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RELATIONS BETWEEN THE GENERALIZED BESSEL FUNCTIONS AND THE JANOWSKI CLASS

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Abstract. We are interested in finding the sufficient conditions on A, B, λ, b and c which ensure that the generalized Bessel functions $u_\lambda := u_{\lambda,b,c}$ satisfies the subordination $u_\lambda(z) \prec (1+Az)/(1+Bz)$. Also, conditions for which $u_\lambda(z)$ to be Janowski convex, and $zu'_\lambda(z)$ to be Janowski starlike in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ are obtained.

1. Introduction

We will denote by $\mathscr{A}$ the set of functions f, analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$, and normalized by the conditions $f(0) = 0 = f'(0) - 1$. If f and g are analytic in $\mathbb{D}$, then f is *subordinate* to g, written $f \prec g$ (or $f(z) \prec g(z)$, $z \in \mathbb{D}$), if there is an analytic self-map w of $\mathbb{D}$, satisfying $w(0) = 0$ and such that $f = g \circ w$. From now on, for $-1 \leqslant B < A \leqslant 1$ the set $\mathscr{P}[A,B]$ denotes a family of functions $p(z) = 1 + c_1 z + \cdots$, analytic in $\mathbb{D}$ and satisfying

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

That family, known as the *Janowski class of functions* [10], contains several other sets. For instance, if $0 \leqslant \beta < 1$, then $\mathscr{P}[1-2\beta, -1]$ is the class of functions $p(z) = 1 + c_1 z + \cdots$ satisfying $\operatorname{Re} p(z) > \beta$ in $\mathbb{D}$ which, in the limiting case $\beta = 0$, reduces to the classical Cárathèodory class $\mathscr{P}$.

In relation to $\mathscr{P}[A,B]$ several subclasses of $\mathscr{A}$ were defined, for example $\mathscr{S}^*[A,B]$, which is called *a class of Janowski starlike functions* [10] and that consists of $f \in \mathscr{A}$ satisfying

$$zf'(z)/f(z) \in \mathscr{P}[A,B].$$

For $0 \leqslant \beta < 1$, $\mathscr{S}^*[1-2\beta, -1] := \mathscr{S}^*(\beta)$ is the usual class of *starlike functions of order* β; $\mathscr{S}^*[1-\beta, 0] := \mathscr{S}^*_\beta = \{f \in \mathscr{A} : |zf'(z)/f(z) - 1| < 1-\beta\}$, and $\mathscr{S}^*[\beta, -\beta] := \mathscr{S}^*[\beta] = \{f \in \mathscr{A} : |zf'(z)/f(z) - 1| < \beta|zf'(z)/f(z) + 1|\}$. These classes have been

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studied, for example, in [1, 2]. A function $f \in \mathscr{A}$ is said to be *close-to-convex of order* β if $\operatorname{Re}(zf'(z)/g(z)) > \beta$ for some $g \in \mathscr{S}^* := \mathscr{S}^*(0)$ [9, 15].

The second order differential equation of a real variable x of a form

$$x^2u'' + xu' + (x^2 - \nu^2)u = 0, \tag{1.1}$$

is known as *the Bessel differential equation*, where the solutions of the Bessel equation yields the Bessel functions J_ν, Y_ν of the first and second kind , and $u = CJ_\nu(x) + DY_\nu(x)$ [16, p. 217]. Here C and D are the arbitrary constants and ν is an arbitrary complex number (the order of Bessel function). The Bessel functions are named for Bessel however Bernoulli is generally credited with being the first who introduced the concept of Bessel's functions in 1732, when solved the hunging chain problem. It is know that the Bessel function of the first kind of order ν is defined by [16]

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{2n+\nu}, \quad x \in \mathbb{R}. \tag{1.2}$$

Bessel's equation arises when solving the Laplace's and Helmholtz equation and are therefore especially important for many problems of wave propagation and static potentials. In finding the solution in cylindrical coordinate systems, one obtains Bessel functions of integer order ($\nu = n$), and in spherical problems one obtains half-integer orders ($\nu = n + 1/2$). There are several interesting facts concerning the Bessel functions, in particular the connections between Bessel functions and Legendre polynomials, hypergeometric functions, the usual trigonometric functions and other.

The Bessel functions are valid for complex arguments x, and an important special case is that of a purely imaginary argument. In this case, the solutions to the Bessel equation are called *the modified Bessel functions* or *the hyperbolic Bessel functions* of the first and second kind. Several applications have an impact of various generalizations and modifications.

A second order differential equation which reduces to (1.1) reads as follows

$$x^2v'' + bxv' + (cx^2 - \nu^2 + (1-b)\nu)u = 0, \tag{1.3}$$

$b, c, \nu \in \mathbb{R}$. A particular solution v_ν has the form

$$v_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!\Gamma(\nu+n+(b+1)/2}) \left(\frac{x}{2}\right)^{2n+\nu}, \tag{1.4}$$

and is called *the generalized Bessel function of the first kind of order* ν [4]. It is readily seen that for $b = 1$ and $c = 1$, v_ν becomes J_ν.

Study of the geometric properties of some cases of Bessel functions, like univalence, starlikeness and convexity were initiated in the sixties by Brown [8], and also by Kreyszig and Todd [11], but a major contributions to the development of a theory in this direction was made by Baricz et al. see, for example [3]–[7]. Motivated by the importance of the Bessel functions and the results in the theory of univalent functions we make a contribution to the subject, by obtaining some necessary and sufficient conditions for Janowski starlikeness and convexity of the generalized Bessel functions of the first kind.

For $b,\lambda,c\in\mathbb{C}$ and κ such that $\kappa=\lambda+(b+1)/2\neq 0,-1,-2,-3,\ldots$ we denote by $u_\lambda=u_{\lambda,b,c}$ the *normalized, generalized Bessel function of of the first kind of order* λ given by the power series

$$u_\lambda(z)=2^\lambda\Gamma(\kappa+1)z^{-\lambda/2}J_\kappa(\sqrt{z})={}_0F_1\left(\kappa,\frac{-c}{4}z\right)=\sum_{k=0}^{\infty}\frac{(-1)^k c^k}{4^k(\kappa)_k}\frac{z^k}{k!}, \tag{1.5}$$

convergent for all z on the complex plane. We note that $u_\lambda(0)=1$, u_λ is analytic in $\mathbb{D}$ and is a solution of the differential equation

$$4z^2u''(z)+4\kappa zu'(z)+czu(z)=0. \tag{1.6}$$

This normalized and generalized Bessel function also satisfy the following recurrence relation [3]

$$4\kappa u'_\lambda(z)=-cu_{\lambda+1}(z), \tag{1.7}$$

which is an useful tool to study several geometric properties of u_λ. There has been several papers, where geometric properties of u_λ such as on a close-to-convexity, starlikeness and convexity, radius of starlikeness and convexity, were studied [3, 4, 6, 7, 20, 21].

In this paper we systematically study the properties of the generalized Bessel function, specially Janowski convexity and Janowski starlikeness of that function.

In the section 2 of this paper, the sufficient conditions on A, B, c, κ are determined that will ensure that u_λ satisfies the subordination $u_\lambda(z)\prec(1+Az)/(1+Bz)$. It is understood that a computationally-intensive methodology is required to obtain the results in this general framework. The benefits of such general results are that by judicious choice of the parameters A and B, they give rise to several interesting applications, which include extension of the results of previous works. Using this subordination result, sufficient conditions are obtained for $(-4\kappa/c)u'_\lambda\in\mathscr{P}[A,B]$, which next readily gives conditions for $(-4\kappa/c)(u_\lambda-1)$ to be close-to-convex. Section 3 gives emphasis to the investigation of u_λ to be Janowski convex as well as of zu'_λ to be Janowski starlike.

The following lemma is needed in the sequel.

LEMMA 1.1. [14, 15] *Let $\Omega\subset\mathbb{C}$, and $\Psi:\mathbb{C}^2\times\mathbb{D}\to\mathbb{C}$ satisfy*

$$\Psi(i\rho,\sigma;z)\notin\Omega, \tag{1.8}$$

for real ρ and σ such that $\sigma\leqslant-(1+\rho^2)/2$, $z\in\mathbb{D}$. If p is analytic in $\mathbb{D}$ with $p(0)=1$, and $\Psi(p(z),zp'(z);z)\in\Omega$ for $z\in\mathbb{D}$, then $\operatorname{Re}p(z)>0$ *in $\mathbb{D}$. In the case $\Psi:\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$, then the condition (1.8) is generalized to*

$$\Psi(i\rho,\sigma,\mu+i\nu;z)\notin\Omega, \tag{1.9}$$

where ρ,σ,μ are real and such that $\sigma+\mu\leqslant 0$ and $\sigma\leqslant-(1+\rho^2)/2$.

2. Membership of the generalized Bessel functions to the Janowski class

In this section we shall discuss the problem of the membership of the generalized Bessel function in the Janowski class. We find the conditions under which $u_\lambda \in \mathscr{P}[A,B]$ and provide several consequences of that fact.

THEOREM 2.1. (Main) *Let* $-1 \leqslant B < A \leqslant 1$. *Suppose* $c, p, b \in \mathbb{C}$ *and* $\kappa = p + (b+1)/2 \neq 0, -1, -2, -3\cdots$, *satisfies*

$$\operatorname{Re}(\kappa - 1) \geqslant \begin{cases} \dfrac{|c|}{4(1+A)}\left(\sqrt{2(1+A^2)} + (1-A)\right) & for\ -1 = B < A \leqslant 3-2\sqrt{2}, \\ \dfrac{|c|(1+A)}{8\sqrt{A}} \quad and \quad \operatorname{Re}(\kappa-1) \leqslant \dfrac{|c|(1+A)}{2(1-A)} & for\quad B = -1, A > 3-2\sqrt{2}, \\ \dfrac{|c|(1+A)(1-B)^2}{4(A-B)(1+B)} - \dfrac{1+B}{(1-B)} & for\quad -1 < B < 0, \\ \dfrac{|c|(1+A)(1+B)}{4(A-B)} - \dfrac{1-B}{1+B} & for\quad B \geqslant 0. \end{cases} \tag{2.1}$$

If $(1+B)u_\lambda \neq (1+A)$, *then* $u_\lambda \in \mathscr{P}[A,B]$.

Proof. Define the analytic function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = -\frac{(1-A) - (1-B)u_\lambda(z)}{(1+A) - (1+B)u_\lambda(z)}, \quad p(0) = 1.$$

Then, a simple computation yields

$$u_\lambda(z) = \frac{(1-A) + (1+A)p(z)}{(1-B) + (1+B)p(z)}, \tag{2.2}$$

$$u'_\lambda(z) = \frac{2(A-B)p'(z)}{((1-B) + (1+B)p(z))^2}, \tag{2.3}$$

and

$$u''_\lambda(z) = \frac{2(A-B)[(1-B)+(1+B)p(z)]p''(z) - 4(1+B)(A-B)p'^2(z)}{((1-B)+(1+B)p(z))^3}. \tag{2.4}$$

Using the identities (2.2)–(2.4), the Bessel differential equation (1.6) can be rewritten as

$$z^2 p''(z) - \frac{2(1+B)}{(1-B)+(1+B)p(z)}(zp'(z))^2 + \kappa z p'(z)$$
$$+ \frac{((1-B)+(1+B)p(z))((1-A)+(1+A)p(z))}{8(A-B)} cz = 0.$$

Suppose $\Omega = \{0\}$, and define $\Psi(r,s,t;z)$ by

$$\Psi(r,s,t;z) := t - \frac{2(1+B)}{(1-B)+(1+B)r}s^2 + \kappa s + \frac{((1-B)+(1+B)r)((1-A)+(1+A)r)}{8(A-B)} cz. \tag{2.5}$$

The equation (2.5) yields that $\Psi(p(z),zp'(z),z^2p''(z);z)\in\Omega$. To ensure $\operatorname{Re}p(z)>0$ for $z\in\mathbb{D}$ we will use the Lemma 1.1. Hence, it suffices to establish $\operatorname{Re}\Psi(i\rho,\sigma,\mu+i\nu;z)\leqslant 0$ in $\mathbb{D}$ for real ρ,σ such that $\sigma\leqslant-(1+\rho^2)/2$, and $\sigma+\mu\leqslant 0$. Applying those inequalities we obtain

$$\begin{aligned}\operatorname{Re}\Psi(i\rho,\sigma,\mu+i\nu;z) &\leqslant -\frac{\operatorname{Re}(\kappa-1)}{2}(1+\rho^2)-\frac{2(1-B^2)\sigma^2}{(1-B)^2+(1+B)^2\rho^2}\\ &\quad+\operatorname{Re}\frac{[(1-B)+(1+B)i\rho][(1-A)+(1+A)i\rho]}{8(A-B)}cz\\ &\leqslant -\frac{\operatorname{Re}(\kappa-1)}{2}(1+\rho^2)-\frac{(1-B^2)(1+\rho^2)^2}{2[(1-B)^2+(1+B)^2\rho^2]}\\ &\quad+\frac{|(1-B)+(1+B)i\rho||(1-A)+(1+A)i\rho||c|}{8(A-B)}.\end{aligned}\tag{2.6}$$

The proof will be divided into four cases. Consider first $B=-1$, $B<A\leqslant 3-2\sqrt{2}$. The inequality (2.6) reduces then to the following

$$\begin{aligned}\operatorname{Re}\Psi(i\rho,\sigma,\mu+i\nu;z) &\leqslant -\frac{\operatorname{Re}(\kappa-1)(1+\rho^2)}{2}+\operatorname{Re}\frac{[(1-A)+(1+A)i\rho]cz}{4(1+A)}\\ &\leqslant -\frac{\operatorname{Re}(\kappa-1)(1+\rho^2)}{2}+\frac{|c|}{4(1+A)}[(1-A)+(1+A)|\rho|]\\ &= -\frac{\operatorname{Re}(\kappa-1)}{2}\rho^2+\frac{|c|}{4}|\rho|+\frac{|c|(1-A)}{4(1+A)}-\frac{\operatorname{Re}(\kappa-1)}{2}\\ &= -\frac{\operatorname{Re}(\kappa-1)}{2}\left(|\rho|-\frac{|c|}{4\operatorname{Re}(\kappa-1)}\right)^2+\frac{|c|^2}{32\operatorname{Re}(\kappa-1)}\\ &\quad+\frac{|c|(1-A)}{4(1+A)}-\frac{\operatorname{Re}(\kappa-1)}{2}\\ &=: G(\rho).\end{aligned}$$

A quadratic function G takes nonpositive values for any ρ, if

$$\frac{|c|^2}{32\operatorname{Re}(\kappa-1)}+\frac{|c|(1-A)}{4(1+A)}-\frac{\operatorname{Re}(\kappa-1)}{2}\leqslant 0.$$

The last inequality may be rewritten as

$$-\operatorname{Re}^2(\kappa-1)+\frac{|c|(1-A)}{2(1+A)}\operatorname{Re}(\kappa-1)+\frac{|c^2|}{16}\leqslant 0,$$

or

$$-\left(\operatorname{Re}(\kappa-1)-\frac{|c|(1-A)}{4(1+A)}\right)^2+\frac{|c|^2(1-A)^2}{16(1+A)^2}+\frac{|c|^2}{16}\leqslant 0,$$

that holds, if

$$\operatorname{Re}(\kappa-1)\geqslant\frac{|c|}{4(1+A)}\left(\sqrt{(1-A)^2+(1+A)^2}+(1-A)\right),$$

which reduces to the assumption. Therefore the assertion follows.

In the second case we consider $B=-1$, $A>3-2\sqrt{2}$. According to (2.6), we have

$$\begin{aligned}\operatorname{Re}\Psi(i\rho,\sigma,\mu+i\nu;z) &\leqslant -\frac{\operatorname{Re}(\kappa-1)(1+\rho^2)}{2}+\frac{|(1-A)+(1+A)i\rho||c|}{4(1+A)}\\ &= -\frac{\operatorname{Re}(\kappa-1)(1+\rho^2)}{2}+\frac{|c|}{4(1+A)}\sqrt{(1-A)^2+(1+A)^2\rho^2}\\ &=: H(\rho).\end{aligned}$$

We note that the function H is even with respect to ρ, and

$$H(0)=\frac{|c|(1-A)}{4(1+A)}-\frac{\operatorname{Re}(\kappa-1)}{2},$$

that satisfies $H(0)\leqslant 0$, if

$$\operatorname{Re}(\kappa-1)\geqslant\frac{|c|(1-A)}{2(1+A)}. \tag{2.7}$$

Moreover $\lim_{\rho\to\infty}H(\rho)=-\infty$, and

$$H'(\rho)=-\operatorname{Re}(\kappa-1)\rho+\frac{|c|(1+A)\rho}{4\sqrt{(1-A)^2+(1+A)^2\rho^2}},$$

with $H'(\rho)=0$ if and only if $\rho=0$ or

$$\rho_0^2=\frac{|c|^2}{16\operatorname{Re}^2(\kappa-1)}-\frac{(1-A)^2}{(1+A)^2}.$$

We observe that $\rho_0^2\geqslant 0$ by the inequality

$$\frac{|c|^2}{16\operatorname{Re}^2(\kappa-1)}\geqslant\frac{(1-A)^2}{(1+A)^2},$$

or

$$\operatorname{Re}(\kappa-1)\leqslant\frac{|c|(1+A)}{4(1-A)}. \tag{2.8}$$

Additionally

$$H''(\rho_0)=-\operatorname{Re}(\kappa-1)+\frac{16\operatorname{Re}^3(\kappa-1)(1-A)^2}{|c|^2(1+A)^2}\leqslant 0,$$

in view of (2.8). Hence $H(\rho_0)=H_{\max}(\rho)$, and

$$H(\rho_0)=\frac{|c|^2}{32\operatorname{Re}(\kappa-1)}-\frac{\operatorname{Re}(\kappa-1)}{2}\left[1-\left(\frac{1-A}{1+A}\right)^2\right]\leqslant 0$$

that holds if

$$\operatorname{Re}(\kappa-1) \geqslant \frac{|c|(1+A)}{8\sqrt{A}}. \tag{2.9}$$

Since

$$\frac{|c|(1+A)}{4(1-A)} \geqslant \frac{|c|(1+A)}{8\sqrt{A}} \geqslant \frac{|c|(1-A)}{2(1+A)}$$

holds for $3-2\sqrt{2} \leqslant A \leqslant 1$, then the conditions (2.7), (2.8) and (2.9) reduce to the assumption (2.1). Therefore the assertion follows.

Let now $-1 < B \leqslant 0$, $A > B$. By the fact $\frac{1-A}{1+A} < \frac{1-B}{1+B}$ we obtain

$$\begin{aligned} &|(1-B)+(1+B)i\rho||(1-A)+(1+A)i\rho| \\ &= (1+A)(1+B)\sqrt{\left(\frac{1-B}{1+B}\right)^2+\rho^2}\sqrt{\left(\frac{1-A}{1+A}\right)^2+\rho^2} \\ &\leqslant (1+A)(1+B)\left[\left(\frac{1-B}{1+B}\right)^2+\rho^2\right]. \end{aligned} \tag{2.10}$$

Also, for $B \leqslant 0$ we have $(1+B)/(1-B) \leqslant 1$, therefore

$$\frac{1+\rho^2}{(1-B)^2+(1+B)^2\rho^2} = \frac{1}{(1-B)^2}\frac{1+\rho^2}{1+\left(\frac{1+B}{1-B}\right)^2\rho^2} \geqslant \frac{1}{(1-B)^2}$$

for any real ρ. Thus

$$\begin{aligned} \operatorname{Re}\Psi(i\rho,\sigma,\mu+i\nu;z) \leqslant& -\frac{\operatorname{Re}(\kappa-1)}{2}(1+\rho^2) - \frac{(1+B)(1+\rho^2)}{2(1-B)} \\ &+\frac{|c|(1+A)(1+B)}{8(A-B)}\left[\left(\frac{1-B}{1+B}\right)^2+\rho^2\right] \\ =& \rho^2\left(-\frac{\operatorname{Re}(\kappa-1)}{2} - \frac{1+B}{2(1-B)} + \frac{|c|(1+A)(1+B)}{8(A-B)}\right) \\ &-\frac{\operatorname{Re}(\kappa-1)}{2} - \frac{1+B}{2(1-B)} + \frac{|c|(1+A)(1-B)^2}{8(A-B)(1+B)}. \end{aligned}$$

Since for $B \leqslant 0$

$$\begin{aligned} &-\frac{\operatorname{Re}(\kappa-1)}{2} - \frac{1+B}{2(1-B)} + \frac{|c|(1+A)(1+B)}{8(A-B)} \\ &\leqslant -\frac{\operatorname{Re}(\kappa-1)}{2} - \frac{1+B}{2(1-B)} + \frac{|c|(1+A)(1-B)^2}{8(A-B)(1+B)}, \end{aligned}$$

and the last expression is nonpositive in view of (2.1) then the assertion follows.

Finally, consider $0 \leqslant B < A \leqslant 1$. In this case $\beta = (1-B)/(1+B) \leqslant 1$. Hence, setting $t = \beta^2 + \rho^2$ with $t \geqslant \beta^2$ and using (2.10), we obtain from (2.6)

$$\begin{aligned}
\operatorname{Re}\Psi(i\rho,\sigma,\mu+iv;z) &\leqslant -\frac{\operatorname{Re}(\kappa-1)}{2}(1-\beta^2+t)-\frac{\beta(1-\beta^2+t)^2}{2t}+\frac{|c|(1+A)(1+B)}{8(A-B)}t \\
&= t\left\{-\frac{\operatorname{Re}(\kappa-1)}{2}-\frac{\beta}{2}+\frac{|c|(1+A)(1+B)}{8(A-B)}\right\} \\
&\quad -\frac{\operatorname{Re}(\kappa-1)}{2}(1-\beta^2)-\frac{\beta(1-\beta^2)^2}{2t}-\beta(1-\beta^2)
\end{aligned}$$

that is nonpositive because of the inequality

$$\operatorname{Re}(\kappa-1) \geqslant \frac{|c|(1+A)(1+B)}{4(A-B)} - \frac{1-B}{1+B},$$

that is equivalent to the assumption (2.1).

Taking into account the above reasoning we see that Ψ satisfies the hypothesis of Lemma 1.1, and thus $\operatorname{Re}\, p(z) > 0$, that is,

$$-\frac{(1-A)-(1-B)u_\lambda(z)}{(1+A)-(1+B)u_\lambda(z)} \prec \frac{1+z}{1-z}.$$

Hence there exists an analytic self-map w of $\mathbb{D}$ with $w(0) = 0$ such that

$$-\frac{(1-A)-(1-B)u_\lambda(z)}{(1+A)-(1+B)u_\lambda(z)} = \frac{1+w(z)}{1-w(z)},$$

which implies that $u_\lambda(z) \prec (1+Az)/(1+Bz)$. $\square$

By the recurrence relation (1.7), we have

$$\operatorname{Re} u_{\lambda+1} = \operatorname{Re}\left(\frac{-4\kappa}{c}u'_\lambda\right),$$

therefore as an immediate consequence of Theorem 2.1 we obtain the following.

THEOREM 2.2. *Let* $-1 \leqslant B < A \leqslant 1$. *Suppose* $c, \lambda, b \in \mathbb{C}$ *and* $\kappa = \lambda + (b+1)/2 \neq 0, -1, -2, -3\cdots$, *satisfy*

$$\operatorname{Re}(\kappa) \geqslant \begin{cases}
\dfrac{|c|}{4(1+A)}\left(\sqrt{2(1+A^2)}+(1-A)\right) & \text{for } -1 = B < A \leqslant 3-2\sqrt{2}, \\
\dfrac{|c|(1+A)}{8\sqrt{A}} \text{ and } \operatorname{Re}(\kappa) \leqslant \dfrac{|c|(1+A)}{4(1-A)} & \text{for } B = -1,\ A > 3-2\sqrt{2}, \\
\dfrac{|c|(1+A)(1-B)^2}{4(A-B)(1+B)} - \dfrac{1+B}{(1-B)} & \text{for } -1 < B < 0, \\
\dfrac{|c|(1+A)(1+B)}{4(A-B)} - \dfrac{1-B}{1+B} & \text{for } B \geqslant 0.
\end{cases}$$

If $(1+B)u_{\lambda+1}(z) \neq (1+A)$, *then* $(-4\kappa/c)u'_\lambda(z) \in \mathscr{P}[A,B]$.

3. Janowski convexity and starlikeness of the generalized Bessel functions

This section is devoted to the study of the Janowski convexity and the Janowski starlikeness of the normalized and generalized Bessel functions zu_λ. We proceed analogously to the proof of Theorem 2.1 applying modification of the Bessel differential equation (1.6) and Lemma 1.1. An application of the Janowski convexity and the relation (1.7) yield conditions for zu_λ to be in $\mathscr{S}^*[A,B]$.

THEOREM 3.1. *Let* $-1 \leqslant B < A \leqslant 1$ *and* $\lambda, b, c \in \mathbb{C}$ *and* $\kappa = \lambda + (b+1)/2 \neq 0, -1, -2 \ldots$ *Suppose that*

$$\operatorname{Re}\kappa \geqslant \frac{(\operatorname{Im}\kappa)^2}{2(2+A)} + \frac{A}{2} + \frac{|c|}{2(A+1)} \quad \textit{for} \quad -1 = B < A \leqslant 1, \tag{3.1}$$

or, for $-1 < B < A \leqslant 1$

$$\frac{A-B-1}{1-B} + \frac{|c|(1-B)}{4(A-B)} \leqslant \operatorname{Re}\kappa < \frac{A-B+1}{1+B} - \frac{|c|(1+B)}{4(A-B)}, \tag{3.2}$$

and

$$\begin{aligned}(B\operatorname{Im}\kappa)^2 \leqslant & \left\{A-B+1-\frac{|c|(1+B)^2}{4(A-B)} - (1+B)\operatorname{Re}\kappa\right\} \\ & \times \left\{(1-B)\operatorname{Re}\kappa - \frac{|c|(1-B)^2}{4(A-B)} - A + B + 1\right\},\end{aligned} \tag{3.3}$$

with

$$|c| < \frac{4(A-B)(1+B^2-AB)}{1-B^2}. \tag{3.4}$$

If $(A-B)u_\lambda'(z) \neq (1+B)zu_\lambda''(z)$, $0 \notin u_\lambda'(\mathbb{D})$ *and* $0 \notin u_\lambda''(\mathbb{D})$, *then*

$$1 + \frac{zu_\lambda''(z)}{u_\lambda'(z)} \prec \frac{1+Az}{1+Bz}.$$

Proof. Define an analytic function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) := \frac{(A-B)u_\lambda'(z) + (1-B)zu_\lambda''(z)}{(A-B)u_\lambda'(z) - (1+B)zu_\lambda''(z)}, \quad p(0) = 1.$$

Then

$$\frac{zu_\lambda''(z)}{u_\lambda'(z)} = \frac{(A-B)(p(z)-1)}{(1-B)+(1+B)p(z)}, \tag{3.5}$$

and

$$\frac{z^2u_\lambda'''(z)+zu_\lambda''(z)}{zu_\lambda''(z)}-\frac{zu_\lambda''(z)}{u_\lambda'(z)}=\frac{zp'(z)}{p(z)-1}-\frac{(1+B)zp'(z)}{(1-B)+(1+B)p(z)}$$
$$=\frac{zp'(z)\left[(1-B)+(1+B)p(z)-(1+B)(p(z)-1)\right]}{(p(z)-1)[(1-B)+(1+B)p(z)]}. \tag{3.6}$$

A rearrangement of (3.6) yields

$$\frac{zu_\lambda'''(z)}{u_\lambda''(z)}=\frac{2zp'(z)}{(p(z)-1)[(1-B)+(1+B)p(z)]}-1+\frac{zu_\lambda''(z)}{u_\lambda'(z)}.$$

Thus,

$$\left(\frac{zu_\lambda'''(z)}{u_\lambda''(z)}\right)\left(\frac{zu_\lambda''(z)}{u_\lambda'(z)}\right)$$
$$=\frac{2(A-B)(p(z)-1)zp'(z)}{(p(z)-1)\left((1-B)+(1+B)p(z)\right)^2}-\frac{(A-B)(p(z)-1)}{(1-B)+(1+B)p(z)}+\frac{(A-B)^2(p(z)-1)^2}{\left((1-B)+(1+B)p(z)\right)^2}. \tag{3.7}$$

Now a differentiation of (1.6) leads to

$$4z^2u_\lambda'''(z)+4(\kappa+1)zu_\lambda''(z)+czu_\lambda'(z)=0,$$

which gives if $u_\lambda'\neq 0,\ u_\lambda''\neq 0$

$$\left(\frac{zu_\lambda'''(z)}{u_\lambda''(z)}\right)\left(\frac{zu_\lambda''(z)}{u_\lambda'(z)}\right)+(\kappa+1)\frac{zu_\lambda''(z)}{u_\lambda'(z)}+\frac{c}{4}z=0. \tag{3.8}$$

Substituting (3.5) and (3.7) into (3.8) we obtain

$$\frac{2(A-B)zp'(z)}{\left((1-B)+(1+B)p(z)\right)^2}+\frac{(A-B)^2(p(z)-1)^2}{\left((1-B)+(1+B)p(z)\right)^2}+\frac{\kappa(A-B)\,(p(z)-1)}{(1-B)+(1+B)p(z)}+\frac{c}{4}z=0,$$

or equivalently

$$zp'(z)+\frac{(A-B)}{2}(p(z)-1)^2+\frac{\kappa\,(p(z)-1)}{2}\left((1-B)+(1+B)p(z)\right)$$
$$+\frac{cz\left((1-B)+(1+B)p(z)\right)^2}{8(A-B)}=0.$$

Set now

$$\Psi(p(z),zp'(z);z):=zp'(z)+\frac{(A-B)}{2}(p(z)-1)^2+\frac{\kappa\,(p(z)-1)}{2}\left((1-B)+(1+B)p(z)\right)$$
$$+\frac{cz\left((1-B)+(1+B)p(z)\right)^2}{8(A-B)}.$$

Then for $\rho \in \mathbb{R}$ and $\sigma \leqslant -(1+\rho^2)/2$ we obtain

$$\begin{aligned}
\operatorname{Re}\Psi(i\rho,\sigma;z) &= \sigma + \frac{(A-B)}{2}\operatorname{Re}(i\rho-1)^2 + \operatorname{Re}\left(\frac{\kappa(i\rho-1)}{2}((1-B)+(1+B)i\rho)\right) \\
&\quad + \operatorname{Re}\left(\frac{cz((1-B)+(1+B)i\rho)^2}{8(A-B)}\right) \\
&\leqslant -\frac{1+\rho^2}{2} + \frac{(A-B)}{2}(1-\rho^2) + \operatorname{Re}\left(\frac{\kappa}{2}(-2Bi\rho-(1-B)-(1+B)\rho^2)\right) \\
&\quad + \frac{|c|\left((1-B)^2+(1+B)^2\rho^2\right)}{8(A-B)} \\
&= -\rho^2\left\{\frac{A-B+1}{2} - \frac{(1+B)\operatorname{Re}\kappa}{2} - \frac{|c|(1+B)^2}{8(A-B)}\right\} + (B\operatorname{Im}\kappa)\rho \\
&\quad + \frac{A-B-1}{2} - \frac{(1-B)\operatorname{Re}\kappa}{2} + \frac{|c|(1-B)^2}{8(A-B)} \\
&:= Q(\rho).
\end{aligned}$$

In order to get the contradiction we need to show $Q(\rho) \leqslant 0$ for $\rho \in \mathbb{R}$. We divide the proof into two cases. Consider first the case $B = -1 < A \leqslant 1$. Then the function Q becomes

$$Q(\rho) = -\frac{2+A}{2}\rho^2 - (\operatorname{Im}\kappa)\rho + \frac{A}{2} - \operatorname{Re}\kappa + \frac{|c|}{2(A+1)},$$

that attains its maximum at $\rho_0 = -\operatorname{Im}\kappa/(2+A)$, and

$$Q(\rho_0) = \frac{(\operatorname{Im}\kappa)^2}{2(2+A)} + \frac{A}{2} - \operatorname{Re}\kappa + \frac{|c|}{2(A+1)}$$

which is nonpositive by the assumption equivalent to (3.1), that is

$$\operatorname{Re}\kappa \geqslant \frac{(\operatorname{Im}\kappa)^2}{2(2+A)} + \frac{A}{2} + \frac{|c|}{2(A+1)}.$$

We now turn to the case $-1 < B < A \leqslant 1$. We rewrite Q in the form

$$Q(\rho) = -P\rho^2 + R\rho - S = -P\left\{\left(\rho - \frac{R}{2P}\right)^2 + \frac{4PS-R^2}{4P^2}\right\},$$

where

$$P = \frac{A-B+1}{2} - \frac{(1+B)\operatorname{Re}\kappa}{2} - \frac{|c|(1+B)^2}{8(A-B)},$$

$$R = B\operatorname{Im}\kappa, \quad S = \frac{(1-B)\operatorname{Re}\kappa}{2} - \frac{|c|(1-B)^2}{8(A-B)} - \frac{A-B-1}{2}.$$

The inequality $Q(\rho) \leqslant 0$ holds for any real ρ, if $P > 0, S \geqslant 0$ and $R^2 \leqslant 4PS$ or, equivalently

$$\begin{cases} \dfrac{A-B+1}{1+B} - \dfrac{|c|(1+B)}{4(A-B)} > \operatorname{Re}\kappa, \\[2ex] \dfrac{A-B-1}{1-B} + \dfrac{|c|(1-B)}{4(A-B)} \leqslant \operatorname{Re}\kappa, \end{cases} \tag{3.9}$$

and

$$(B\operatorname{Im}\kappa)^2 \leqslant \left\{A-B+1-\frac{|c|(1+B)^2}{4(A-B)}-(1+B)\operatorname{Re}\kappa\right\} \times\left\{(1-B)\operatorname{Re}\kappa-\frac{|c|(1-B)^2}{4(A-B)}-A+B+1\right\},$$

that holds by the hypothesis (3.2) and (3.3). The inequalities (3.9) can be satisfied only if

$$|c|<\frac{4(A-B)(1+B^2-AB)}{1-B^2}$$

that is equivalent to (3.4). Therefore, in both cases the function Ψ satisfies the hypothesis of Lemma 1.1, and hence $\operatorname{Re}\ p(z)>0$, or equivalently

$$\frac{(A-B)u_\lambda'+(1-B)zu_\lambda''}{(A-B)u_\lambda'-(1+B)zu_\lambda''}\prec\frac{1+z}{1-z}.$$

By definition of subordination, there exists an analytic self-map w of $\mathbb{D}$ with $w(0)=0$, and

$$\frac{(A-B)u_\lambda'(z)+(1-B)zu_\lambda''(z)}{(A-B)u_\lambda'(z)-(1+B)zu_\lambda''(z)}=\frac{1+w(z)}{1-w(z)},$$

that gives the equality

$$1+\frac{zu_\lambda''(z)}{u_\lambda'(z)}=\frac{1+Aw(z)}{1+Bw(z)}.$$

Hence

$$1+\frac{zu_\lambda''(z)}{u_\lambda'(z)}\prec\frac{1+Az}{1+Bz},$$

which is the desired conclusion. □

Based on the relation (1.7) we also show that

$$\frac{z\left(zu_\lambda'(z)\right)'}{zu_\lambda'(z)}=1+\frac{zu_{\lambda-1}''(z)}{u_{\lambda-1}'(z)}.$$

Applying the above and Theorem 3.1, the following result for $zu_\lambda'(z)\in\mathscr{S}^*[A,B]$ immediately follows.

THEOREM 3.2. *Let* $-1\leqslant B<A\leqslant 1$ *and* $\lambda,b,c\in\mathbb{C}$ *and* $\kappa=\lambda+(b+1)/2\neq 0,-1,-2\ldots$ *Suppose that*

$$\operatorname{Re}\kappa\geqslant\frac{(\operatorname{Im}\kappa)^2}{2(2+A)}+\frac{A}{2}+\frac{|c|}{2(A+1)}\quad \textit{for}\quad -1=B<A\leqslant 1, \tag{3.10}$$

or, for $-1 < B < A \leqslant 1$

$$\frac{A-B-1}{1-B} + \frac{|c|(1-B)}{4(A-B)} \leqslant \operatorname{Re}\kappa < \frac{A-B+1}{1+B} - \frac{|c|(1+B)}{4(A-B)}, \tag{3.11}$$

and

$$(B\operatorname{Im}\kappa)^2 \leqslant \left\{A-B+1-\frac{|c|(1+B)^2}{4(A-B)} - (1+B)\operatorname{Re}\kappa\right\} \times \left\{(1-B)\operatorname{Re}\kappa - \frac{|c|(1-B)^2}{4(A-B)} - A+B+1\right\}, \tag{3.12}$$

with

$$|c| < \frac{4(A-B)(1+B^2-AB)}{1-B^2}. \tag{3.13}$$

If $(A-B)u'_\lambda(z) \neq (1+B)zu''_\lambda(z)$, $0 \notin u'_\lambda(\mathbb{D})$ *and* $0 \notin u''_\lambda(\mathbb{D})$, *then* $zu'_\lambda(z) \in \mathscr{S}^*[A,B]$.

In the special case $B=-1$ and $A = 1-2\gamma$ we have from Theorem 3.1

COROLLARY 3.1. *Let* $\gamma \in [0,1)$ *and* $\lambda, b, c \in \mathbb{C}$ *and* $\kappa = \lambda + (b+1)/2 \neq 0, -1, -2\ldots$, *and*

$$\operatorname{Re}\kappa \geqslant \frac{(\operatorname{Im}\kappa)^2}{2(3-2\gamma)} + \frac{1-2\gamma}{2} + \frac{|c|}{4(1-\gamma)}. \tag{3.14}$$

If $0 \notin u'_\lambda(\mathbb{D})$ *then*

$$\operatorname{Re}\left(1 + \frac{zu''_\lambda(z)}{u'_\lambda(z)}\right) > \gamma.$$

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