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NORM INEQUALITIES AND CHARACTERIZATIONS OF INNER PRODUCT SPACES

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Abstract. Let $(X, \|.\|)$ be a real normed space and let $\theta : (0,\infty) \to (0,\infty)$ be an increasing function such that $t \mapsto \frac{t}{\theta(t)}$ is non-decreasing on $(0,\infty)$. For such function, we introduce the notion of θ-angular distance $\alpha_\theta[x,y]$, where $x,y \in X \setminus \{0\}$, and show that X is an inner product space if and only if $\alpha_\theta[x,y] \leqslant 2\frac{\|x-y\|}{\theta\|x\|+\theta\|y\|}$ for each $x,y \in X \setminus \{0\}$. Then, in order to generalize the Dunkl-Williams constant of X [10], we introduce a new geometric constant $C_F(X)$ for X wrt F, where $F : (0,\infty)\times(0,\infty) \to (0,\infty)$ is a given function, and obtain some characterizations of inner product spaces related to the constant $C_F(X)$. Our results generalize and extend various known results in the literature.

1. Introduction and preliminaries

In 1935, Jordan and von Neumann [11] characterized inner product spaces as normed spaces satisfying the parallelogram law. In 1948, Lorch [13] presented several characterizations of inner product spaces. Since then, the problem of finding necessary and sufficient conditions for a normed space to be an inner product space has been investigated by many mathematicians by considering some geometric aspects of underlying spaces.

There are interesting norm inequalities connected with the characterizations of inner product spaces [2]. One of celebrated characterizations of inner product spaces has been based on the so-called Dunkl-Williams inequality.

In 1936, Clarkson [4] introduced the concept of angular distance between nonzero elements x and y in a normed space $(X, \|.\|)$ as $\alpha[x,y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$. In 1964, Dunkl and Williams [8] showed that for any nonzero elements x,y in a normed space $(X, \|.\|)$, $\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant 4\frac{\|x-y\|}{\|x\|+\|y\|}$. In the same paper, the authors proved that the constant 4 can be replaced by 2 if X is an inner product space. Then Kirk and Smiley [12] completed this result by showing that the above inequality with 2 in place of 4 in fact characterizes

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the inner product spaces. Motivated by this fact, Jiménez-Melado et al. [10] defined the Dunkl-Williams constant $DW(X)$ of a normed space X, that is,

$$DW(X) := \sup\left\{ \frac{\|x\|+\|y\|}{\|x-y\|}\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| : \quad x, y \in X\setminus\{0\},\ x \neq y \right\}.$$

By the above mentioned result of Kirk and Smiley, $(X,\|.\|)$ is an inner product space if and only if $DW(X) = 2$. In 1993, Al-Rashed [1] generalized the work of Kirk and Smiley. His result can be reformulated as follows: The normed space $(X,\|.\|)$ is an inner product space if and only if

$$\alpha[x,y] \leqslant 2^{1/q} \frac{\|x-y\|}{(\|x\|^q + \|y\|^q)^{1/q}}, \quad \text{for each } x,y \in X\setminus\{0\},$$

where $q \in (0,1]$. As a generalization of the concept of angular distance, Maligranda [14] introduced the p-angular distance $\alpha_p[x,y]$ as follows:

$$\alpha_p[x,y] = \left\| \frac{x}{\|x\|^{1-p}} - \frac{y}{\|y\|^{1-p}} \right\|, \text{ where } p \geqslant 0 \text{ and } x,y \in X\setminus\{0\}.$$

In 2010, Dadipour and Moslehian [5] presented a characterization of inner product spaces related to the p-angular distance which is a generalization of the above mentioned results of Kirk and Smiley [12] and Al-Rashed [1]. In 2014, Tanaka, Ohwada and Saito [16] obtained a new characterization of inner product spaces related to norm inequalities.

Now, we recall some definitions and facts which will be needed in the next sections. For $x,y \in X$, x is said to be BJ-orthogonal to y, denoted by $x \perp_B y$, if $\|x\| \leqslant \|x+\gamma y\|$ for all $\gamma \in \mathbb{R}$. The BJ-orthogonality is homogeneous, that is, $x \perp_B y$ implies $\lambda x \perp_B \mu y$ for all $\lambda, \mu \in \mathbb{R}$. However, it is not symmetric in general, that is, $x \perp_B y$ does not necessarily imply $y \perp_B x$. It is known that if $\dim(X) \geqslant 3$, then BJ-orthogonality is symmetric if and only if X is an inner product space [3, 9].

The rest of the paper is organized as follows: In section 2, for an increasing function $\theta : (0,\infty) \to (0,\infty)$ for which $t \mapsto \frac{t}{\theta(t)}$ is non-decreasing on $(0,\infty)$, we introduce the notion of θ-angular distance $\alpha_\theta[x,y] = \left\| \frac{x}{\theta\|x\|} - \frac{y}{\theta\|y\|} \right\|$ between nonzero elements x and y in X and show that $(X,\|.\|)$ is an inner product space if and only if $\alpha_\theta[x,y] \leqslant 2\frac{\|x-y\|}{\theta\|x\|+\theta\|y\|}$ for each $x,y \in X\setminus\{0\}$. In section 3, in order to generalize the Dunkl-Williams constant of a normed space [10], we introduce a new geometric constant $C_F(X)$ of the normed space X wrt F, where $F : (0,\infty)\times(0,\infty) \to (0,\infty)$ is a given function satisfying $F^{-1}\{1\} \neq \emptyset$ and $\sigma_F = \inf_{r,s>0:\ F(r,s)=1}(r+s) \in (0,\infty)$. We show that if X is an inner product space then $C_F(X) = \frac{2}{\sigma_F}$, and by giving an example we prove the converse implication does not hold in general. We also study some conditions on F under which, the equality $C_F(X) = \frac{2}{\sigma_F}$ characterize inner product spaces among all normed spaces. Our results generalize some well known characterizations of inner product spaces due to Lorch [13], Kirk and Smiley [12], Al-Rashed [1], Dadipour and Moslehian [5], Dehghan [7], and Tanaka, Ohwada and Saito [16].

2. θ-angular distance

Throughout the paper, let $(X,\|.\|)$ be a real normed space and let S_X denote the unit sphere of X. Let Θ denote the set of all increasing functions $\theta:(0,\infty)\to(0,\infty)$ such that $t\mapsto\frac{t}{\theta(t)}$ is non-decreasing on $(0,\infty)$. For a given $\theta\in\Theta$, the θ-angular distance between nonzero elements $x,y\in X$ is denoted by $\alpha_\theta[x,y]$ and defined as $\alpha_\theta[x,y]=\left\|\frac{x}{\theta\|x\|}-\frac{y}{\theta\|y\|}\right\|$.

Note that if we take $\theta(t)=t^{1-p}$ for each $t>0$, where $p\in[0,1)$, then $\alpha_\theta[x,y]$ reduces to the concept of p-angular distance between x and y which was introduced and studied by Maligranda [14].

Here is our first result in this section.

THEOREM 1. *Let $(X,\|.\|)$ be an inner product space and let $\theta\in\Theta$. Then the following inequality holds*

$$\alpha_\theta[x,y]\leqslant 2\frac{\|x-y\|}{\theta\|x\|+\theta\|y\|},\ \text{for each } x,y\in X\setminus\{0\}. \tag{2.1}$$

Proof. Notice first that since θ is non-decreasing then $\frac{\theta\|y\|}{\theta\|x\|+\theta\|y\|}\leqslant\frac{1}{2}$ provided that $\|y\|\leqslant\|x\|$. Then, we have

$$\begin{aligned}\sup_{\substack{x,y\in X\setminus\{0\}\\ x\neq y}}\frac{\alpha_\theta[x,y]^2}{\left(\frac{\|x-y\|}{\theta\|x\|+\theta\|y\|}\right)^2}&=\sup_{\substack{x,y\in X\setminus\{0\}\\ x\neq y,\ \|y\|\leqslant\|x\|}}\frac{\left\|\frac{x}{\theta\|x\|}-\frac{y}{\theta\|y\|}\right\|^2}{\left\|\frac{\theta\|x\|}{\theta\|x\|+\theta\|y\|}\frac{x}{\theta\|x\|}-\frac{\theta\|y\|}{\theta\|x\|+\theta\|y\|}\frac{y}{\theta\|y\|}\right\|^2}\\ &\leqslant\sup_{\substack{x,y\in X\setminus\{0\}\\ x\neq y}}\sup_{t\in[0,\frac{1}{2}]}\frac{\left\|\frac{x}{\theta\|x\|}-\frac{y}{\theta\|y\|}\right\|^2}{\left\|(1-t)\frac{x}{\theta\|x\|}-t\frac{y}{\theta\|y\|}\right\|^2}\\ &=\sup_{\substack{x,y\in X\setminus\{0\}\\ x\neq y}}\frac{\left\|\frac{x}{\theta\|x\|}-\frac{y}{\theta\|y\|}\right\|^2}{\inf_{t\in[0,\frac{1}{2}]}\left\|(1-t)\frac{x}{\theta\|x\|}-t\frac{y}{\theta\|y\|}\right\|^2}.\end{aligned}$$

Then

$$\sup_{\substack{x,y\in X\setminus\{0\}\\ x\neq y}}\frac{\alpha_\theta[x,y]^2}{\left(\frac{\|x-y\|}{\theta\|x\|+\theta\|y\|}\right)^2}\leqslant\sup_{\substack{x,y\in X\setminus\{0\}\\ x\neq y}}\frac{\left\|\frac{x}{\theta\|x\|}-\frac{y}{\theta\|y\|}\right\|^2}{\inf_{t\in[0,\frac{1}{2}]}\left\|(1-t)\frac{x}{\theta\|x\|}-t\frac{y}{\theta\|y\|}\right\|^2}. \tag{2.2}$$

Let $f:[0,1]\to\mathbb{R}$ be defined by

$$\begin{aligned}f(t)&=\left\|(1-t)\frac{x}{\theta\|x\|}-t\frac{y}{\theta\|y\|}\right\|^2\\ &=\left\{\left(\frac{\|x\|}{\theta\|x\|}\right)^2+\left(\frac{\|y\|}{\theta\|y\|}\right)^2+2\left\langle\frac{x}{\theta\|x\|},\frac{y}{\theta\|y\|}\right\rangle\right\}t^2\\ &\quad-2\left\{\left(\frac{\|x\|}{\theta\|x\|}\right)^2+\left\langle\frac{x}{\theta\|x\|},\frac{y}{\theta\|y\|}\right\rangle\right\}t+\left(\frac{\|x\|}{\theta\|x\|}\right)^2.\end{aligned}$$

Since

$$f''(t) = 2\left\{\left(\frac{\|x\|}{\theta\|x\|}\right)^2 + \left(\frac{\|y\|}{\theta\|y\|}\right)^2 + 2\left\langle \frac{x}{\theta\|x\|}, \frac{y}{\theta\|y\|}\right\rangle\right\}$$
$$= 2\left\langle \frac{x}{\theta\|x\|} + \frac{y}{\theta\|y\|}, \frac{x}{\theta\|x\|} + \frac{y}{\theta\|y\|}\right\rangle \geqslant 0,$$

for each $t \in [0,1]$, then $f'(t)$ is non-decreasing on $[0,1]$. Since the function $t \mapsto \frac{t}{\theta(t)}$ is non-decreasing and $\|y\| \leqslant \|x\|$ then $f'(\frac{1}{2}) = \left(\frac{\|y\|}{\theta\|y\|}\right)^2 - \left(\frac{\|x\|}{\theta\|x\|}\right)^2 \leqslant 0$. Hence $f'(t) \leqslant 0$ for each $t \in [0,\frac{1}{2}]$ and so f is non-increasing on $[0,\frac{1}{2}]$. Thus

$$\inf_{t\in[0,\frac{1}{2}]} \left\|(1-t)\frac{x}{\theta\|x\|} - t\frac{y}{\theta\|y\|}\right\|^2 = \frac{1}{4}\left\|\frac{x}{\theta\|x\|} - \frac{y}{\theta\|y\|}\right\|^2, \tag{2.3}$$

from (2.2) and (2.3), we get that

$$\sup_{\substack{x,y\in X\setminus\{0\}\\ x\neq y}} \frac{\alpha_\theta[x,y]}{\frac{\|x-y\|}{\theta\|x\|+\theta\|y\|}} \leqslant 2,$$

and the proof is complete. □

The next result provides a reverse of Theorem 1.

THEOREM 2. *Let* $(X,\|.\|)$ *be a normed space with* $dim(X) \geqslant 3$ *and* $\theta \in \Theta$. *If for some* $q > 0$

$$\alpha_\theta[x,y] \leqslant 2^{\frac{1}{q}} \frac{\|x-y\|}{\left((\theta\|x\|)^q + (\theta\|y\|)^q\right)^{\frac{1}{q}}}, \quad \textit{for each } x,y \in X\setminus\{0\}. \tag{2.4}$$

Then X *is an inner product space.*

Proof. We will show that the $BJ-$orthognality is symmetric. Let $x,y \in X\setminus\{0\}$ be such that $x \perp_B y$. Then

$$\|\alpha x\| \leqslant \|\alpha x + \beta y\|, \quad \text{for any real numbers } \alpha,\beta. \tag{2.5}$$

To show that $y \perp_B x$, on the contrary assume that

$$\|\gamma x + y\| < \|y\|, \quad \text{for some } \gamma \in \mathbb{R}. \tag{2.6}$$

From (2.5), we obtain (note that from (2.5) and (2.6) we have that $\|\gamma x + y\| > 0$)

$$\left\|\frac{\gamma x + y}{\theta\|\gamma x + y\|} - \frac{y}{\theta\|y\|}\right\| = \left\|\frac{\gamma x}{\theta\|\gamma x + y\|} + \left(\frac{1}{\theta\|\gamma x + y\|} - \frac{1}{\theta\|y\|}\right)y\right\| \geqslant \frac{\|\gamma x\|}{\theta\|\gamma x + y\|}. \tag{2.7}$$

Then from (2.4) and (2.7), we obtain

$$\begin{aligned} 2^{\frac{1}{q}} &\geqslant \frac{\left((\theta\|\gamma x+y\|)^q+(\theta\|y\|)^q\right)^{\frac{1}{q}}}{\|\gamma x\|}\left\|\frac{\gamma x+y}{\theta\|\gamma x+y\|}-\frac{y}{\theta\|y\|}\right\| \\ &\geqslant \frac{\left((\theta\|\gamma x+y\|)^q+(\theta\|y\|)^q\right)^{\frac{1}{q}}}{\|\gamma x\|}\frac{\|\gamma x\|}{\theta\|\gamma x+y\|} \\ &\geqslant \left(1+\left(\frac{\theta\|y\|}{\theta\|\gamma x+y\|}\right)^q\right)^{\frac{1}{q}}. \end{aligned}$$

From the above inequality, we get that $\theta\|y\| \leqslant \theta\|\gamma x+y\|$. Since θ is increasing, we deduce $\|y\| \leqslant \|\gamma x+y\|$, which contradicts (2.6). □

Now we are ready to state our characterization of inner product spaces related to the θ-angular distance.

THEOREM 3. *Let $(X,\|.\|)$ be a normed space with $dim(X) \geqslant 3$ and let $\theta \in \Theta$. Then the following statements are equivalent:*

(i) For all $q \in (0,1]$, $\alpha_\theta[x,y] \leqslant 2^{\frac{1}{q}} \frac{\|x-y\|}{\left((\theta\|x\|)^q+(\theta\|y\|)^q\right)^{\frac{1}{q}}}$, for each $x,y \in X\setminus\{0\}$.

(ii) For some $q>0$, $\alpha_\theta[x,y] \leqslant 2^{\frac{1}{q}} \frac{\|x-y\|}{\left((\theta\|x\|)^q+(\theta\|y\|)^q\right)^{\frac{1}{q}}}$, for each $x,y \in X\setminus\{0\}$.

(iii) X is an inner product space.

Proof. (i)⇒ (ii) is trivial.

(ii) ⇒ (iii) is the same as Theorem 2.

(iii)⇒ (i) Let $q \in (0,1]$, $a := \theta\|x\|$, $b := \theta\|y\|$ and $t := q$ in the known inequality

$$a^t+b^t \leqslant 2^{1-t}(a+b)^t, \text{ for } a,b \geqslant 0, \ 0<t\leqslant 1,$$

we get

$$(\theta\|x\|)^q+(\theta\|y\|)^q \leqslant 2^{1-q}(\theta\|x\|+\theta\|y\|)^q,$$

and so

$$2\frac{\|x-y\|}{\theta\|x\|+\theta\|y\|} \leqslant 2^{\frac{1}{q}}\frac{\|x-y\|}{\left((\theta\|x\|)^q+(\theta\|y\|)^q\right)^{\frac{1}{q}}}, \quad \text{for each} \quad x,y \in X\setminus\{0\}. \tag{2.8}$$

Since X is an inner product space then by Theorem 1 we have

$$\alpha_\theta[x,y] \leqslant 2\frac{\|x-y\|}{\theta\|x\|+\theta\|y\|}, \quad \text{for each} \quad x,y \in X\setminus\{0\}. \tag{2.9}$$

From (2.8) and (2.9), we get the conclusion. □

If we put $\theta(t) = t^{1-p}$, $p \in [0,1)$, then the above theorem reduces to the following characterization of inner product spaces due to Dadipour and Moslehian [5].

COROLLARY 1. *Let $(X,\|.\|)$ be a normed space and let $p\in[0,1)$. Then the following statements are equivalent:*

(i) For all $q\in(0,1]$, $\alpha_p[x,y]\leqslant 2^{\frac{1}{q}}\frac{\|x-y\|}{\left(\|x\|^{(1-p)q}+\|y\|^{(1-p)q}\right)^{\frac{1}{q}}}$, *for each $x,y\in X\setminus\{0\}$.*

(ii) For some $q>0$, $\alpha_p[x,y]\leqslant 2^{\frac{1}{q}}\frac{\|x-y\|}{\left(\|x\|^{(1-p)q}+\|y\|^{(1-p)q}\right)^{\frac{1}{q}}}$, *for each $x,y\in X\setminus\{0\}$.*

(iii) X is an inner product space.

3. A new geometric constant of a normed space

We begin with the following definition.

DEFINITION 1. Let $(X,\|.\|)$ be a normed space. Let $F:(0,\infty)\times(0,\infty)\to(0,\infty)$ be a function satisfying $F^{-1}\{1\}\neq\emptyset$ and $\sigma_F=\inf_{r,s>0:F(r,s)=1}\left(r+s\right)\in(0,\infty)$. We introduce a new geometric constant $C_F(X)$ of X wrt F as follows:

$$C_F(X):=\sup\left\{\frac{\|u+v\|}{\|ru+sv\|}:\ u,v\in S_X,\ u\neq -v,\ \ r,s\in(0,\infty)\text{ with }F(r,s)=1\right\}.$$

Note that by using the Dunkl-Williams inequality [15, Proposition 2.1], we have

$$\frac{\|u+v\|}{\|ru+sv\|}\leqslant\frac{1}{r+s}\frac{\|u+v\|}{\|\frac{r}{r+s}u+\frac{s}{r+s}v\|}\leqslant\frac{DW(X)}{\sigma_F},$$

for each $u,v\in S_X$, $u\neq -v$, $r,s\in(0,\infty)$ with $F(r,s)=1$, and so $C_F(X)\leqslant\frac{DW(X)}{\sigma_F}<\infty$.

If we take $F_S(r,s)=r+s$ for each $r,s\in(0,\infty)$, then $C_{F_S}(X)=DW(X)$, where $DW(X)$ denotes the Dunkl-Williams constant of X which was introduced by Jiménez-Melado et al. [10].

The following lemma is useful for calculating $C_F(X)$, in the case of F is a homogeneous function of degree 1.

LEMMA 1. *Let $(X,\|.\|)$ be a normed space and let $F:(0,\infty)\times(0,\infty)\to(0,\infty)$ be a function satisfying $F^{-1}\{1\}\neq\emptyset$ and $\sigma_F\in(0,\infty)$. Assume that $F(\lambda x,\lambda y)=\lambda F(x,y)$, for each $\lambda,x,y\in(0,\infty)$. Then*

$$C_F(X)=\sup\left\{F(\|x\|,\|y\|)\frac{\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|}{\|x-y\|}:\ x,y\in X\setminus\{0\},\ x\neq y\right\}.$$

Proof. Since $F(\lambda x,\lambda y)=\lambda F(x,y)$, for each $\lambda,x,y\in(0,\infty)$, then

$$\left\{(r,s):r,s\in(0,\infty)\ \text{ and }\ F(r,s)=1\right\}=\left\{\left(\frac{r}{F(r,s)},\frac{s}{F(r,s)}\right):r,s\in(0,\infty)\right\},$$

and so

$$\begin{aligned}
C_F(X) &= \sup\left\{ \frac{\|u+v\|}{\|ru+sv\|} : u,v \in S_X,\ u \neq -v,\ \ r,s \in (0,\infty) \text{ with } F(r,s)=1 \right\} \\
&= \sup\left\{ \frac{\|u+v\|}{\|\frac{r}{F(r,s)}u + \frac{s}{F(r,s)}v\|} : u,v \in S_X,\ u \neq -v,\ \ r,s \in (0,\infty) \right\} \\
&= \sup\left\{ \frac{\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|}{\|\frac{\|x\|}{F(\|x\|,\|y\|)}\frac{x}{\|x\|} - \frac{\|y\|}{F(\|x\|,\|y\|)}\frac{y}{\|y\|}\|} : x,y \in X\setminus\{0\},\ x\neq y \right\} \\
&= \sup\left\{ \frac{\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|}{\|\frac{x}{F(\|x\|,\|y\|)} - \frac{y}{F(\|x\|,\|y\|)}\|} : x,y \in X\setminus\{0\},\ x\neq y \right\} \\
&= \sup\left\{ F(\|x\|,\|y\|)\frac{\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|}{\|x-y\|} : x,y \in X\setminus\{0\},\ x\neq y \right\}. \quad \square
\end{aligned}$$

Now, we calculate the geometric constant $C_F(X)$ of an inner product space X.

THEOREM 4. *Let* $(X,\|.\|)$ *be an inner product space and let* $F:(0,\infty)\times(0,\infty)\to(0,\infty)$ *be a function satisfying* $F^{-1}\{1\}\neq\emptyset$ *and* $\sigma_F\in(0,\infty)$*. Then*

$$C_F(X) = \frac{2}{\sigma_F}.$$

Proof. Since X is an inner product space then, we have

$$\begin{aligned}
[C_F(X)]^2 &= \sup\left\{ \frac{\|u+v\|^2}{\|ru+sv\|^2} : u,v \in S_X,\ u \neq -v,\ \ r,s \in (0,\infty) \ \text{ with } \ F(r,s)=1 \right\} \\
&= \sup_{\substack{r,s\in(0,\infty)\\ F(r,s)=1}} \sup\left\{ \frac{\|u+v\|^2}{\|ru+sv\|^2} : \ u,v \in S_X,\ u\neq -v \right\} \\
&= \sup_{\substack{r,s\in(0,\infty)\\ F(r,s)=1}} \sup\left\{ \frac{2+2\langle u,v\rangle}{r^2+s^2+2rs\langle u,v\rangle} : u,v \in S_X,\ u\neq -v \right\}.
\end{aligned}$$

Since the function $t\mapsto \frac{2+2t}{r^2+s^2+2rst}$ is non-decreasing on $(-1,1]$ for each $r,s\in(0,\infty)$ then, we have (note that by the Cauchy-Schwartz inequality $|\langle u,v\rangle| \leqslant \|u\|\|v\| = 1$)

$$\sup\left\{ \frac{2+2\langle u,v\rangle}{r^2+s^2+2rs\langle u,v\rangle} : u,v \in S_X,\ u\neq -v \right\} = \frac{4}{(r+s)^2}.$$

Therefore

$$C_F(X) = \sup_{\substack{r,s\in(0,\infty)\\ F(r,s)=1}} \left\{ \frac{2}{r+s} \right\} = \frac{2}{\sigma_F}. \quad \square$$

The following example shows that the converse of Theorem 4 does not hold, in general.

EXAMPLE 1. Let $(X, \|.\|)$ be a normed space and let $F_M : (0,\infty)\times(0,\infty) \to (0,\infty)$ be defined by $F_M(r,s) = \max(r,s)$. Then

$$C_{F_M}(X) = \sup\left\{ \frac{\|u+v\|}{\|ru+sv\|} : \ u,v \in S_X, u \neq -v, \ \ r,s \in (0,\infty), \ \max(r,s) = 1 \right\} = 2$$
$$= \frac{2}{\inf\left\{ r+s : \ r,s \in (0,\infty), \ \max(r,s) = 1 \right\}}.$$

To show the claim, note that for each $r \in [0,1]$ and $u,v \in S_X$ with $u \neq -v$,

$$\|u+v\| \leqslant \|u-ru\| + \|ru+v\| = (1-r) + \|ru+v\|,$$

and

$$\|ru+v\| \geqslant \|v\| - \|ru\| = (1-r).$$

Then

$$\frac{\|u+v\|}{\|ru+v\|} \leqslant 2, \text{ for each } r \in [0,1] \text{ and } u,v \in S_X \text{ with } u \neq -v.$$

Thus

$$2 \geqslant \sup\left\{ \frac{\|u+v\|}{\|ru+sv\|} : \ u,v \in S_X, u \neq -v, \ \ r>0, \ s>0\ , \ \max(r,s) = 1 \right\}$$
$$= \sup\left\{ \frac{\|u+v\|}{\|ru+v\|} : \ u,v \in S_X, u \neq -v, \ \ 0 < r \leqslant 1 \right\} \geqslant \frac{\|u+u\|}{\|u\|} = 2,$$

and the proof is complete.

Now the following problem naturally arise:

PROBLEM 1. Find necessary and sufficient conditions on F such that the equality $C_F(X) = \frac{2}{\sigma_F}$ characterize inner product spaces among all normed spaces.

Here is our first partial answer to this problem.

THEOREM 5. *Let $(X, \|.\|)$ be a normed space with dim $(X) \geqslant 3$. Let $F : (0,\infty)\times(0,\infty) \to (0,\infty)$ be a continuous function which is increasing in each variable and F satisfying the conditions $F^{-1}\{1\} \neq \emptyset$ and $\sigma_F \in (0,\infty)$. Assume that*

$$\lim_{t\to\infty} F(tx,ty) > 1 \ \ \textit{and} \ \ \lim_{t\to 0^+} F(tx,ty) < 1, \ \ \textit{for each} \ \ x,y \in (0,\infty), \tag{3.1}$$

and

$$F\left(\frac{\sigma_F}{2}, \frac{\sigma_F}{2}\right) \geqslant 1. \tag{3.2}$$

Then, X is an inner product space if and only if

$$C_F(X) = \frac{2}{\sigma_F}.$$

Proof. If X is an inner product space then by Theorem 4, $C_F(X) = \frac{2}{\sigma_F}$. Now, assume that $C_F(X) = \frac{2}{\sigma_F}$ and will show that X is an inner product space. Since F is continuous, then by (3.1) there exists a function $\varphi : (0,\infty)\times(0,\infty) \to (0,\infty)$ satisfying

$$F\left(\frac{r}{\varphi(r,s)}, \frac{s}{\varphi(r,s)}\right) = 1, \quad \text{for each } r,s \in (0,\infty). \tag{3.3}$$

Assume that $x \perp_B y$, where $x,y \neq 0$ are elements in X. Then,

$$\|\alpha x\| \leqslant \|\alpha x + \beta y\|, \quad \text{for each } \alpha,\beta \in \mathbb{R}. \tag{3.4}$$

To prove that X is an inner product space it is necessary and sufficient to show that $y \perp_B x$, that is for any $\beta \in \mathbb{R}$, $\|y\| \leqslant \|\beta x + y\|$. On the contrary assume that

$$\|\gamma x + y\| < \|y\|, \quad \text{for some } \gamma \in \mathbb{R}. \tag{3.5}$$

By the assumption

$$\|u+v\| \leqslant \frac{2}{\sigma_F}\|ru+sv\|, \quad \text{for each } u,v \in S_X \text{ and } r,s \in (0,+\infty) \text{ with } F(r,s) = 1. \tag{3.6}$$

From (3.3) and (3.6), we obtain (note that from (3.4) and (3.5) we have that $\|\gamma x + y\| > 0$)

$$\left\| \frac{\gamma x + y}{\|\gamma x + y\|} - \frac{y}{\|y\|} \right\| \leqslant \frac{2}{\sigma_F} \left\| \frac{\|\gamma x + y\|}{\varphi(\|\gamma x + y\|, \|y\|)} \frac{\gamma x + y}{\|\gamma x + y\|} - \frac{\|y\|}{\varphi(\|\gamma x + y\|, \|y\|)} \frac{y}{\|y\|} \right\|,$$

for each $x,y \in X \setminus \{0\}$. Since $x \perp_B y$, from (2.7) and the above inequality, we get

$$\begin{aligned} \frac{2}{\sigma_F} &\geqslant \frac{\varphi(\|\gamma x + y\|, \|y\|)}{\|\gamma x\|} \left\| \frac{\gamma x + y}{\|\gamma x + y\|} - \frac{y}{\|y\|} \right\| \\ &\geqslant \frac{\varphi(\|\gamma x + y\|, \|y\|)}{\|\gamma x\|} \frac{\|\gamma x\|}{\|\gamma x + y\|}. \end{aligned}$$

and so

$$\frac{2}{\sigma_F} \geqslant \frac{\varphi(\|\gamma x + y\|, \|y\|)}{\|\gamma x + y\|}. \tag{3.7}$$

From (3.5) and (3.7), we obtain

$$\frac{\|y\|}{\varphi(\|\gamma x + y\|, \|y\|)} > \frac{\|\gamma x + y\|}{\varphi(\|\gamma x + y\|, \|y\|)} \geqslant \frac{\sigma_F}{2}. \tag{3.8}$$

Since F is increasing in each variable then from (3.7) and (3.8), we get

$$1 = F\left(\frac{\|\gamma x + y\|}{\varphi(\|\gamma x + y\|, \|y\|)}, \frac{\|y\|}{\varphi(\|\gamma x + y\|, \|y\|)}\right) > F\left(\frac{\sigma_F}{2}, \frac{\sigma_F}{2}\right),$$

a contradiction. □

Now, we get the following characterizations of inner product spaces due to Al-Rashed [1, Theorem 2.3 and Corollary 2.4] (the case of $p = 1$ was obtained by Kirk and Smiley [12]).

COROLLARY 2. *Let* $(X,\|.\|)$ *be a normed space with dim* $(X)\geqslant 3$ *and let* $p\in(0,1]$. *Then* X *is an inner product space if and only if*

$$\sup\left\{\frac{(\|x\|^p+\|y\|^p)^{\frac{1}{p}}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|}{\|x-y\|}:\ x,y\in X\setminus\{0\},\ x\neq y\right\}=2^{1/p}. \tag{3.9}$$

Proof. Let $F(r,s):=(r^p+s^p)^{\frac{1}{p}}$, for each $r,s\in(0,\infty)$. Then F is continuous and increasing in each variable. It is easy to see that $\sigma_F=2^{1-\frac{1}{p}}$ and so $F(\frac{\sigma_F}{2},\frac{\sigma_F}{2})=1$. Then all the assumptions of Theorem 5 are satisfied. Then we get the conclusion if we show that $C_F(X)=\frac{2}{\sigma_F}=2^{\frac{1}{p}}$. From Lemma 1 and (3.9), we deduce that

$$C_F(X)=\sup\left\{\frac{(\|x\|^p+\|y\|^p)^{\frac{1}{p}}\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|}{\|x-y\|}:\ x,y\in X\setminus\{0\},\ x\neq y\right\}=2^{1/p},$$

and the proof is complete. □

Now we get the following characterization of inner product spaces due to Lorch [13].

COROLLARY 3. *Let* $(X,\|.\|)$ *be a normed space with dim* $(X)\geqslant 3$. *Then* $(X,\|.\|)$ *is an inner product space if and only if for all* $x,y\in X$ *satisfying* $\|x\|=\|y\|$ *the inequality* $\|x+y\|\leqslant\|\gamma x+\gamma^{-1}y\|$ *holds for all real* $\gamma\neq 0$.

Proof. Let $F_P(r,s)=\sqrt{rs}$ for each $r,s\in(0,\infty)$. Then F is continuous, increasing in each variable, and $\sigma_{F_P}=2$. So $F_P(\frac{\sigma_{F_P}}{2},\frac{\sigma_{F_P}}{2})=1$. Then from Theorem 5, X is an inner product space if and only if $C_{F_P}(X)=\frac{2}{\sigma_{F_P}}=1$. Now the assumption holds if and only if

$$\begin{aligned}C_{F_P}(X)&=\sup\left\{\frac{\|u+v\|}{\|\gamma u+\gamma^{-1}v\|}:\ u,v\in S_X,\ u\neq -v,\ \ \gamma\in(0,\infty)\right\}\\&=\sup\left\{\frac{\|x+y\|}{\|\gamma x+\gamma^{-1}y\|}:\ x,y\in X,\ \|x\|=\|y\|,\ x\neq -y,\ \ \gamma\in(0,\infty)\right\}=1.\end{aligned}$$ □

Here is a characterization of inner product spaces due to Dehghan [7, Theorem 3.2].

COROLLARY 4. *Let* $(X,\|.\|)$ *be a normed space with dim* $(X)\geqslant 3$. *Then* $(X,\|.\|)$ *is an inner product space if and only if*

$$\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|\leqslant\left\|\frac{x}{\|y\|}-\frac{y}{\|x\|}\right\|,\ \textit{for each}\ x,y\in X\setminus\{0\}. \tag{3.10}$$

Proof. Let $F_P(r,s) := \sqrt{rs}$, for each $r,s \in (0,\infty)$. Then (3.10) holds if and only if

$$\begin{aligned} C_{F_P}(X) &= \sup\left\{ \frac{\|u+v\|}{\|ru+sv\|} : u,v \in S_X,\ u \neq -v,\ \ r,s \in (0,\infty) \text{ with } rs = 1 \right\} \\ &= \sup\left\{ \frac{\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|}{\left\| \frac{\|x\|}{\|y\|}\frac{x}{\|x\|} - \frac{\|y\|}{\|x\|}\frac{y}{\|y\|} \right\|} : x,y \in X\setminus\{0\},\ \frac{x}{\|x\|} \neq \frac{y}{\|y\|} \right\} \\ &= \sup\left\{ \frac{\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|}{\left\| \frac{x}{\|y\|} - \frac{y}{\|x\|} \right\|} : \ x,y \in X\setminus\{0\},\ \frac{x}{\|x\|} \neq \frac{y}{\|y\|} \right\} = 1 = \frac{2}{\sigma_{F_P}}. \end{aligned}$$

Now the conclusion follows from Theorem 5. $\square$

Now we give lower and upper bounds for the Dunkl-Williams constant of X wrt F.

THEOREM 6. *Let $(X,\|.\|)$ be a normed space and let $F:(0,\infty)\times(0,\infty)\to(0,\infty)$ be a function such that $F(\lambda x,\lambda y) = \lambda F(x,y)$ for any $\lambda > 0$. Then*

$$2\sup_{t\in(0,1)} F(1-t,t) \leqslant C_F(X) \leqslant DW(X) \sup_{t\in(0,1)} F(1-t,t). \tag{3.11}$$

In particular, if X is an inner product space then $C_F(X) = 2\sup_{t\in(0,1)} F(1-t,t)$.

Proof. For each $t \in (0,1)$ we have

$$\begin{aligned} C_F(X) &\geqslant \frac{\left\| \frac{x}{\|x\|} + \frac{\frac{t}{1-t}x}{\frac{t}{1-t}\|x\|} \right\|}{\left\| \frac{\|x\|}{F(\|x\|,\frac{t}{1-t}\|x\|)}\frac{x}{\|x\|} + \frac{\frac{t}{1-t}\|x\|}{F(\|x\|,\frac{t}{1-t}\|x\|)}\frac{\frac{t}{1-t}x}{\frac{t}{1-t}\|x\|} \right\|} \\ &= 2\frac{F(\|x\|,\frac{t}{1-t}\|x\|)}{\|x+\frac{t}{1-t}x\|} = 2F(1-t,t), \end{aligned}$$

and so

$$C_F(X) \geqslant \sup_{t\in(0,1)} 2F(1-t,t).$$

On the other hand

$$\begin{aligned} C_F(X) &= \sup\left\{ \frac{\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|}{\left\| r\frac{x}{\|x\|} - s\frac{y}{\|y\|} \right\|} : x,y \in X\setminus\{0\}, \frac{x}{\|x\|} \neq \frac{y}{\|y\|}, r,s \in (0,\infty) \text{ with } F(r,s) = 1 \right\} \\ &= \sup\left\{ \frac{\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|}{\left\| \frac{r}{F(r,s)}\frac{x}{\|x\|} - \frac{s}{F(r,s)}\frac{y}{\|y\|} \right\|} : x,y \in X\setminus\{0\}, \frac{x}{\|x\|} \neq \frac{y}{\|y\|}, r,s \in (0,\infty) \right\} \\ &= \sup\left\{ \frac{\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|}{\left\| \frac{r}{r+s}\frac{x}{\|x\|} - \frac{s}{r+s}\frac{y}{\|y\|} \right\|}\frac{F(r,s)}{r+s} : x,y \in X\setminus\{0\}, \frac{x}{\|x\|} \neq \frac{y}{\|y\|}, r,s \in (0,\infty) \right\} \end{aligned}$$

$$= \sup\left\{ \frac{\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|}{\left\| \frac{r}{r+s}\frac{x}{\|x\|} - \frac{s}{r+s}\frac{y}{\|y\|} \right\|} F\left(\frac{r}{r+s}, \frac{s}{r+s}\right) : \right.$$

$$\left. x, y \in X \setminus \{0\}, \frac{x}{\|x\|} \neq \frac{y}{\|y\|}, r, s \in (0,\infty) \right\}$$

$$\leqslant DW(X) \sup_{t\in(0,1)} F(1-t,t). \quad \square$$

Here is our second partial answer to the Problem 1, which gives another characterization of inner product spaces.

THEOREM 7. *Let* $(X, \|.\|)$ *be a normed space with dim* $(X) \geqslant 3$ *and let* $F : (0,\infty)\times(0,\infty) \to (0,\infty)$ *be a function with* $\sigma_F \in (0,\infty)$ *such that* $F(\lambda x, \lambda y) = \lambda F(x,y)$ *for any* $\lambda, x, y \in (0,\infty)$. *Moreover, assume that*

$$\frac{2}{\sigma_F} = F(1,1) \;\; and \;\; F(1-t,1-t) < F(1-t,t), \;\; for \;\; each \;\; t \in \left(\frac{1}{2},1\right). \tag{3.12}$$

Then X *is an inner product space if (and only if)* $C_F(X) = \frac{2}{\sigma_F}$.

Proof. Let $x, y \in X \setminus \{0\}$ such that $x \perp_B y$. Hence $\|\alpha x + \beta y\| \geqslant \|\alpha x\|$ for any real numbers α, β and so for each real number γ

$$\left\| \frac{\gamma x + y}{\|\gamma x + y\|} - \frac{y}{\|y\|} \right\| \geqslant \frac{\|\gamma x\|}{\|\gamma x + y\|}. \tag{3.13}$$

Since

$$F\left(\frac{\|\gamma x + y\|}{F(\|\gamma x + y\|, \|y\|)}, \frac{\|y\|}{F(\|\gamma x + y\|, \|y\|)}\right) = \frac{F(\|\gamma x + y\|, \|y\|)}{F(\|\gamma x + y\|, \|y\|)} = 1,$$

from (3.12),

$$\begin{aligned}
F(1,1) &= \frac{2}{\sigma_F} \\
&= C_F(X) \\
&\geqslant \frac{\left\| \frac{\gamma x + y}{\|\gamma x + y\|} - \frac{y}{\|y\|} \right\|}{\left\| \frac{\|\gamma x + y\|}{F(\|\gamma x + y\|, \|y\|)} \frac{\gamma x + y}{\|\gamma x + y\|} - \frac{\|y\|}{F(\|\gamma x + y\|, \|y\|)} \frac{y}{\|y\|} \right\|} \\
&\geqslant \frac{F(\|\gamma x + y\|, \|y\|)}{\|\gamma x\|} \left\| \frac{\gamma x + y}{\|\gamma x + y\|} - \frac{y}{\|y\|} \right\| \\
&\geqslant \frac{F(\|\gamma x + y\|, \|y\|)}{\|\gamma x + y\|} \\
&= \frac{F\left(\frac{\|\gamma x + y\|}{\|\gamma x + y\| + \|y\|}, \frac{\|y\|}{\|\gamma x + y\| + \|y\|}\right)}{\frac{\|\gamma x + y\|}{\|\gamma x + y\| + \|y\|}},
\end{aligned}$$

and so we have $\frac{\|y\|}{\|\gamma x+y\|+\|y\|} \leqslant \frac{1}{2}$. Thus $\|y\| \leqslant \|\gamma x+y\|$ holds for all $\gamma \in \mathbb{R}$, and so $y \perp_B x$. $\square$

Let Φ_2 denote the family of all continuous concave function $\psi : [0,1] \to \mathbb{R}$ such that $\psi(0)=\psi(1)=1$. For each $\psi \in \Phi_2$, define the function $\|.\|_\psi$ on $\mathbb{R}^2$ by

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\psi(\frac{|y|}{|x|+|y|}) & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

For each $x,y \in X$, define $\|(x,y)\|_\psi = \|(\|x\|,\|y\|)\|_\psi$. Let X be a real Banach space and let $\psi \in \Phi_2$. Defined [16]

$$C_\psi(X) := \sup\left\{ \frac{\|(x,y)\|_\psi}{\|x-y\|}\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \ : \ \ x,y \in X \setminus \{0\}\ ,\ x \neq y \right\}.$$

The following is a characterization of inner product spaces due to Tanaka et al [16].

COROLLARY 5. *Let* $(X,\|.\|)$ *be normed space with dim* $(X) \geqslant 3$. *Let* $\psi \in \Phi_2$ *such that* $\max_{t\in[0,1]} \psi(t) = \psi(\frac{1}{2})$. *Then* $(X,\|.\|)$ *is an inner product space if and only if* $C_\psi(X) = 2\psi(\frac{1}{2})$.

Proof. Let $F(r,s) = \|(r,s)\|_\psi$ for each $r,s \in (0,\infty)$. Then from Lemma 1, we have

$$C_F(X) = \sup\left\{ \frac{\|(x,y)\|_\psi}{\|x-y\|}\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \ : \ \ x,y \in X \setminus \{0\}\ ,\ x \neq y \right\} = C_\psi(X).$$

We show that all the assumptions of Theorem 7 and then the conclusion follows.

$$\begin{aligned} \frac{2}{\sigma_F} &= \frac{2}{\inf\{r+s:\ r,s>0\ , \|(r,s)\|_\psi = (r+s)\psi(\frac{s}{r+s}) = 1\}} \\ &= 2\sup\left\{ \psi\Big(\frac{s}{r+s}\Big) :\ r,s>0 \right\} = 2\psi\Big(\frac{1}{2}\Big). \end{aligned}$$

Then

$$F(1,1) = 2\psi\Big(\frac{1}{2}\Big) = \frac{2}{\sigma_F}.$$

From Lemma 2 in [16], we also have the function $t \mapsto \frac{F(1-t,t)}{1-t} = \frac{\psi(t)}{1-t}$ is increasing on $[0,1)$. $\square$

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