$S_X \subseteq \bigcup_{i \in [m]} (p_i + \gamma B_X).$$

By Lemma 3, there exist two points $u_1, v_1 \in S_X$ such that

$$(p_1 + \gamma B_X) \cap S_X = \overrightarrow{\text{arc}}(u_1, v_1) = \text{arc}(u_1, v_1).$$

There exists an equilateral Minkowski m -gon $P = x_1 x_2 \cdots x_m$ inscribed in S_X such that $x_1 = u_1$, and that, for each $i \in [m - 1]$, the orientation from x_i to x_{i+1} is counter-clockwise. We may also require that

$$\overrightarrow{\text{arc}}(x_i, x_{i+1}) = \text{arc}(x_i, x_{i+1}), \quad \forall i \in [m].$$

Moreover, the side length of this equilateral Minkowski m -gon is strictly greater than 2γ . Then, for each $i \in [m]$, $p_i + \gamma B_X$ can only cover at most one (and, consequently, precisely one) vertex of P . Thus we may assume without loss of generality that, for each $i \in [m]$, there exist two points $u_i, v_i \in S_X$ such that $x_i \in (p_i + \gamma B_X) \cap S_X = \overrightarrow{\text{arc}}(u_i, v_i) = \text{arc}(u_i, v_i)$.

Next we show that, for each $i \in [m]$, $v_i \in \text{arc}(x_i, x_{i+1}) \setminus \{x_{i+1}\}$ (we put $x_{m+1} = x_1$ and $x_0 = x_m$). Since

$$\|u_1 - v_1\| \leq 2\gamma < \|x_1 - x_2\| = \|u_1 - x_2\|,$$

Lemma 1 shows that $v_1 \in \text{arc}(x_1, x_2) \setminus \{x_2\}$. Now suppose that $v_k \in \text{arc}(x_k, x_{k+1}) \setminus \{x_{k+1}\}$ for some $k \in [m - 1]$. It is clear that there exists a $j \in [m] \setminus \{k\}$ such that

$$v_k \in (p_j + \gamma B_X) \cap S_X = \text{arc}(u_j, v_j).$$

If x_j lies in the open semicircle connecting v_k with $-v_k$ and containing x_k , then, since $\|u_k - v_k\| < \|x_k - x_j\|$, we have $x_j \in \text{arc}(u_k, -v_k)$. It follows that

$$\|p_j - v_k\| \geq \|v_k - x_j\| - \|x_j - p_j\| \geq \|x_k - x_j\| - \gamma > \gamma,$$

which is impossible. Therefore, x_j has to be in the semicircle connecting v_k with $-v_k$ and containing x_{k+1} . We claim that $j = k + 1$, since otherwise, by using the fact that $x_j \notin \text{arc}(x_k, x_{k+1})$, we would have

$$\|p_j - v_k\| \geq \|v_k - x_j\| - \|x_j - p_j\| \geq \|x_{k+1} - x_j\| - \gamma > \gamma,$$

a contradiction. Now it is clear that $\text{arc}(v_k, x_{k+1}) \subseteq \text{arc}(u_{k+1}, v_{k+1})$. If x_{k+2} lies in the semicircle connecting u_{k+1} and $-u_{k+1}$ and containing v_{k+1} , then

$$\|u_{k+1} - v_{k+1}\| \leq 2\gamma < \|x_{k+1} - x_{k+2}\| \leq \|u_{k+1} - x_{k+2}\|.$$

Otherwise, v_{k+1} lies in $\text{arc}(x_{k+1}, -u_{k+1})$. In both cases we have

$$v_{k+1} \in \text{arc}(x_{k+1}, x_{k+2}) \setminus \{x_{k+2}\}.$$

It follows by induction that

$$v_m \in \text{arc}(x_m, x_{m+1}) \setminus \{x_{m+1}\} = \text{arc}(x_m, x_1) \setminus \{x_1\}.$$

Thus the relative interior of $\text{arc}(v_m, x_1)$ is not contained in $\cup_{i \in [m]} (p_i + \gamma B_X)$, a contradiction. \square

REMARK 10. Proposition 14 in [12] proved a result similar to Theorem 9 for planar convex bodies that are strictly convex and smooth.

LEMMA 11. Let X be a Minkowski plane, $u \in S_X$, and $[a, b]$ and $[s, t]$ be two chords of S_X parallel to $\langle -u, u \rangle$ such that $\langle s, t \rangle$ lies strictly between $\langle a, b \rangle$ and $\langle -u, u \rangle$. Then

$$\|s + t\| < \|a + b\|. \quad (3)$$

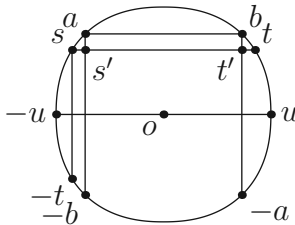


Figure 4: The proof of Lemma 11

Proof. It is not difficult to see that we only need to consider the case when $\|a + b\| < 2$. Without loss of generality we may assume that $t - s$ and $b - a$ are both positive scalar multiples of u . Since $\langle s, t \rangle$ lies strictly between $\langle a, b \rangle$ and $\langle -u, u \rangle$, the lines $\langle a, -b \rangle$ and $\langle -a, b \rangle$ intersect $\langle s, t \rangle$ in a point s' and t' , respectively; see Figure 4. Clearly, $[s', t'] \subseteq [s, t]$. If one of the two points s' and t' is in S_X (note that these two points lie in the relative interior of $[a, -b]$ and $[-a, b]$), then both $[a, -b]$ and $[-a, b]$ are contained in S_X . It follows that $[s, -t]$ and $[-s, t]$ are contained in the relative interiors of $[a, -b]$ and $[-a, b]$, respectively. Hence, (3) holds. In the following we assume that s' and t' are both interior points of B_X . Then s and $-t$ is contained in the open halfplane bounded by $\langle a, -b \rangle$ not containing o . By Lemma 4 and the fact that $\|a + b\| < 2$, we have (3). \square

Let $x, y \in X$. If $\|x + y\| = \|x - y\|$, then we say that x is *isosceles orthogonal* to y , denoted $x \perp_I y$, see [9].

COROLLARY 12. *Let X be a Minkowski plane, $u \in S_X$. Then there exists a unique pair of chords $[a, b]$ and $[-a, -b]$ parallel to $\langle -u, u \rangle$ such that $\|a + b\| = \|a - b\|$.*

Proof. By the uniqueness of isosceles orthogonality (see [1, Theorem 4.35]), there exists a unique $v \in S_X$ (except for the sign) such that $u \perp_I v$. Put

$$a = \frac{-u + v}{\|-u + v\|} \quad \text{and} \quad b = \frac{u + v}{\|u + v\|}.$$

Then

$$a, b \in S_X, \quad b - a = \frac{2}{\|u + v\|}u, \quad a + b = \frac{2}{\|u + v\|}v, \quad \|a - b\| = \frac{2}{\|u + v\|} = \|a + b\|.$$

Thus, $[a, b]$ and $[-a, -b]$ is a pair of chords having the desired properties.

In the following we show the uniqueness. Otherwise, there exists another pair of chords $[s, t]$ and $[-s, -t]$ having the required properties. Without loss of generality we may assume that $\langle s, t \rangle$ lies strictly between $\langle a, b \rangle$ and $\langle -u, u \rangle$, and that $t - s$ is a positive scalar multiple of u . Suppose that $s \in [a, -b]$ or $t \in [b, -a]$. Then $-t \in [a, -b] \subset S_X$, $-s \in [b, -a] \subset S_X$. It follows that

$$\|s + t\| < \|a + b\| = \|a - b\| = \|s - t\|,$$

a contradiction. Similarly, $s \notin [a, -b]$, $-s \notin [b, -a]$. Lemma 11 and Lemma 4 show that

$$\|s + t\| < \|a + b\| = \|a - b\| \leq \|s - t\|,$$

a contradiction. \square

LEMMA 13. *Let X be a Minkowski plane, and T be an equilateral triangle having a, b, c as vertices which is inscribed in S_X so that $o \in T$. Then*

$$\|a + b\| \leq \|a - b\|.$$

Proof. The case $\|a - b\| = 2$ is clear. In the rest of the proof we assume that

$$\|a - b\| = \|a - c\| = \|b - c\| < 2.$$

In this case, o is an interior point of T , which implies that $-c$ is a relatively interior point of arc (a, b) . Put

$$u = \frac{a - b}{\|a - b\|}.$$

Suppose the contrary, that $\|a + b\| > \|a - b\|$. Then there exists a unique pair of points $s, t \in S_X$ such that

1. $\langle s, t \rangle$ lies strictly between $\langle a, b \rangle$ and $\langle -u, u \rangle$,
2. $t - s$ is a positive scalar multiple of $-u$, and

$$3. \|s+t\| = \|s-t\|.$$

Since o is an interior point of T , c lies in the open halfplane bounded by $\langle -u, u \rangle$ not containing $[a, b]$. If $c \in \text{arc}(-u, -s)$, then T is completely contained in the open halfplane bounded by $\langle -s, s \rangle$ containing $[a, b]$, which is in contradiction to $o \in \text{int} T$. Thus $c \notin \text{arc}(-u, -s)$. Similarly, $c \notin \text{arc}(u, -t)$. It follows that c is a relatively interior point of $\text{arc}(-s, -t)$. By Lemma 1 and Lemma 4, we have

$$\begin{aligned} \|c-b\| &\geq \|-s-t\| = \|s+t\| = \|s-t\| > \|a-b\|, \\ \|c-a\| &\geq \|-t-s\| = \|s+t\| = \|s-t\| > \|a-b\|. \end{aligned}$$

These are in contradiction to the fact that T is equilateral. \square

THEOREM 14. *Let X be a Minkowski plane. Then*

$$\gamma_3(X) = \Gamma_3(X) = \frac{1}{2}S(3, X).$$

Proof. When S_X is a parallelogram, it is clear that

$$\gamma_3(X) = \Gamma_3(X) = \frac{1}{2}S(3, X) = 1.$$

In the following we assume that S_X is not a parallelogram. In this case we have $S(3, X) < 2$, and o is the interior of each equilateral triangle inscribed in S_X .

By Theorem 9, $\gamma_3(X) = (1/2)S(3, X)$. Let $\{a_1, a_2, a_3\}$ be the vertex set of an equilateral triangle inscribed in S_X whose side length is $\gamma = S(3, X)$. Let c_1, c_2, c_3 be the midpoint of $[a_1, a_2]$, $[a_2, a_3]$, and $[a_3, a_1]$, respectively. Then

$$S_X \subseteq \bigcup_{i \in [3]} \left(c_i + \frac{1}{2}S(3, m) \right),$$

and, by Lemma 13,

$$o \in \bigcap_{i \in [3]} \left(c_i + \frac{1}{2}S(3, m) \right).$$

It follows that

$$B_X \subseteq \bigcup_{i \in [3]} \left(c_i + \frac{1}{2}S(3, m) \right).$$

Thus $\Gamma_3(X) \leq (1/2)S(3, X)$, which implies that $\Gamma_3(X) = (1/2)S(3, X)$. \square

REMARK 15. The situation is much more complicated when $m \geq 5$. On the one hand, if S_X is a parallelogram, then we have

$$\frac{1}{2} = \gamma_4(X) \geq \gamma_5(X) \geq \gamma_6(X) = \frac{1}{2}$$

and

$$\gamma_6(X) \leq \Gamma_6(X) \leq \Gamma_5(X) \leq \Gamma_4(X) = \frac{1}{2}.$$

It follows that

$$\gamma_5(X) = \gamma_6(X) = \Gamma_5(X) = \Gamma_6(X) = \frac{1}{2}.$$

On the other hand, when X is the Euclidean plane we clearly have

$$\Gamma_5(X) > \gamma_5(X).$$

4. A better estimation for $\Gamma_m(X)$

Let X be a Minkowski space, $u \in S_X$ and $\varepsilon \in [0, 2]$. The *directional modulus of convexity* $\delta_X(u, \varepsilon)$ is defined by

$$\delta_X(u, \varepsilon) := \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in S_X, \exists \lambda \in \mathbb{R} \text{ s.t. } x - y = \lambda u \notin \text{int } \varepsilon B_X \right\}.$$

For each $u \in S_X$ and each $\lambda > 0$, we put

$$I(u, \lambda) = \{z \in S_X : z + \lambda u \in B_X\},$$

$$\mathcal{U}_m(X) = \left\{ \{u_i : i \in [m]\} \subset S_X : \exists \lambda > 0 \text{ such that } S_X \subseteq \bigcup_{i \in [m]} I(u_i, \lambda) \right\},$$

and, for each $U = \{u_i : i \in [m]\} \in \mathcal{U}_m(X)$,

$$\lambda(U) = \sup \left\{ \lambda > 0 : S_X \subseteq \bigcup_{i \in [m]} I(u_i, \lambda) \right\},$$

and

$$\delta(U) = 1 - \min \{ \delta_X(u_i, \lambda(U)) : i \in [m] \}.$$

Note that, for a particular choices of $m \in \mathbb{Z}^+$, $\mathcal{U}_m(X)$ might be empty. By standard compactness arguments, one can show that “sup” in the definition of $\lambda(U)$ can be replaced by “max”.

Now we improve the estimation of $\Gamma_m(X)$ provided in [8].

THEOREM 16. *Suppose that $m \in \mathbb{Z}^+$ and $\mathcal{U}_m(X) \neq \emptyset$. Then*

$$\Gamma_m(X) \leq \inf \left\{ \max \left\{ \delta(U), \frac{1}{2} \lambda(U) \right\} : U \in \mathcal{U}_m(X) \right\}.$$

Proof. Let $U = \{u_i : i \in [m]\}$ be an arbitrary element in $\mathcal{U}_m(X)$. We show that

$$\Gamma_m(X) \leq \max \left\{ \delta(U), \frac{1}{2} \lambda(U) \right\}.$$

The case when $\delta(U) = 1$ or $\lambda(U) = 2$ is clear. In the following we assume that $\delta(U) < 1$ and $\lambda(U) < 2$. In this case we have

$$\delta_X(u_i, \lambda(U)) \geq 1 - \delta(U) > 0, \forall i \in [m]. \quad (4)$$

Therefore, if $[a, b]$ is a chord of S_X whose length is not smaller than $\lambda(U)$ and which is parallel to $\langle -u_i, u_i \rangle$ for some $i \in [m]$, then $[a, b] \setminus \{a, b\} \subset \text{int} B_X$.

Let i be an arbitrary integer in $[m]$. Put

$$x_i = -\frac{\lambda(U)}{2} u_i.$$

We show that $I(u_i, \lambda(U)) \subseteq B_X(x_i, \delta(U))$, where $B_X(x_i, \delta(U))$ is the ball centered at x_i whose radius is $\delta(U)$.

For each $x \in I(u_i, \lambda(U))$, there exists a point $y \in S_X$ such that

$$[x, y] = B_X \cap (x + \langle -u_i, u_i \rangle).$$

Clearly, $\|x - y\| \geq \lambda(U)$. Whether x and y are linearly independent or not, there exist two points $u, v \in S_X$ such that

$$v - u = \lambda(U) u_i, \quad x, y \in \text{span}\{u, v\}, \quad [u, v] = \langle u, v \rangle \cap B_X,$$

and that $\langle x, y \rangle$ lies between $\langle -u_i, u_i \rangle$ and $\langle u, v \rangle$ (cf. the proof of Theorem 12 in [8]). Then

$$\|u - x_i\| = \left\| u + \frac{\lambda(U)}{2} u_i \right\| = \left\| \frac{1}{2}(u + v) \right\| \leq 1 - \delta_X(u_i, \lambda(U)) \leq \delta(U).$$

Lemma 1 shows that

$$\|x - x_i\| \leq \|u - x_i\| \leq \delta(U).$$

Therefore $I(u_i, \lambda(U)) \subseteq B_X(x_i, \delta(U))$.

Moreover,

$$o \in \bigcap_{i \in [m]} B_X \left(x_i, \frac{1}{2} \lambda(U) \right).$$

It follows that

$$B_X \subseteq \bigcup_{i \in [m]} B_X \left(x_i, \max \left\{ \delta(U), \frac{1}{2} \lambda(U) \right\} \right),$$

which completes the proof. \square

REMARK 17. The estimation of $\Gamma_m(X)$ above is better than the estimation given by Theorem 12 in [8]. Take, for example, $l_\infty^2 = (\mathbb{R}^2, \|\cdot\|_\infty)$. Theorem 12 in [8] gives $\Gamma_4(l_\infty^2) \leq 1$. Let $U = \{(1, 1), (-1, 1), (-1, -1), (1, -1)\}$. It is not difficult to see that $\lambda(U) = 1$ and $\delta(U) = 1/2$. Theorem 16 yields $\Gamma_4(l_\infty^2) \leq 1/2 = \Gamma_4(l_\infty^2)$, an estimation which is best possible.

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