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HARDY TYPE INEQUALITIES AND COMPACTNESS OF A CLASS OF INTEGRAL OPERATORS WITH LOGARITHMIC SINGULARITIES

AKBOTA M. ABYLAYEVA AND LARS-ERIK PERSSON

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Abstract. We establish criteria for both boundedness and compactness for some classes of integral operators with logarithmic singularities in weighted Lebesgue spaces for cases $1 < p \leqslant q < \infty$ and $1 < q < p < \infty$. As corollaries some corresponding new Hardy inequalities are pointed out.

1. Introduction

Let $0 < q < \infty$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $R_+ = (0, \infty)$. Moreover, let $u : R_+ \to R$ and $v : R_+ \to R$ be weight functions, i.e. non-negative measurable functions on R_+.

Since the 70-s of the last century weighted estimates of the form

$$\|vKf\|_q \leqslant C\|uf\|_p \tag{1}$$

are intensively studied in the literature for different classes of the operators K, where $\|\cdot\|_p$ is the usual norm of the space $L_p \equiv L_p(R_+)$. Review of research in the period $1970 - 1982$, where estimates of the form (1) are given, can be found in [5]. Some directions of research of the estimate (1) until 2009 for integral operators are summarized in the books [6, 11, 12, 14]. Estimates of the form (1) are considered not only in Lebesgue spaces but also in other function spaces (see. e.g. [4, 8, 17] and Chapter 11 of the book [11]). Moreover, in [18] a sequence of classes of non-negative functions $K(\cdot,\cdot)$ was considered and when the kernels $K(x,s)$ of an integral operator

$$Kf(x) = \int_0^x K(x,s) f(s) ds, \tag{2}$$

belong to these classes, a full description of weights v and u was given, so that, the estimate (1) holds for the operator K defined by (2). However, these results do not

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include the operator in the form of (2), when the kernel $K(\cdot,\cdot)$ have a singularity, for example, the Riemann-Liouville operator

$$R_\alpha f(x)=\int_0^x \frac{f(s)ds}{(x-s)^{1-\alpha}}, \tag{3}$$

when $0<\alpha<1$. The estimate of the form (1) remains open for the operator (3) in the general case. However, the following cases are studied: $v\equiv u$ in [3], $u\equiv 1$ in [15, 20] and u is non-decreasing in [7] and when one of the weighted functions v,u is non-increasing in [21].

The estimate (1) for a singular operator in a form

$$Kf(x)=\int_0^x s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds, \tag{4}$$

is equivalent to an estimate

$$\|K_\gamma f\|_q \leqslant C\|f\|_p \tag{5}$$

for the operator

$$K_\gamma f(x)=v(x)\int_0^x u(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds. \tag{6}$$

The estimate (5) is equivalent to the boundedness of the operator (6) from L_p to L_q with the norm $\|K_\gamma\|=C$, where C is the best constant in (5). The operator (4) in the case $\gamma=0$ is called a fractional integration operator of infinitesimal order [16].

The operator

$$K^*_\gamma f(s)=u(s)s^{\gamma-1}\int_s^\infty v(x)\ln\frac{x}{x-s}f(x)dx, \quad s>0, \tag{7}$$

is dual to the operator K_γ with respect to the scalar product $\int_0^\infty f(x)g(x)dx$.

The main purpose of this paper is to establish the boundedness and compactness of the operator (6) and the dual operator (7) from L_p to L_q.

In the case $u(x)\equiv 1$ of boundedness and of compactness from L_p to L_q of the operator (6) was studied in [1] and [2], respectively.

The main results (Theorems 1–4) are presented in Section 3. As corollaries some corresponding new Hardy type inequalities (Corollaries 1–4) are pointed out. The detailed proofs are given in Section 4 and in order not to disturb the argumentations in these proofs some auxiliary results are collected in Section 2.

CONVENTIONS. *Uncertainties of the type* $0\cdot\infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ *are assumed to be zero. The inequality of the form* $A\leqslant\beta B$ *is written in the form* $A\ll B$, *where the positive constant* β *may be dependent on the parameters* p, q, γ, *and the relation* $A\approx B$ *means that* $A\ll B\ll A$. $\chi_{(a,b)}(\cdot)$ *denotes a characteristic function of the interval* (a,b), Z *is the set of integer numbers. The notations* $\sum_k$, $\sup_k$ *mean* $\sum_{k\in Z}$, $\sup_{k\in Z}$, *respectively.*

2. Auxiliary results

Since

$$\ln\frac{x}{x-s}=\int_0^s\frac{dt}{x-t}\quad \text{for } x>s\geqslant 0, \tag{8}$$

the following inequalities

$$\frac{s}{x-s}>\ln\frac{x}{x-s}>\frac{s}{x},\quad x>s>0 \tag{9}$$

hold. The function $\ln\frac{x}{x-s}$ decreases with respect to x and increases with respects to s when $x>s\geqslant 0$, and from the inequality (9) it follows that the functions $x\ln\frac{x}{x-s}$, $\frac{1}{s}\ln\frac{x}{x-s}$ also decreases with respect to x and increases with respects to s when $x>s>0$. Indeed,

$$\frac{\partial}{\partial x}\left(x\ln\frac{x}{x-s}\right)=\ln\frac{x}{x-s}-\frac{s}{x-s}<0,$$

and

$$\frac{\partial}{\partial s}\left(\frac{1}{s}\ln\frac{x}{x-s}\right)=\frac{1}{s^2}\left(\frac{s}{x-s}-\ln\frac{x}{x-s}\right)>0$$

for $x>s>0$.

From (8) we have

$$\int_0^x\ln\frac{x}{x-s}f(s)ds=\int_0^x\int_0^s\frac{dt}{x-t}f(s)ds=\int_0^x\frac{1}{x-t}\int_t^x f(s)dsdt. \tag{10}$$

In the case when the function u is positive a.e. in R_+ we put $u(s)s^{\gamma-1}f(s)=g'(s)$. Then from (10) and (6) it follows that the inequality (5) is equivalent to the inequality

$$\left(\int_0^\infty\left|v(x)\int_0^x\frac{g(x)-g(s)}{x-s}ds\right|^q dx\right)^{\frac{1}{q}}\leqslant C\left(\int_0^\infty|g'(x)u^{-1}(x)x^{1-\gamma}|^p dx\right)^{\frac{1}{p}} \tag{11}$$

for the differentiable functions g.

Similarly, if the function v is positive a.e. in R_+, then the inequality (5) for the operator (7) is equivalent to the inequality

$$\left(\int_0^\infty\left|u(s)s^\gamma\int_s^\infty\frac{f(x)-f(s)}{x-s}\frac{dx}{x}\right|^q ds\right)^{\frac{1}{q}}\leqslant C\left(\int_0^\infty|f'(x)v^{-1}(x)|^p dx\right)^{\frac{1}{p}} \tag{12}$$

for any differentiable functions f. In this case we have that

$$\int_s^\infty\ln\frac{x}{x-s}f(x)dx=\int_s^\infty f(x)\int_x^\infty\frac{sdt}{t(t-s)}dx=s\int_s^\infty\frac{1}{t-s}\int_t^x f(s)ds\frac{dt}{t}.$$

Along with the operator K_γ defined by (6) we consider the operator H_γ defined by

$$H_\gamma f(x) = \frac{v(x)}{x}\int_0^x u(s)s^\gamma f(s)ds, \ \ x>0.$$

It is easy to see that

$$K_\gamma f \geqslant H_\gamma f \tag{13}$$

for $f \geqslant 0$. Let

$$A(x) = \left(\int_0^x u^{p'}(s)s^{\gamma p'}ds\right)^{\frac{1}{p'}}\left(\int_x^\infty \frac{v^q(t)}{t^q}dt\right)^{\frac{1}{q}}, \quad A = \sup_{x>0} A(x).$$

For the operator H_γ the following theorem holds [11, 12, 19]:

THEOREM A. *Let* $1<p\leqslant q<\infty$. *Then the operator* H_γ *is bounded from* L_p *to* L_q *if and only if* $A<\infty$. *Moreover,* $\|H_\gamma\| \approx A$.

REMARK 1. Here and below for any operator T the value $\|T\|$ denotes the norm of the operator T from L_p to L_q.

The corresponding result for the case $q<p$ reads:

THEOREM B. *Let* $0<q<p<\infty$, $p>1$. *The operator* H_γ *is bounded from* L_p *to* L_q *if and only if*

$$B = \left(\int_0^\infty\left(\int_x^\infty \frac{v^q(t)}{t^q}dt\right)^{\frac{p}{p-q}}\left(\int_0^x u^{p'}(s)s^{p'\gamma}ds\right)^{\frac{p(q-1)}{p-q}} u^{p'}(x)x^{p'\gamma}dx\right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\|H_\gamma\| \approx B$.

REMARK 2. In the case $1<q<p<\infty$, the constant B is equivalent to the constant

$$\widetilde{B} = \left(\int_0^\infty\left(\int_x^\infty \frac{v^q(t)}{t^q}dt\right)^{\frac{q}{p-q}}\left(\int_0^x u^{p'}(s)s^{p'\gamma}ds\right)^{\frac{q(p-1)}{p-q}} \frac{v^q(x)}{x^q}dx\right)^{\frac{p-q}{pq}}.$$

3. The main results

Our first main result reads:

THEOREM 1. *Let* $1<p\leqslant q<\infty$, $\gamma>\frac{1}{p}$, *and* $u(x)$ *be a non-increasing function. Then the operator* K_γ *defined by* (6)

i) is bounded from L_p *to* L_q *if and only if* $A<\infty$ *and, moreover,* $\|K_\gamma\|\approx A$,

ii) is compact from L_p *to* L_q *if and only if* $A<\infty$ *and* $\lim\limits_{x\to 0^+}A(x)=\lim\limits_{x\to\infty}A(x)=0$.

COROLLARY 1. *Let the function* u *be positive a.e. on* R_+ *and the conditions of Theorem* 1 *be fulfilled. Then the Hardy type inequality* (11) *holds if and only if* $A<\infty$. *Moreover,* $A\approx C$, *where* C *is the best constant in* (11).

The corresponding result for the case $q<p$ reads:

THEOREM 2. *Let* $p>1$, $0<q<p<\infty$ *and* $\gamma>\frac{1}{p}$. *Let* u *be a non-increasing function on* R_+. *Then the operator* K_γ *defined by* (6)

i) is bounded from L_p *to* L_q *if and only if* $B<\infty$ *and, moreover,* $\|K_\gamma\|\approx B$,

ii) is compact from L_p *to* L_q *if and only if* $B<\infty$ *when ever* $q>1$.

COROLLARY 2. *Let* $0<q<p<\infty$. *Let the function* u *be positive a.e. in* R_+ *and the conditions of Theorem* 2 *be fulfilled. Then the Hardy type inequality* (11) *holds if and only if* $B<\infty$. *Moreover,* $B\approx C$ *for the best constant* C *in* (11).

We define

$$A^*(x)=\left(\int_x^\infty \frac{v^{p'}(t)}{t^{p'}}dt\right)^{\frac{1}{p'}}\left(\int_o^x s^{q\gamma}u^q(s)ds\right)^{\frac{1}{q}},\quad A^*=\sup_{x>0}A^*(x),$$

and

$$B^*=\left(\int_0^\infty\left(\int_x^\infty \frac{v^q(t)}{t^{p'}}dt\right)^{\frac{q(p-1)}{p-q}}\left(\int_0^x s^{q\gamma}u^q(s)ds\right)^{\frac{q}{p-q}}x^{q\gamma}u^q(x)dx\right)^{\frac{p-q}{pq}}.$$

We consider the operator K_γ^* (defined by (7)) and its action from L_p to L_q. If $1<p,q<\infty$, then the operator K_γ^* is bounded (compact) from L_p to L_q if and only if the operator K_γ is bounded (compact) from $L_{q'}$ to $L_{p'}$. In this case the conditions $1<p\leqslant q<\infty$ and $1<q<p<\infty$ are equivalent to the conditions $1<q'\leqslant p'<\infty$ and $1<p'<q'<\infty$, respectively. Therefore from Theorems 1 and 2, we have the following:

THEOREM 3. *Let $1 < p \leqslant q < \infty$ and $\gamma > \frac{1}{p}$. Then the operator K_γ^* defined by* (7)

i) is bounded from L_p to L_q if only if $A^* < \infty$ and, moreover, $\|K_\gamma^*\| \approx A^*$,

ii) is compact from L_p to L_q if only if $A^* < \infty$ and $\lim\limits_{x\to 0^+} A^*(x) = \lim\limits_{x\to\infty} A^*(x) = 0$.

COROLLARY 3. *Let the function v be positive a.e. on R_+ and the conditions of Theorem 3 be fulfilled. Then the Hardy type inequality* (12) *holds if and only if $A^* < \infty$. Moreover, $A^* \approx C$, where C is the best constant in* (12).

THEOREM 4. *Let $1 < q < p < \infty$ and $\gamma > \frac{1}{p}$. Then the operator K_γ^* defined by* (7) *is bounded and compact from L_p to L_q if only if $B^* < \infty$ and, moreover, $\|K_\gamma^*\| \approx B^*$.*

COROLLARY 4. *Let the function v be positive a.e. on R_+ and the conditions of Theorem 4 be fulfilled. Then the Hardy type inequality* (12) *holds if and only if $B^* < \infty$. Moreover, $B^* \approx C$ for the best constant C in* (12).

4. Proofs of the main results

Proof of Theorem 1. *Proof of i)*. *Necessity.* Let the operator (6) be bounded from L_p to L_q. Then, in view of (13), the operator H_γ is bounded from L_p to L_q and $\|K_\gamma\| \geqslant \|H_\gamma\|$. Therefore, by Theorem A the value $A < \infty$ and

$$\|K_\gamma\| \gg A. \tag{14}$$

Sufficiency. Let $A < \infty$. Since $\ln\frac{x}{x-s} \geqslant 0$ when $x > s \geqslant 0$, then it is enough to prove the inequality (5) for $f \geqslant 0$. Let $0 \leqslant f \in L_p$. Then we have

$$\begin{aligned} \|K_\gamma f\|_q^q &= \sum_k \int_{2^k}^{2^{k+1}} v^q(x)\left(\int_0^x u(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right)^q dx \\ &\ll \sum_k \int_{2^k}^{2^{k+1}} v^q(x)\left(\int_0^{2^{k-1}} u(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right)^q dx \\ &\quad + \sum_k \int_{2^k}^{2^{k+1}} v^q(x)\left(\int_{2^{k-1}}^x u(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right)^q dx := I_1 + I_2. \end{aligned} \tag{15}$$

We estimate I_1 and I_2 separately. Using the monotonicity of the function $\frac{1}{s}\ln\frac{x}{x-s}$ with respect to the variables x and s, we obtain that for $x > s \geqslant 0$

$$
\begin{aligned}
I_1 &\leqslant \sum_k \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_0^{2^{k-1}} u(s) s^\gamma \frac{1}{2^{k-1}} \ln \frac{2^k}{2^k - 2^{k-1}} f(s) ds \right)^q dx \\
&\leqslant (\ln 2)^q \sum_k \int_{2^k}^{2^{k+1}} \frac{v^q(x)}{(2^{k-1})^q} \left(\int_0^{2^{k-1}} u(s) s^\gamma f(s) ds \right)^q dx \\
&\ll \int_0^\infty \frac{v^q(x)}{x^q} \left(\int_0^x u(s) s^\gamma f(s) ds \right)^q dx = \|H_\gamma f\|_q^q. \qquad (16)
\end{aligned}
$$

In view of Theorem A from (16) it follows that

$$
I_1 \ll A^q \|f\|_q^q. \qquad (17)
$$

By now using the fact that the function u is increasing, applying Hölder's and Jensen's inequalities and making the change of the variable $s = xt$ in the integral below, we have

$$
\begin{aligned}
I_2 &\leqslant \sum_k u^q(2^{k-1}) \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_0^x s^{p'(\gamma-1)} \ln^{p'} \frac{x}{x-s} ds \right)^{\frac{q}{p'}} dx \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t) dt \right)^{\frac{q}{p}} \\
&\leqslant \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t) dt \right)^{\frac{q}{p}} u^q(2^{k-1}) \int_{2^k}^{2^{k+1}} v^q(x) x^{q(\gamma-1)} \left(\int_0^x \ln^{p'} \frac{x}{x-s} ds \right)^{\frac{q}{p'}} dx \\
&= \beta^{\frac{q}{p'}} \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t) dt \right)^{\frac{q}{p}} u^q(2^{k-1}) \int_{2^k}^{2^{k+1}} v^q(x) x^{q(\gamma-1)+\frac{q}{p'}} dx \\
&\ll \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t) dt \right)^{\frac{q}{p}} u^q(2^{k-1}) \left[2^{(k-1)(\gamma+\frac{1}{p'})} \left(\int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q} dx \right)^{\frac{1}{q}} \right]^q \\
&\ll \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t) dt \right)^{\frac{q}{p}} \left[u(2^{k-1}) \left(\int_0^{2^{k-1}} s^{p'\gamma} ds \right)^{\frac{1}{p'}} \left(\int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q} dx \right)^{\frac{1}{q}} \right]^q \\
&\ll \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t) dt \right)^{\frac{q}{p}} \left[\left(\int_0^{2^{k-1}} s^{p'\gamma} u^{p'}(s) ds \right)^{\frac{1}{p'}} \left(\int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q} dx \right)^{\frac{1}{q}} \right]^q \\
&\leqslant A^q \left(\sum_k \int_{2^{k-1}}^{2^{k+1}} f^p(t) dt \right)^{\frac{q}{p}} \ll A^q \|f\|_p^q, \qquad (18)
\end{aligned}
$$

where $\beta = \int_0^1 t^{p'(\gamma-1)} \ln^{p'} \frac{1}{1-t} dt$. The finiteness of β follows from the estimate

$$\beta \leqslant \ln^{p'} 2 \int_0^{\frac{1}{2}} s^{p'(\gamma-1)} ds + \max\{1, 2^{-p'(\gamma-1)}\} \int_{\ln 2}^{\infty} t^{p'} e^{-t} dt$$

and from the condition $\gamma > \frac{1}{p}$.

From (15), (17) and (18) it follows that

$$\|K_\gamma f\|_q \ll A \|f\|_p.$$

Hence, $\|K_\gamma\| \ll A$. This relation together with (14) gives $\|K_\gamma\| \approx A$. The statement *i*) of Theorem 1 is proved.

Proof of ii). *Necessity.* Let the operator K_γ be compact from L_p to L_q. Then the operator is bounded and therefore, by assertion *i*), $A < \infty$. First, we prove that $\lim\limits_{z \to 0^+} A(z) = 0$.

Consider the family of functions $\{f_t\}_{t \in I}$, where

$$f_t(x) = \chi_{(0,t)}(x) u^{p'-1}(x) x^{(p'-1)\gamma} \left(\int_0^t u^{p'}(s) s^{p'\gamma} ds \right)^{-\frac{1}{p}}. \tag{19}$$

Then

$$\int_0^\infty |f_t(x)|^p dx = \left(\int_0^t u^{p'}(s) s^{p'\gamma} ds \right)^{-1} \int_0^t u^{p'}(x) x^{p'\gamma} dx \equiv 1. \tag{20}$$

Next we show that the family of functions $\{f_t\}$ converges weakly to zero in L_p. Let $g \in L_{p'} = (L_p)^*$.

Applying the Hölder inequality and using (20) we have that

$$\int_0^\infty f_t(x) g(x) dx \leqslant \left(\int_0^t |f_t(x)|^p dx \right)^{\frac{1}{p}} \left(\int_0^t |g(x)|^{p'} dx \right)^{\frac{1}{p'}} = \left(\int_0^t |g(x)|^{p'} dx \right)^{\frac{1}{p'}}.$$

Since $g \in L_{p'}$, then the last integral converges to zero as $t \to 0^+$, which means the weak convergence to zero for the family of functions $\{f_t\}$. Then, by the compactness of the operator K_γ from L_p to L_q

$$\lim_{t \to 0^+} \|K_\gamma f_t\|_q = 0. \tag{21}$$

Since $\ln\frac{x}{x-s} \geqslant \frac{s}{x}$ for $x > s > 0$ we find that

$$\|K_\gamma f_t\|_q^q = \int_0^\infty v^q(x)\left(\int_0^x u(s)s^{\gamma-1}\ln\frac{x}{x-s}f_t(s)ds\right)^q dx$$
$$\geqslant \int_t^\infty \frac{v^q(x)}{x^q}\left(\int_0^t u(s)s^\gamma f_t(s)ds\right)^q dx$$
$$= \left(\int_0^t u^{p'}(s)s^{p'\gamma}ds\right)^{-\frac{q}{p}}\left(\int_0^t u^{p'}(s)s^{p'\gamma}ds\right)^q \int_t^\infty \frac{v^q(x)}{x^q}dx = (A(t))^q. \tag{22}$$

By combining (21) and (22) we obtain that $\lim\limits_{t\to 0^+} A(t) = 0$.

Now we prove that $\lim\limits_{t\to\infty} A(t) = 0$.

The compactness of the operator $K_\gamma : L_p \to L_q$ implies the compactness of the dual operator (7) from $L_{q'}$ to $L_{p'}$.

We introduce the family of functions $\{g_t\}_{t\in I}$, where

$$g_t(x) = \chi_{(t,\infty)}(x)\left(\int_t^\infty \frac{v^q(x)}{x^q}dx\right)^{-\frac{1}{q'}} \frac{v^{q-1}(x)}{x^{q-1}}.$$

Since $A < \infty$, then the function g_t is well defined.

In view of the equality

$$\int_0^\infty |g_t(x)|^{q'}dx = \left(\int_t^\infty \frac{v^q(x)}{x^q}dx\right)^{-1}\left(\int_t^\infty \frac{v^q(x)}{x^q}dx\right) = 1$$

for $f \in L_q = (L_{q'})^*$ we see that

$$\int_0^\infty f(x)g_t(x)dx \leqslant \left(\int_t^\infty |f(x)|^q dx\right)^{\frac{1}{q}}\left(\int_t^\infty |g_t(x)|^{q'}dx\right)^{\frac{1}{q'}} = \left(\int_t^\infty |f(x)|^q dx\right)^{\frac{1}{q}}.$$

Consequently, $\lim\limits_{t\to\infty}\int_0^\infty f(x)g_t(x)dx = 0$ for any $f \in L_q$, which means the weak convergence to zero of the family of functions $\{g_t\}$. Then, by the compactness of the operator K_γ^* from $L_{q'}$ to $L_{p'}$, it follows that

$$\lim_{t\to\infty}\|K_\gamma^* g_t\|_{p'} = 0. \tag{23}$$

Again using that $\ln\frac{x}{x-s} \geqslant \frac{s}{x}$ for $x > s > 0$, we obtain that

$$\|K_\gamma^* g_t\|_{p'}^{p'} \geqslant \int_0^t |u(s)s^{\gamma-1}|^{p'} \left(\int_t^\infty v(x) \ln \frac{x}{x-s} g_t(x) dx \right)^{p'} ds$$

$$\geqslant \int_0^t u^{p'}(s) s^{p'\gamma} ds \left(\int_t^\infty \frac{v^q(x)}{x^q} dx \right)^{-\frac{p'}{q'}} \left(\int_a^t \frac{v^q(x)}{x^q} dx \right)^{p'} = A^{p'}(t). \quad (24)$$

By combining (23) and (24) it follows that $\lim_{t\to\infty} A(t) = 0$. The necessity of statement *ii*) is proved.

Sufficiency. Let $A < \infty$ and $\lim_{z\to 0^+} A(z) = \lim_{z\to\infty} A(z) = 0$.

For $0 < c < d < \infty$ we define

$$P_c f = \chi_{(0,c]} f, \qquad P_{cd} f = \chi_{(c,d]} f, \qquad Q_d f = \chi_{(d,\infty)} f.$$

Then $f = P_c f + P_{cd} f + Q_d f$ and since $P_c K_\gamma P_{cd} \equiv 0$, $P_c K_\gamma Q_d \equiv 0$, $P_{cd} K_\gamma Q_d \equiv 0$, we have that

$$K_\gamma f = P_{cd} K_\gamma P_{cd} f + P_c K_\gamma P_c f + P_{cd} K_\gamma P_c f + Q_d K_\gamma f. \quad (25)$$

We show that the operator $P_{cd} K_\gamma P_{cd}$ is compact from L_p to L_q. Since $P_{cd} K_\gamma P_{cd} f(x) = 0$ for $x \in I \setminus (c,d)$, then it is enough to show that the operator $P_{cd} K_\gamma P_{cd}$ is compact from $L_p(c,d)$ to $L_q(c,d)$. This, in turn, is equivalent to compactness of the operator

$$Tf(x) = \int_c^d K(x,s) f(s) ds$$

from $L_p(c,d)$ to $L_q(c,d)$ with the kernel

$$K(x,s) = u(s)s^{\gamma-1} v(x) \chi_{(c,d)}(x-s) \ln \frac{x}{x-s}.$$

Next we note that there are the points $2^i, 2^n$, $n > i$ such that $2^i \leqslant c < 2^{i+1}$, $2^{n-1} < d \leqslant 2^n$. We assume that the numbers c and d are chosen so that $2^{i+1} < 2^{n-1}$. Then arguing as in the estimates of I_1 and I_2 in Theorem 1, we find that

$$\int_c^d \left(\int_c^d |K(x,s)|^{p'} ds \right)^{\frac{q}{p'}} dx = \int_c^d v^q(x) \left(\int_c^x u^{p'}(s) s^{p'(\gamma-1)} \left(\ln \frac{x}{x-s} \right)^{p'} ds \right)^{\frac{q}{p'}} dx$$

$$\ll \sum_{k=i}^{n-1} \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_0^{2^{k-1}} u^{p'}(s) s^{p'(\gamma-1)} \left(\ln \frac{x}{x-s} \right)^{p'} ds \right)^{\frac{q}{p'}} dx$$

$$+ \sum_{k=i}^{n-1} \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_{2^{k-1}}^x u^{p'}(s) s^{p'(\gamma-1)} \left(\ln \frac{x}{x-s} \right)^{p'} ds \right)^{\frac{q}{p'}} dx$$

$$\leqslant \mu(n-i+1) A < \infty,$$

where the constant μ does not depend on i and n. Therefore, on the basis of Kantorovich condition [9] (page 589), the operator T is compact from $L_p(c,d)$ to $L_q(c,d)$, which is equivalent to the compactness of the operator $P_{cd}K_\gamma P_{cd}$ from L_p to L_q .

From (25) it follows that

$$\|K_\gamma - P_{cd}K_\gamma P_{cd}\| \leqslant \|P_cK_\gamma P_c\| + \|P_{cd}K_\gamma P_c\| + \|Q_dK_\gamma\|. \tag{26}$$

We show that the right side of (26) tends to zero at $c \to 0^+$ and $d \to \infty$. Then it follows that the operator K_γ as the uniform limit of compact operators is compact from L_p to L_q.

By statement $i)$ we have that

$$\begin{aligned}
\|P_cK_\gamma P_cf\|_q &= \left(\int_0^c v^q(x)\left|\int_0^x u(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right|^q dx\right)^{\frac{1}{q}} \\
&\ll \sup_{0<z<c}\left(\int_0^z u^{p'}(s)s^{p'\gamma}ds\right)^{\frac{1}{p'}}\left(\int_z^c v^q(x)x^{-q}dx\right)^{\frac{1}{q}}\|f\|_p \\
&\leqslant \sup_{0<z<c} A(z)\|f\|_p.
\end{aligned}$$

Consequently, $\|P_cK_\gamma P_c\| \ll \sup\limits_{0<z<c} A(z)$. Hence,

$$\lim_{c\to 0^+}\|P_cK_\gamma P_c\| \ll \lim_{c\to 0^+}\sup_{0<z<c} A(z) = \lim_{c\to 0^+} A(c) = 0. \tag{27}$$

Let $v_d = Q_dv$. Then, by using statement $i)$, we find that

$$\begin{aligned}
\|Q_dK_\gamma f\|_q &= \left(\int_0^\infty v_d^q(x)\left|\int_0^x u(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right|^q dx\right)^{\frac{1}{q}} \\
&\ll \sup_{0<z}\left(\int_0^z u^{p'}(s)s^{p'\gamma}ds\right)^{\frac{1}{p'}}\left(\int_z^\infty v_d^q(x)x^{-q}dx\right)^{\frac{1}{q}}\|f\|_p \\
&\leqslant \sup_{d<z} A(z)\|f\|_p.
\end{aligned}$$

Therefore,

$$\lim_{d\to\infty}\|Q_dK_\gamma\| \ll \lim_{d\to\infty} A(d) = 0. \tag{28}$$

Now we will prove that

$$\lim_{c\to 0^+}\|P_{cd}K_\gamma P_c\| = 0. \tag{29}$$

We put $v_{cd} = P_{cd}v$ and $u_c = P_c u$. It is obvious that the function u_c is non-increasing. Therefore, according to statement $i)$, we get that

$$\begin{aligned}\|P_{cd}K_\gamma P_c f\|_q &= \left(\int_0^\infty v_{cd}^q(x)\left|\int_0^x u_c(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right|^q dx\right)^{\frac{1}{q}} \\ &\ll \sup_{0<z}\left(\int_0^z u_c^{p'}(s)s^{p'\gamma}ds\right)^{\frac{1}{p'}}\left(\int_z^\infty v_{cd}^q(x)x^{-q}dx\right)^{\frac{1}{q}}\|f\|_p \\ &\leqslant A(c)\|f\|_p.\end{aligned}$$

and we conclude that equality (29) holds.

From (27), (28) and (29) it follows that the right side of (26) tends to zero at $c \to 0^+$ and $d \to \infty$. Hence, also the sufficiency of $ii)$ is proved. The proof is complete. □

Proof of Theorem 2. Proof of statement $i)$. Necessity. Let the operator (6) be bounded from L_p to L_q. Then, in view of (13), the operator H_γ is bounded from L_p to L_q and $\|K_\gamma\| \geqslant \|H_\gamma\|$. Therefore, by Theorem B the value $B < \infty$ and

$$\|K_\gamma\| \gg B. \tag{30}$$

Sufficiency. Let $B < \infty$. We have the estimate (15) for $0 \leqslant f \in L_p$. In view of Theorem B and from (16) we have that

$$I_1 \ll B^q\|f\|_q^q. \tag{31}$$

Moreover, from the estimate I_2 in the proof of $i)$ of Theorem 1 it follows that

$$\begin{aligned} I_2 &\ll \sum_k\left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt\right)^{\frac{q}{p}} u^q(2^{k-1})2^{k\frac{q}{p'}(p'\gamma+1)}\int_{2^k}^{2^{k+1}}\frac{v^q(x)}{x^q}dx \\ &\ll \sum_k\left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt\right)^{\frac{q}{p}}\left(u^{p'}(2^{k-1})\int_{2^{k-2}}^{2^{k-1}} t^{p'\gamma}dt\right)^{\frac{q}{p'}}\int_{2^k}^{2^{k+1}}\frac{v^q(x)}{x^q}dx \\ &\leqslant \sum_k\left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt\right)^{\frac{q}{p}}\left(\int_{2^{k-2}}^{2^{k-1}} u^{p'}(t)t^{p'\gamma}dt\right)^{\frac{q}{p'}}\int_{2^k}^{2^{k+1}}\frac{v^q(x)}{x^q}dx. \end{aligned} \tag{32}$$

By now using the Hölder inequality with exponents $\frac{p}{q}$, $\frac{p}{p-q}$ and the estimate

$$\left(\int_{2^{k-2}}^{2^{k-1}} u^{p'}(t)t^{\gamma p'}dt\right)^{\frac{q(p-1)}{p-q}} \ll \int_{2^{k-2}}^{2^{k-1}}\left(\int_{2^{k-2}}^{x} u^{p'}(s)s^{\gamma p'}ds\right)^{\frac{p(q-1)}{p-q}} u^{p'}(x)x^{\gamma p'}dx$$

in (32) we find that

$$I_2 \ll \left(\sum_k \left(\int_{2^{k-2}}^{2^{k-1}} u^{p'}(t) t^{\gamma p'} dt \right)^{\frac{q(p-1)}{p-q}} \left(\int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q} dx \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{q}} \left(\sum_k \int_{2^{k-1}}^{2^{k+1}} f^p(t) dt \right)^{\frac{q}{p}}$$

$$\ll \left(\sum_k \int_{2^{k-2}}^{2^{k-1}} \left(\int_0^x u^{p'}(s) s^{\gamma p'} ds \right)^{\frac{p(q-1)}{p-q}} \left(\int_x^\infty \frac{v^q(t)}{t^q} dt \right)^{\frac{p}{p-q}} u^{p'}(x) x^{\gamma p'} dx \right)^{\frac{p-q}{q}}$$

$$\times \left(\sum_k \int_{2^{k-1}}^{2^{k+1}} f^p(t) dt \right)^{\frac{q}{p}}$$

$$\ll B^q \|f\|_p^q. \tag{33}$$

From (16), (31) and (33) we obtain the estimate

$$\|K_\gamma f\|_q \ll B \|f\|_p ,$$

which together with (30) gives $\|K_\gamma\| \approx B$. The statement $i)$ is proved.

Proof of $ii)$. *Necessity.* Let the operator K_γ be compact from L_p to L_q. Then the operator is bounded and therefore, by assertion $i)$, $B < \infty$.

Sufficiency. Let $A < \infty$. Here we have $K_\gamma f = P_d K_\gamma P_d f + P_d K_\gamma Q_d f + Q_d K_\gamma f$. Therefore

$$\|K_\gamma - P_d K_\gamma P_d\| \leqslant \|P_d K_\gamma Q_d\| + \|Q_d K_\gamma\|. \tag{34}$$

Since $d < \infty$, then from the Ando theorem and its generalizations (see e.g. [10]) the operator $P_d K_\gamma P_d$ is compact from $L_p(0,d)$ to $L_q(0,d)$, which is equivalent to the compactness of it from L_p to L_q. We show that the right-hand side (34) tends to zero as $d \to \infty$. Then the operator K_γ is compact from L_p to L_q as the uniform limit of compact operators. Similarly as in the proof of $ii)$ of Theorem 1 we find that

$$\|Q_d K_\gamma f\|_q = \left(\int_0^\infty v_d^q(x) \left| \int_0^x u(s) s^{\gamma-1} \ln \frac{x}{x-s} f(s) ds \right|^q dx \right)^{\frac{1}{q}}.$$

Then, in view of the statement $i)$,

$$\|Q_d K_\gamma\| \ll \left(\int_d^\infty \left(\int_z^\infty u^{p'}(s) s^{p'\gamma} ds \right)^{\frac{q(p-1)}{p-q}} \left(\int_d^z v^q(x) x^{-q} dx \right)^{\frac{q}{p-q}} v^q(z) z^{-q} dz \right)^{\frac{(p-q)}{pq}}.$$

From this estimate and the fact that $B < \infty$ it follows that

$$\lim_{d \to \infty} \|Q_d K_\gamma\| = 0. \tag{35}$$

Let $v_{dd} = P_d v$ and $u_d = Q_d u$. Then, using again statement $i)$, we obtain that

$$\begin{aligned}\|P_d K_\gamma Q_d f\|_q &= \left(\int_0^\infty v_{dd}^q(x)\left|\int_0^x u_d(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right|^q dx\right)^{\frac{1}{q}} \\ &\ll \left(\int_d^\infty u^{p'}(s)s^{p'\gamma}ds\right)^{\frac{1}{p'}}\left(\int_0^d v^q(x)x^{-q}dx\right)^{\frac{1}{q}}\|f\|_p \\ &= A(d)\|f\|_p. \qquad (36)\end{aligned}$$

We also note that, by Remark 2, $B \approx \widetilde{B}$. Since

$$\begin{aligned}A(d) &\ll \widetilde{B}(d,\infty) \\ &= \left(\int_d^\infty \left(\int_x^\infty \frac{v^q(t)}{t^q}dt\right)^{\frac{q}{p-q}}\left(\int_0^x u^{p'}(s)s^{p'\gamma}ds\right)^{\frac{q(p-1)}{p-q}}\frac{v^q(x)}{x^q}dx\right)^{\frac{p-q}{pq}}\end{aligned}$$

then from (36) we have that $\lim\limits_{d\to\infty}\|P_d K_\gamma Q_d\| = 0$. From this and from (35) it follows that the right-hand side of (34) tends to zero at $d \to \infty$. Therefore also the sufficiency part of $ii)$ is proved. The proof is complete. □

Finally, we remark that as mentioned before the proofs of Theorem 3 and 4 follows by using Theorems 1 and 2, respectively, and a standard duality argument.

REMARK 3. The current status of the mentioned open question to characterize the Hardy type inequality (1) - (2) without restriction on the kernel $\mathscr{K}(x,s)$ was recently described in [13]. However, the cases considered in this paper are new and can not be found there.

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Akbota M. Abylayeva
L. N. Gumilev Eurasian National University
Kazakhstan
e-mail: abylayeva_b@mail.ru

Lars-Erik Persson
Luleå University of Technology
Sweden
and
UiT, The Artic University of Norway
Norway
and
RUDN University
Moscow, Russia
e-mail: larserik@ltu.se