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MORE ACCURATE CLASSES OF JENSEN–TYPE INEQUALITIES FOR CONVEX AND OPERATOR CONVEX FUNCTIONS

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Abstract. Motivated by a recent refinement of the scalar Jensen inequality obtained via linear interpolation, in this paper we develop a general method for improving two classes of Jensen-type inequalities for bounded self-adjoint operators. The first class refers to a usual convexity, while the second one deals with the operator convexity. The general results are then applied to quasi-arithmetic and power operator means. As a consequence, we obtain strengthened forms of the inequalities between arithmetic, geometric and harmonic operator means. We also obtain more accurate Young-type inequalities for unitarily invariant norms as well as more precise relations for some important jointly concave mappings.

1. Introduction

Throughout the paper, let H be a Hilbert space and let $\mathscr{B}_h(H)$ be the semi-space of all bounded self-adjoint operators on H. Further, let $\mathscr{B}^+(H)$ and $\mathscr{B}^{++}(H)$ respectively denote the sets of all positive and positive invertible operators in $\mathscr{B}_h(H)$. The weighted operator arithmetic mean ∇_t, geometric mean $\sharp_t$, and harmonic mean $!_t$, for $t \in [0,1]$ and $A,B \in \mathscr{B}^{++}(H)$, are defined as follows:

$$\begin{aligned} A\nabla_t B =& (1-t)A+tB, \\ A\sharp_t B =& A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}, \\ A!_t B =& \left((1-t)A^{-1}+tB^{-1}\right)^{-1}. \end{aligned}$$

If $t=\frac{1}{2}$, we write $A!B$, $A\sharp B$, $A\nabla B$ for brevity.

Like in the real case, the arithmetic-geometric-harmonic operator mean inequality asserts that

$$A\nabla_t B \geqslant A\sharp_t B \geqslant A!_t B, \quad t\in[0,1], \tag{1}$$

with respect to the operator order. Both real and operator mean inequalities lie in the field of interest of numerous mathematicians. In the last ten years, a considerable attention has been given to developing methods for improving these inequalities. In 2011,

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Furuichi [8] (see also Kittaneh *et.al.* [15]), established the following refinement of the operator arithmetic-geometric mean inequality in a difference form:

$$A\nabla_t B - A\sharp_t B \geqslant 2r_0(t)\,(A\nabla B - A\sharp B), \quad r_0(t) = \min\{t, 1-t\}. \tag{2}$$

Moreover, Zhao and Wu [25], derived a more accurate estimate for the inequality (2): If $0 < t \leqslant \frac{1}{2}$ and $r_1(t) = \min\{2r_0(t), 1-2r_0(t)\}$, then

$$A\nabla_t B - A\sharp_t B \geqslant 2t\,(A\nabla B - A\sharp B) + r_1(t)\big(A\sharp B - 2A\sharp_{\frac{1}{4}} B + A\big), \tag{3}$$

while for $\frac{1}{2} < t < 1$, one has

$$A\nabla_t B - A\sharp_t B \geqslant 2(1-t)\,(A\nabla B - A\sharp B) + r_1(t)\big(A\sharp B - 2A\sharp_{\frac{3}{4}} B + B\big). \tag{4}$$

For some related refinements of mean inequalities, the reader is also referred to recent papers [3], [22], [23], and references therein.

Inequalities (1), (2), (3) and (4) are established via improved versions of the scalar inequality

$$(1-t)a + tb \geqslant a^{1-t}b^t, \quad a,b > 0,\ 0 \leqslant t \leqslant 1, \tag{5}$$

usually referred to as the Young inequality, and by virtue of monotonicity principle for bounded self-adjoint operators on a Hilbert space: If $X \in \mathscr{B}_h(H)$ with a spectrum $\mathrm{Sp}(X)$, then $f(t) \geqslant g(t),\ t \in \mathrm{Sp}(X) \Longrightarrow f(X) \geqslant g(X)$, provided that f and g are real valued continuous functions (for more details, see [9]).

On the other hand, utilizing a suitable linear interpolation of a convex function, Choi *et.al.* [4], obtained a general refinement of the scalar Jensen inequality. Recall that a function $f : I \to \mathbb{R}$ is said to be convex on interval I if for all $x, y \in I$ and all $t \in [0,1]$

$$(1-t)f(x) + tf(y) \geqslant f((1-t)x + ty) \tag{6}$$

holds. If the inequality in (6) is reversed, then f is said to be concave. In this article, the inequality (6) will be referred to as the scalar Jensen inequality.

As we previously announced, we quote the refinement of the scalar Jensen inequality derived in [4]. If $f : [0,1] \to \mathbb{R}$ is a convex function and N is a nonnegative integer, then

$$(1-t)f(0) + tf(1) - f(t) \geqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(t), \tag{7}$$

where

$$\Delta_f(n,k) = f\left(\frac{k-1}{2^n}\right) + f\left(\frac{k}{2^n}\right) - 2f\left(\frac{2k-1}{2^{n+1}}\right),$$

the functions $r_n(t)$ are defined recursively by

$$\begin{aligned} r_0(t) &= \min\{t, 1-t\} \\ r_n(t) &= \min\{2r_{n-1}(t), 1 - 2r_{n-1}(t)\}, \end{aligned}$$

and where χ stands for a characteristic function of the corresponding interval. Any summation having $\sum_{n=0}^{N-1}$ is assumed to be zero for $N=0$, therefore in this case the inequality (7) coincides with (6). Moreover, since f is convex, it follows that $\Delta_f(n,k) \geqslant 0$, therefore (7) represents the refinement of the inequality (6). In particular, if $N=1$, the right-hand side of the inequality (7) becomes $r_0(t)\Delta_f(0,1)$ providing the well-known refinement of the Jensen inequality (for more details, see [11]):

$$(1-t)f(0)+tf(1)-f(t) \geqslant r_0(t)\left(f(0)+f(1)-2f\left(\tfrac{1}{2}\right)\right).$$

It has been shown in [4] that the functions r_n can be rewritten in an explicit form

$$r_n(t)=\begin{cases} 2^n t-k+1, & \frac{k-1}{2^n} \leqslant t \leqslant \frac{2k-1}{2^{n+1}}, \\ k-2^n t, & \frac{2k-1}{2^{n+1}} < t \leqslant \frac{k}{2^n}, \end{cases}$$

for $k=1,2,\ldots,2^n$.

In particular, applying the inequality (7) to a convex function $f(t)=a^{1-t}b^t$, $a,b>0$, $t\in[0,1]$, one obtains

$$\begin{aligned}&(1-t)a+tb\\ &\quad\geqslant a^{1-t}b^t+\sum_{n=0}^{N-1} r_n(t)\sum_{k=1}^{2^n}\left(a^{\frac{1}{2}-\frac{k-1}{2^{n+1}}}b^{\frac{k-1}{2^{n+1}}}-a^{\frac{1}{2}-\frac{k}{2^{n+1}}}b^{\frac{k}{2^{n+1}}}\right)^2\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t),\end{aligned} \tag{8}$$

which represents a refinement of the Young inequality (5).

Motivated by the inequality (7), in this paper we develop a general method for improving two classes of Jensen-type inequalities for bounded self-adjoint operators on a Hilbert space. The first class refers to mere convexity, while the second one deals with the operator convexity.

The paper is divided into five sections as follows: After this Introduction, in Section 2 we obtain the improved class of Jensen-type inequalities for convex functions. The main result is then applied to quasi-arithmetic and power operator means. As a consequence, we obtain improved forms of inequalities between arithmetic, geometric and harmonic operator means, presented in the Introduction. In Section 3 we derive a similar refinement for a class of Jensen-type inequalities regarding operator convexity. The main result is also applied to quasi-arithmetic and power operator means. In Section 4 we consider some mappings possessing the so-called joint concavity property. The most important are connections, the heart of the famous theory developed by Kubo and Ando [18]. Namely, the operator means are defined via connections and there is an one-to-one correspondence between connections and nonnegative operator monotone functions on $\mathbb{R}_+$. By virtue of the improved Jensen-type inequality from Section 3 we obtain the strengthened form of the joint concavity property, and as an application, we obtain refinements for the weighted operator versions of the Hölder and Minkowski inequalities. Based on the refined Young inequality (8), in the last section we give several strengthened Young-type inequalities for unitarily invariant norms.

2. A unified treatment of the improved Jensen operator inequality for convex functions

Based on the inequality (7), in this section we provide a unified approach to the operator Jensen-type inequality referring to a usual convexity. In this regard, our first step is to extend relation (7) to hold for an arbitrary interval. We have the following simple result.

LEMMA 1. *Suppose $f:[a,b]\to\mathbb{R}$ is a convex function and let N ba a nonnegative integer. Then the inequality*

$$\begin{aligned}(1-t)f(a)+tf(b)-f((1-t)a+tb)&\\ \geqslant \sum_{n=0}^{N-1} r_n(t)\sum_{k=1}^{2^n}\Delta_f(a,b,n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t)&\end{aligned} \tag{9}$$

holds for all $t\in[0,1]$, where

$$\begin{aligned}\Delta_f(a,b,n,k)=&f\left(\frac{2^n-k+1}{2^n}a+\frac{k-1}{2^n}b\right)+f\left(\frac{2^n-k}{2^n}a+\frac{k}{2^n}b\right)\\ &-2f\left(\frac{2^{n+1}-2k+1}{2^{n+1}}a+\frac{2k-1}{2^{n+1}}b\right).\end{aligned}$$

If $f:[a,b]\to\mathbb{R}$ is a concave function, then the sign of inequality (9) is reversed.

Proof. The inequality (9) follows directly from (7) by replacing $f(t)$ with a function $f((1-t)a+tb)$, which is obviously convex on $[a,b]$. The reversed inequality for the case of a concave function f follows by using the fact that the function $-f$ is convex. □

REMARK 1. Due to the Jensen inequality, it follows that $\Delta_f(a,b,n,k)\geqslant 0$. Therefore (9) represents the improvement of the Jensen scalar inequality (6).

REMARK 2. According to the inequality (9), in this paper we deal with Jensen-type inequalities including two points (or two operators in the operator case). On the other hand, it has been shown in [11] that if $f:I\to\mathbb{R}$ is convex function and $\sum_{i=1}^k w_i=1$, $w_i\geqslant 0$, then the relation

$$\max_{1\leqslant i\leqslant k}\{w_i\}\,j_k(f,\mathbf{x})\geqslant\sum_{i=1}^k w_if(x_i)-f\left(\sum_{i=1}^k w_ix_i\right)\geqslant\min_{1\leqslant i\leqslant k}\{w_i\}\,j_k(f,\mathbf{x}), \tag{10}$$

where $j_k(f,\mathbf{x})=\sum_{i=1}^k f(x_i)-kf\left(\frac{1}{k}\sum_{i=1}^k x_i\right)$, holds for all $\mathbf{x}=(x_1,x_2,\dots,x_k)\in I^k$. The first inequality in the above relation represents the converse while the second one provides the refinement of the Jensen inequality. It should be noticed here that the inequality (9) with $N=1$ and the second inequality in (10) with $k=2$ coincide.

Now, by virtue of the functional calculus we obtain the following Jensen-type inequality for bounded self-adjoint operators on a Hilbert space. Throughout, 1_H stands for an identity operator on a Hilbert space H.

THEOREM 1. *Suppose I is an interval in $\mathbb{R}$, let $d \in I$, and let N be a nonnegative integer. If $f : I \to \mathbb{R}$ is a continuous convex function and $X \in \mathscr{B}_h(H)$ such that* $\mathrm{Sp}(X) \subseteq$ *I, then the inequality*

$$
\begin{aligned}
&(1-t)f(d1_H)+tf(X)-f((1-t)d1_H+tX) \\
&\qquad \geqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} \Delta_f(d1_H,X,n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t),
\end{aligned} \tag{11}
$$

where

$$
\Delta_f(d1_H,X,n,k) = f\big(d1_H \nabla_{\frac{k-1}{2^n}} X\big) + f\big(d1_H \nabla_{\frac{k}{2^n}} X\big) - 2f\big(d1_H \nabla_{\frac{2k-1}{2^{n+1}}} X\big),
$$

holds for all $t \in [0,1]$. If $f : I \to \mathbb{R}$ is a concave function, then the sign of inequality (11) is reversed.

Proof. Let $x \in I$. We will first show that the scalar inequality

$$
(1-t)f(d)+tf(x)-f((1-t)d+tx) \geqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} \Delta_f(d,x,n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t) \tag{12}
$$

holds for all $x \in I$ and $t \in [0,1]$. If $x \geqslant d$, then the above inequality holds due to (9). Otherwise, if $x \leqslant d$, then, employing (9) with $a = x$ and $b = d$ yields the relation

$$
(1-t)f(x)+tf(d)-f((1-t)x+td) \geqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} \Delta_f(x,d,n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t).
$$

Now, let $u = 1-t$, $t \in [0,1]$. Then $\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(1-u) = \chi_{\left(\frac{2^n-k}{2^n},\frac{2^n-k+1}{2^n}\right)}(u)$ holds for all $u \in [0,1]$. Moreover, since $\Delta_f(x,d,n,k) = \Delta_f(d,x,n,2^n-k+1)$ and $r_n(u) = r_n(1-u)$, due to symmetry, the previous inequality reduces to

$$
\begin{aligned}
&(1-u)f(d)+uf(x)-f((1-u)d+ux) \\
&\qquad \geqslant \sum_{n=0}^{N-1} r_n(u) \sum_{k=1}^{2^n} \Delta_f(d,x,n,2^n-k+1)\chi_{\left(\frac{2^n-k}{2^n},\frac{2^n-k+1}{2^n}\right)}(u),
\end{aligned}
$$

which clearly coincides with (12). Hence the inequality (12) holds for all $x \in I$ and $t \in [0,1]$.

Now, continuous functional calculus provides for the function f, which is continuous on I to act on the self-adjoint operator X. Order preserving property for operator functions provides that (12) holds if we substitute x by X. Hence the statement of the theorem is true. $\square$

Our next step is to give a form of Theorem 1 which will be more suitable for our applications.

COROLLARY 1. *Let I be an interval in $\mathbb{R}$ and let $d \in I$. Suppose $A \in \mathscr{B}^{++}(H)$ and $B \in \mathscr{B}_h(H)$ are such that $\operatorname{Sp}\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big) \subseteq I$. If $f : I \to \mathbb{R}$ is continuous convex function, then the inequality*

$$\begin{aligned}(1-t)f(d)A + tA^{\frac{1}{2}}f\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big)A^{\frac{1}{2}} - A^{\frac{1}{2}}f((1-t)d1_H + tA^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \\ \geqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} A^{\frac{1}{2}}\Delta_f(d, A^{-\frac{1}{2}}BA^{-\frac{1}{2}}, n, k)A^{\frac{1}{2}}\chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(t),\end{aligned} \tag{13}$$

where

$$\begin{aligned}\Delta_f(d, A^{-\frac{1}{2}}BA^{-\frac{1}{2}}, n, k) =& f\left(A^{-\frac{1}{2}}\left(dA\nabla_{\frac{k-1}{2^n}}B\right)A^{-\frac{1}{2}}\right) + f\left(A^{-\frac{1}{2}}\left(dA\nabla_{\frac{k}{2^n}}B\right)A^{-\frac{1}{2}}\right) \\ &- 2f\left(A^{-\frac{1}{2}}\left(dA\nabla_{\frac{2k-1}{2^{n+1}}}B\right)A^{-\frac{1}{2}}\right),\end{aligned}$$

holds for all $t \in [0,1]$. If $f : I \to \mathbb{R}$ is a concave function, then the sign of inequality (13) is reversed.

Proof. We utilize relation (11) with $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. In addition, multiplying the inequality by $A^{\frac{1}{2}}$ both-sidedly, which preserves the operator order, we obtain (13). $\square$

Our first application of Corollary 1 refers to quasi-arithmetic means. Let $\varphi : I \to \mathbb{R}$ be a continuous strictly monotone function. We define

$$M_\varphi(A, B; t) = \varphi^{-1}\left((1-t)\varphi(A) + t\varphi(B)\right), \qquad t \in [0,1],$$

where $A, B \in \mathscr{B}_h(H)$ are such that their spectra are contained in the interval I. In this regard, Corollary 1 can be rewritten in the following form:

COROLLARY 2. *Suppose $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions and let $1 \in I$. Further, let $A \in \mathscr{B}^{++}(H)$ and $B \in \mathscr{B}_h(H)$ be such that $\operatorname{Sp}\big(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\big) \subseteq I$. If N is a nonnegative integer and $\varphi \circ \psi^{-1}$ is well-defined and convex on $\psi(I)$, then the inequality*

$$\begin{aligned}A^{\frac{1}{2}}\varphi\left(M_\varphi(1_H, A^{-\frac{1}{2}}BA^{-\frac{1}{2}}; t)\right)A^{\frac{1}{2}} - A^{\frac{1}{2}}\varphi\left(M_\psi(1_H, A^{-\frac{1}{2}}BA^{-\frac{1}{2}}; t)\right)A^{\frac{1}{2}} \\ \geqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} A^{\frac{1}{2}}\Delta_{\varphi,\psi}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}, n, k)A^{\frac{1}{2}}\chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(t),\end{aligned} \tag{14}$$

where

$$\Delta_{\varphi,\psi}(X, n, k) = \varphi\left(M_\psi(1_H, X; \tfrac{k-1}{2^n})\right) + \varphi\left(M_\psi(1_H, X; \tfrac{k}{2^n})\right) - 2\varphi\left(M_\psi(1_H, X; \tfrac{2k-1}{2^{n+1}})\right),$$

holds for all $t \in [0,1]$. If $\varphi \circ \psi^{-1}$ is a concave function, then the sign of inequality (14) is reversed.

Proof. It follows from the inequality (13) with the function $\varphi \circ \psi^{-1} : \psi(I) \to \mathbb{R}$ and with $\psi\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)$ and $\psi(1)$ instead of $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and d respectively. □

The most common example of a quasi-arithmetic mean is a power operator mean defined by

$$M_r(A,B;t) = \begin{cases} ((1-t)A^r + tB^r)^{\frac{1}{r}}, & r \neq 0 \\ \exp((1-t)\log A + t\log B), & r = 0, \end{cases} \tag{15}$$

where $A,B \in \mathscr{B}^{++}(H)$.

Now, taking into account the Corollary 2 we obtain the improved series of inequalities for power operator means.

COROLLARY 3. *Let $A,B \in \mathscr{B}^{++}(H)$ and let $t \in [0,1]$.*

(i) If either $s \leqslant 0 \leqslant r$ or $r \leqslant 0 \leqslant s$ or $0 \leqslant r \leqslant s$ or $s \leqslant r \leqslant 0$, then holds the inequality

$$\begin{aligned} &A^{\frac{1}{2}}M_s^s\left(1_H, A^{-\frac{1}{2}}BA^{-\frac{1}{2}};t\right)A^{\frac{1}{2}} - A^{\frac{1}{2}}M_r^s\left(1_H, A^{-\frac{1}{2}}BA^{-\frac{1}{2}};t\right)A^{\frac{1}{2}} \\ &\qquad \geqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} A^{\frac{1}{2}}\Delta_{s,r}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}, n, k\right)A^{\frac{1}{2}}\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t), \end{aligned} \tag{16}$$

where

$$\Delta_{s,r}(X,n,k) = M_r^s\left(1_H, X; \tfrac{k-1}{2^n}\right) + M_r^s\left(1_H, X; \tfrac{k}{2^n}\right) - 2M_r^s\left(1_H, X; \tfrac{2k-1}{2^{n+1}}\right).$$

Further, if $0 \leqslant s \leqslant r$ or $r \leqslant s \leqslant 0$, then the sign of inequality in (16) is reversed.

(ii) If $r < 0$, then

$$\begin{aligned} &A^{\frac{1}{2}}\log M_0\left(1_H, A^{-\frac{1}{2}}BA^{-\frac{1}{2}};t\right)A^{\frac{1}{2}} - A^{\frac{1}{2}}\log M_r\left(1_H, A^{-\frac{1}{2}}BA^{-\frac{1}{2}};t\right)A^{\frac{1}{2}} \\ &\qquad \geqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} A^{\frac{1}{2}}\Delta_r\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}, n, k\right)A^{\frac{1}{2}}\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t), \end{aligned} \tag{17}$$

where

$$\begin{aligned} \Delta_r(X,n,k) = \log M_r\left(1_H, X; \tfrac{k-1}{2^n}\right) + \log M_r\left(1_H, X; \tfrac{k}{2^n}\right) \\ - 2\log M_r\left(1_H, X; \tfrac{2k-1}{2^{n+1}}\right), \end{aligned}$$

while for $r > 0$ the sign of inequality is reversed.

Proof. The proof is a consequence of Corollary 2, accompanied with particular choices of functions φ and ψ.

First, set $\varphi(t) = t^s$ and $\psi(t) = t^r$, where s and r are real parameters such that $r \neq 0$. The function $\left(\varphi \circ \psi^{-1}\right)(t) = t^{\frac{s}{r}}$ is convex on $\mathbb{R}_+$ if $\frac{s}{r} \leqslant 0$ or $\frac{s}{r} \geqslant 1$, which is possible in each of the following four cases: $s \leqslant 0 < r$ or $r < 0 \leqslant s$ or $0 < r \leqslant s$ or $s \leqslant r < 0$. Therefore we obtain (16).

Conversely, the function $\left(\varphi\circ\psi^{-1}\right)(t)=t^{\frac{s}{r}}$ is concave on $\mathbb{R}_+$ provided that $0\leqslant\frac{s}{r}\leqslant 1$, therefore, if $0\leqslant s\leqslant r\neq 0$ or $0\neq r\leqslant s\leqslant 0$, we have (16) with reversed signs of the inequality.

It remains to consider non-trivial cases when one of the parameters r and s is equal to zero. If $r=0$, then, setting $\varphi(t)=t^s$ and $\psi(t)=\log t$, it follows that the function $\left(\varphi\circ\psi^{-1}\right)(t)=\exp(st)$ is convex for every $s\in\mathbb{R}$, that is, we obtain that the inequality (16) with $r=0$ holds for all $s\in\mathbb{R}$.

Finally, if $s=0$, then, putting $\varphi(t)=\log t$ and $\psi(t)=t^r$, it follows that $\left(\varphi\circ\psi^{-1}\right)(t)=\frac{1}{r}\log t$. Obviously, this function is convex (concave) for $r<0$ ($r>0$), which yields (17) and the corresponding reversed inequality. □

By virtue of Corollary 3 we can improve the arithmetic-geometric harmonic operator mean inequality (1) as well as its refinements presented in the Introduction.

REMARK 3. Since

$$\begin{aligned} M_1\left(1_H,A^{-\frac{1}{2}}BA^{-\frac{1}{2}};t\right) &=(1-t)1_H+tA^{-\frac{1}{2}}BA^{-\frac{1}{2}},\\ M_{-1}\left(1_H,A^{-\frac{1}{2}}BA^{-\frac{1}{2}};t\right) &=\left((1-t)1_H+tA^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\right)^{-1}, \end{aligned}$$

and so,

$$\begin{aligned} A^{\frac{1}{2}}M_1\left(1_H,A^{-\frac{1}{2}}BA^{-\frac{1}{2}};t\right)A^{\frac{1}{2}} &=A\nabla_t B,\\ A^{\frac{1}{2}}M_0\left(1_H,A^{-\frac{1}{2}}BA^{-\frac{1}{2}};t\right)A^{\frac{1}{2}} &=A!_t B, \end{aligned}$$

the inequality (16) with $s=1$ and $r=-1$ reduces to

$$\begin{aligned} &A\nabla_t B-A!_t B\\ &\quad\geqslant\sum_{n=0}^{N-1}r_n(t)\sum_{k=1}^{2^n}\left(A!_{\frac{k-1}{2^n}}B+A!_{\frac{k}{2^n}}B-2A!_{\frac{2k-1}{2^{n+1}}}B\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t), \end{aligned} \tag{18}$$

where $0\leqslant t\leqslant 1$. If $N=1$, the inequality (18) reduces to the well-known relation in which the difference between the weighted arithmetic and harmonic mean is bounded by the difference of the corresponding nonweighted means (see [24]):

$$A\nabla_t B-A!_t B\geqslant 2r_0(t)\left(A\nabla B-A!B\right).$$

REMARK 4. Since $M_0\left(1_H,A^{-\frac{1}{2}}BA^{-\frac{1}{2}};t\right)=\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t$, it follows that

$$A^{\frac{1}{2}}M_0\left(1_H,A^{-\frac{1}{2}}BA^{-\frac{1}{2}};t\right)A^{\frac{1}{2}}=A\natural_t B.$$

Therefore the inequality (16) with $s=1$ and $r=0$ reduces to

$$\begin{aligned} &A\nabla_t B-A\natural_t B\\ &\quad\geqslant\sum_{n=0}^{N-1}r_n(t)\sum_{k=1}^{2^n}\left(A\natural_{\frac{k-1}{2^n}}B+A\natural_{\frac{k}{2^n}}B-2A\natural_{\frac{2k-1}{2^{n+1}}}B\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t), \end{aligned} \tag{19}$$

where $0 \leqslant t \leqslant 1$. This inequality provides more accurate arithmetic-geometric mean inequality than the refinements presented in the Introduction. Namely, if $N = 1$, the inequality (19) reduces to (2), while for $N = 2$ we obtain inequalities (3) and (4) established in [25].

REMARK 5. By virtue of Corollary 3, we can also improve the geometric-harmonic mean inequality. Namely, since the function $g(t) = -\frac{1}{t}$ is operator monotone on $\mathbb{R}_+$ (see [9], p.9), it follows that $\left(A\sharp_t B\right)^{-1} \leqslant \left(A!_t B\right)^{-1}$. Therefore, considering (16) with $s = -1$, $r = 0$ and multiplying the inequality by A^{-1} both-sidedly, we obtain the corresponding refinement:

$$\begin{aligned}&\left(A!_t B\right)^{-1} - \left(A\sharp_t B\right)^{-1}\\ &\geqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} \left(\left(A\sharp_{\frac{k-1}{2^n}} B\right)^{-1} + \left(A\sharp_{\frac{k}{2^n}} B\right)^{-1} - 2\left(A\sharp_{\frac{2k-1}{2^{n+1}}} B\right)^{-1} \right) \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(t).\end{aligned}$$

3. Extension to operator convexity

The method for improving Jensen-type inequalities presented in the previous section can also be applied to operator convex functions. Recall that a real valued continuous function f on an interval I is said to be operator convex if

$$(1-t)f(A) + tf(B) \geqslant f((1-t)A + tB) \tag{20}$$

holds for all $t \in [0,1]$ and for every pair of selfadjoint operators A and B on a Hilbert space H whose spectra are contained in I. If the sign of inequality (20) is reversed then f is an operator concave function.

It has been shown in [17] that for an operator convex function there is a more accurate version of (20). More precisely, if $f : I \to \mathbb{R}$ is operator convex function, then

$$\begin{aligned}&(1-t)f(A) + tf(B) - f((1-t)A + tB)\\ &\qquad \geqslant r_0(t)\left(f(A) + f(B) - 2f\left(\frac{A+B}{2}\right)\right),\end{aligned} \tag{21}$$

assuming that A and B are selfadjoint operators with spectra contained in the interval I. It should be noticed here that the right-hand side of (21) represents the positive operator due to operator convexity of f.

Now, motivated by the techniques presented in [4], we can obtain even more precise estimate than the relation (21).

THEOREM 2. *Let $f : I \to \mathbb{R}$ be an operator convex function and let N ba a non-negative integer. If $A, B \in \mathscr{B}_h(H)$ are such that their spectra are contained in I, then the inequality*

$$\begin{aligned}&(1-t)f(A) + tf(B) - f((1-t)A + tB)\\ &\qquad \geqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} \Delta_f(A,B,n,k)\chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(t),\end{aligned} \tag{22}$$

where

$$\Delta_f(A,B,n,k) = f\left(A\nabla_{\frac{k-1}{2^n}}B\right) + f\left(A\nabla_{\frac{k}{2^n}}B\right) - 2f\left(A\nabla_{\frac{2k-1}{2^{n+1}}}B\right),$$

holds for all $t\in[0,1]$. *If* $f: I\to\mathbb{R}$ *is operator concave function, then the sign of inequality* (22) *is reversed.*

Proof. Let N be a nonnegative integer. Denote by $\varphi_N(t)$, $t\in[0,1]$, a parametric function

$$\varphi_N(t) = (1-t)f(A)+tf(B) - \sum_{n=0}^{N-1} r_n(t)\sum_{k=1}^{2^n}\Delta_f(A,B,n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t).$$

The starting point in our proof is to find a more suitable form of the above function. We will show that the relation

$$\begin{aligned}\varphi_N(t) = &\left(k-2^N t\right) f\left(\frac{2^N-k+1}{2^N}A+\frac{k-1}{2^N}B\right)\\ &+\left(2^N t-k+1\right) f\left(\frac{2^N-k}{2^N}A+\frac{k}{2^N}B\right)\end{aligned} \tag{23}$$

holds for $\frac{k-1}{2^N}\leqslant t\leqslant\frac{k}{2^N}$ and $k=1,2,\ldots,2^N$. In order to prove our assertion, we use induction on N. Obviously, if $N=0$, then $\varphi_0(t)=(1-t)f(A)+tf(B)$ so that (23) holds.

Further, suppose that (23) holds for N and let $\frac{m-1}{2^{N+1}}\leqslant t\leqslant\frac{m}{2^{N+1}}$, $m=1,2,\ldots,2^{N+1}$. We consider two cases depending on whether m is an odd or an even integer. If $m=2l-1$, $l\in\mathbb{N}$, then $\frac{l-1}{2^N}\leqslant t\leqslant\frac{2l-1}{2^{N+1}}<\frac{l}{2^N}$ and therefore

$$\begin{aligned}\varphi_{N+1}(t) &= \varphi_N(t) - r_N(t)\sum_{k=1}^{2^N}\Delta_f(A,B,N,k)\chi_{\left(\frac{k-1}{2^N},\frac{k}{2^N}\right)}(t)\\ &= \varphi_N(t) - r_N(t)\Delta_f(A,B,N,l).\end{aligned}$$

Clearly, since $\frac{l-1}{2^N}\leqslant t\leqslant\frac{2l-1}{2^{N+1}}$, it follows that $r_N(t)=2^N t-l+1$. Finally, rewriting expressions for $\varphi_N(t)$ and $\Delta_f(A,B,N,l)$, we have

$$\begin{aligned}\varphi_{N+1}(t) &= \left(2l-2^{N+1}t-1\right) f\left(\frac{2^N-l+1}{2^N}A+\frac{l-1}{2^N}B\right)\\ &\quad+\left(2^{N+1}t-2l+2\right) f\left(\frac{2^{N+1}-2l+1}{2^{N+1}}A+\frac{2l-1}{2^{N+1}}B\right)\\ &= \left(m-2^{N+1}t\right) f\left(\frac{2^{N+1}-m+1}{2^{N+1}}A+\frac{m-1}{2^{N+1}}B\right)\\ &\quad+\left(2^{N+1}t-m+1\right) f\left(\frac{2^{N+1}-m}{2^{N+1}}A+\frac{m}{2^{N+1}}B\right).\end{aligned}$$

The case of an even integer is treated in the similar way. Namely, if $m = 2l$, then $\frac{l-1}{2^N} < \frac{2l-1}{2^{N+1}} \leqslant t \leqslant \frac{l}{2^N}$, which implies that $r_N(t) = l - 2^N t$. Therefore, as in the previous case, we have

$$\begin{aligned}
\varphi_{N+1}(t) &= \varphi_N(t) - r_N(t)\Delta_f(A,B,N,l) \\
&= \left(2^{N+1}t - 2l + 1\right) f\left(\frac{2^N - l}{2^N}A + \frac{l}{2^N}B\right) \\
&\quad + \left(2l - 2^{N+1}t\right) f\left(\frac{2^{N+1} - 2l + 1}{2^{N+1}}A + \frac{2l-1}{2^{N+1}}B\right) \\
&= \left(2^{N+1}t - m + 1\right) f\left(\frac{2^{N+1} - m}{2^{N+1}}A + \frac{m}{2^{N+1}}B\right) \\
&\quad + \left(m - 2^{N+1}t\right) f\left(\frac{2^{N+1} - m + 1}{2^{N+1}}A + \frac{m-1}{2^{N+1}}B\right),
\end{aligned}$$

so the relation (23) holds for all nonnegative integers N.

Now, let $t \in [0,1]$ and let $k \in \left\{1, 2, \ldots, 2^N\right\}$ be such that $\frac{k-1}{2^N} \leqslant t \leqslant \frac{k}{2^N}$. A straightforward computation shows that the convex combination of operators A and B, that is, $(1-t)A + tB$, can be rewritten as a convex combination of operators $\frac{2^N-k+1}{2^N}A + \frac{k-1}{2^N}B$ and $\frac{2^N-k}{2^N}A + \frac{k}{2^N}B$ in the following way:

$$\begin{aligned}
(1-t)A + tB = &\left(k - 2^N t\right)\left(\frac{2^N - k + 1}{2^N}A + \frac{k-1}{2^N}B\right) \\
&+ \left(2^N t - k + 1\right)\left(\frac{2^N - k}{2^N}A + \frac{k}{2^N}B\right).
\end{aligned}$$

Finally, applying the operator convexity of the function f to the above convex combination and taking into account relation (23), it follows that

$$f((1-t)A + tB) \leqslant \varphi_N(t),$$

which yields the inequality (22).

The reversed inequality for the case of operator concave function f follows by using the fact that $-f$ is an operator convex function. $\square$

REMARK 6. Since $\Delta_f(A,B,n,k) \geqslant 0$ for an operator convex function f, the inequality (22) provides the refinement of (20). Moreover, if $N = 0$, the inequality (22) coincides with (20), while for $N = 1$, the right-hand side of (22) becomes $r_0(t)\Delta_f(A,B,0,1)$, that is, we obtain the inequality (21).

Similarly to the previous section, we first give a variant of Theorem 2 which refers to quasi-arithmetic means.

COROLLARY 4. *Suppose $\varphi, \psi : I \to \mathbb{R}$ are continuous strictly monotone functions and let $A, B \in \mathscr{B}_h(H)$ be such that their spectra are contained in I. If N is a* *nonnegative integer and* $\varphi \circ \psi^{-1}$ *is well-defined and operator convex on* $\psi(I)$, *then the inequality*

$$\begin{aligned} &\varphi\left(M_{\varphi}(A,B;t)\right) - \varphi\left(M_{\psi}(A,B;t)\right) \\ &\qquad \geqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} \Delta_{\varphi,\psi}(A,B,n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t), \end{aligned} \tag{24}$$

where

$$\Delta_{\varphi,\psi}(A,B,n,k) = \varphi\left(M_{\psi}(A,B;\tfrac{k-1}{2^n})\right) + \varphi\left(M_{\psi}(A,B;\tfrac{k}{2^n})\right) - 2\varphi\left(M_{\psi}(A,B;\tfrac{2k-1}{2^{n+1}})\right),$$

holds for all $t \in [0,1]$. *If* $\varphi \circ \psi^{-1}$ *is operator concave, then the sign of inequality* (24) *is reversed.*

Proof. It follows from the inequality (22) accompanied with the function $\varphi \circ \psi^{-1}$: $\psi(I) \to \mathbb{R}$ and with operators $\psi(A)$, $\psi(B)$ instead of A, B respectively. $\square$

In addition, the Corollary 4 can also be rewritten in terms of power operator means.

COROLLARY 5. *Let* $A,B \in \mathscr{B}^{++}(H)$ *and let* $t \in [0,1]$.

(i) If either $0 < r \leqslant s \leqslant 2r$ or $2r \leqslant s \leqslant r < 0$ or $0 \leqslant s+r \leqslant r$ or $r \leqslant r+s \leqslant 0$, *then holds the inequality*

$$\begin{aligned} &M_s^s\left(A,B;t\right) - M_r^s\left(A,B;t\right) \\ &\qquad \geqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} \Delta_{s,r}\left(A,B,n,k\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t), \end{aligned} \tag{25}$$

where

$$\Delta_{s,r}(A,B,n,k) = M_r^s\left(A,B;\tfrac{k-1}{2^n}\right) + M_r^s\left(A,B;\tfrac{k}{2^n}\right) - 2M_r^s\left(A,B;\tfrac{2k-1}{2^{n+1}}\right).$$

Further, if $r \leqslant s \leqslant 0$ or $0 \leqslant s \leqslant r$, *then the sign of inequality* (25) *is reversed.*

(ii) If $r < 0$, *then*

$$\begin{aligned} &\log M_0\left(A,B;t\right) - \log M_r\left(A,B;t\right) \\ &\qquad \geqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} \Delta_r\left(A,B,n,k\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t), \end{aligned} \tag{26}$$

where

$$\begin{aligned} \Delta_r(A,B,n,k) = \log M_r\left(A,B;\tfrac{k-1}{2^n}\right) + \log M_r\left(A,B;\tfrac{k}{2^n}\right) \\ - 2\log M_r\left(A,B;\tfrac{2k-1}{2^{n+1}}\right), \end{aligned}$$

while for $r > 0$ *the sign of inequality is reversed.*

Proof. We follow the same procedure as in the proof of Corollary 3 except that we utilize operator convexity instead of mere convexity.

Setting $\varphi(t)=t^s$ and $\psi(t)=t^r$, where s and r are real parameters such that $r\neq 0$, it follows that $\left(\varphi\circ\psi^{-1}\right)(t)=t^{\frac{s}{r}}$. Now, the inequality (25) follows due to the fact that the function $\varphi\circ\psi^{-1}$ is operator convex on $\mathbb{R}_+$ if either $1\leqslant\frac{s}{r}\leqslant 2$ or $-1\leqslant\frac{s}{r}\leqslant 0$ and is operator concave if $0\leqslant\frac{s}{r}\leqslant 1$ (see [9], p.17).

The inequality (26) follows by substituting $\varphi(t)=\log t$ and $\psi(t)=t^r$ in (24) and by noting that the function $\left(\varphi\circ\psi^{-1}\right)(t)=\frac{1}{r}\log t$ is operator convex (operator concave) for $r<0$ ($r>0$). $\square$

REMARK 7. Contrary to (16), the inequality (25) does not hold in general if $r=0$. The reason for this lies in the fact that the function $f(x)=\exp x$ is not operator convex (see [9], p.17).

Corollaries 4 and 5 provide inequalities for quasi-arithmetic and power means of two operators. The quasi-arithmetic mean of k self-adjoint operators is defined analogously. Let $\varphi: I\to\mathbb{R}$ be a continuous strictly monotone function. We define

$$M_\varphi(\mathbf{A};\mathbf{w})=\varphi^{-1}\left(\sum_{i=1}^{k}w_i\varphi(A_k)\right),$$

where $\sum_{i=1}^k w_i=1$, $w_i\geqslant 0$, and $\mathbf{A}=(A_1,A_2,\ldots,A_k)$ is a k-tuple of bounded self-adjoint operators whose spectra are contained in I. The corresponding power mean is defined as

$$M_r(\mathbf{A};\mathbf{w})=\begin{cases}\left(\sum_{i=1}^k w_iA_i^r\right)^{\frac{1}{r}}, & r\neq 0\\ \exp\left(\sum_{i=1}^k w_i\log A_i\right), & r=0.\end{cases}$$

In the following remark we show that corollaries 4 and 5 can be extended to a k-tuple of positive invertible operators for the case when $N=1$. If $\mathbf{w}=(\frac{1}{k},\frac{1}{k},\ldots,\frac{1}{k})$, we denote $M_\varphi(\mathbf{A};\mathbf{w})$, $M_r(\mathbf{A};\mathbf{w})$ respectively by $M_\varphi(\mathbf{A})$, $M_r(\mathbf{A})$, for brevity.

REMARK 8. It has been shown in [17] that if $f: I\to\mathbb{R}$ is an operator convex function and $\sum_{i=1}^k w_i=1$, $w_i\geqslant 0$, then the relation

$$\max_{1\leqslant i\leqslant k}\{w_i\}\,\mathscr{J}_k(f,\mathbf{A})\geqslant\sum_{i=1}^k w_if(A_i)-f\left(\sum_{i=1}^k w_iA_i\right)\geqslant\min_{1\leqslant i\leqslant k}\{w_i\}\,\mathscr{J}_k(f,\mathbf{A}),\tag{27}$$

where $\mathscr{J}_k(f,\mathbf{A})=\sum_{i=1}^k f(A_i)-kf\left(\frac{1}{k}\sum_{i=1}^k A_i\right)$, holds for all operators $A_i\in\mathscr{B}_h(H)$, $i=1,2,\ldots,k$, whose spectra are contained in I. The first inequality in the above relation provides the converse while the second one represents the refinement of the Jensen inequality for operator convex functions. Similarly to the real case, the inequality (22) with $N=1$ and the second inequality in (27) with $k=2$ coincide.

Now, with the assumptions as in Corollary 4, the series of inequalities in (27) reduces to

$$\begin{aligned}k\max_{1\leqslant i\leqslant k}\{w_i\}\left(\varphi\left(M_\varphi(\mathbf{A})\right)-\varphi\left(M_\psi(\mathbf{A})\right)\right)&\geqslant\varphi\left(M_\varphi(\mathbf{A};\mathbf{w})\right)-\varphi\left(M_\psi(\mathbf{A};\mathbf{w})\right)\\&\geqslant k\min_{1\leqslant i\leqslant k}\{w_i\}\left(\varphi\left(M_\varphi(\mathbf{A})\right)-\varphi\left(M_\psi(\mathbf{A})\right)\right).\end{aligned}$$

In addition, substituting the same power functions φ and ψ as in the proof of Corollary 5, we obtain the corresponding relations for the power operator means. More precisely, the series of inequalities

$$\begin{aligned} k \max_{1\leqslant i\leqslant k}\{w_i\}\left(M_s^s(\mathbf{A})-M_r^s(\mathbf{A})\right) \geqslant & M_s^s(\mathbf{A};\mathbf{w})-M_r^s(\mathbf{A};\mathbf{w}) \\ \geqslant & k \min_{1\leqslant i\leqslant k}\{w_i\}\left(M_s^s(\mathbf{A})-M_r^s(\mathbf{A})\right) \end{aligned}$$

holds if either $0<r\leqslant s\leqslant 2r$ or $2r\leqslant s\leqslant r<0$ or $0\leqslant s+r\leqslant r$ or $r\leqslant r+s\leqslant 0$, while for $r\leqslant s\leqslant 0$ or $0\leqslant s\leqslant r$, the signs of inequalities are reversed. In addition, if $r<0$, then

$$\begin{aligned} k \max_{1\leqslant i\leqslant k}\{w_i\}\left(\log M_0(\mathbf{A})-\log M_r(\mathbf{A})\right) \geqslant & \log M_0(\mathbf{A};\mathbf{w})-\log M_r(\mathbf{A};\mathbf{w}) \\ \geqslant & k \min_{1\leqslant i\leqslant k}\{w_i\}\left(\log M_0(\mathbf{A})-\log M_r(\mathbf{A})\right), \end{aligned}$$

while for $r>0$ the signs of inequalities are reversed. These series of inequalities provide more accurate relations between power operator means than the corresponding ones in [9] (see Chapter 4.).

Theorem 2 can naturally be extended to the case of an operator convex function in m variables. Let H, H_i, $i=1,2,\ldots,m$, be Hilbert spaces and let $X\subseteq\prod_{i=1}^m \mathscr{B}_h(H_i)$ be a convex set. A function $F: X\to\mathscr{B}_h(H)$ is operator convex in m variables if for all $\mathbf{A}=(A_1,A_2,\ldots,A_m)$, $\mathbf{B}=(B_1,B_2,\ldots,B_m)\in X$ and for $0\leqslant t\leqslant 1$ is

$$(1-t)F(\mathbf{A})+tF(\mathbf{B})\geqslant F\left((1-t)\mathbf{A}+t\mathbf{B}\right). \tag{28}$$

If the reverse inequality holds in (28), then the function F is operator concave in m variables. With the above definition, it is obvious that the proof of Theorem 2 can be extended to the just described multidimensional setting.

COROLLARY 6. *Let H, H_i, $i=1,2,\ldots,m$, be Hilbert spaces, let $X\subseteq\prod_{i=1}^m \mathscr{B}_h(H_i)$ be a convex set and let $F: X\to\mathscr{B}_h(H)$ be an operator convex function. If $\mathbf{A}=(A_1,A_2,\ldots,A_m)$, $\mathbf{B}=(B_1,B_2,\ldots,B_m)\in X$, then the inequality*

$$\begin{aligned} &(1-t)F(\mathbf{A})+tF(\mathbf{B})-F((1-t)\mathbf{A}+t\mathbf{B}) \\ &\quad\geqslant \sum_{n=0}^{N-1} r_n(t)\sum_{k=1}^{2^n}\Delta_F(\mathbf{A},\mathbf{B},n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t), \end{aligned} \tag{29}$$

where

$$\begin{aligned} \Delta_F(\mathbf{A},\mathbf{B},n,k) =& F\left(\frac{2^n-k+1}{2^n}\mathbf{A}+\frac{k-1}{2^n}\mathbf{B}\right)+F\left(\frac{2^n-k}{2^n}\mathbf{A}+\frac{k}{2^n}\mathbf{B}\right) \\ &-2F\left(\frac{2^{n+1}-2k+1}{2^{n+1}}\mathbf{A}+\frac{2k-1}{2^{n+1}}\mathbf{B}\right), \end{aligned}$$

holds for all $t\in[0,1]$. If $F: X\to\mathscr{B}_h(H)$ is an operator concave function, then the sign of inequality (29) is reversed.

This corollary will be exploited in the next section where we are going to establish more precise relations for some significant jointly concave mappings.

4. Applications to some jointly concave mappings

Our aim now is to apply just presented concept of operator convexity of several variables to some interesting mappings.

The theory of operator means for positive operators on a Hilbert space was established and for most part developed by Kubo and Ando [18]. Operator means are defined via connections. A binary operation $(A,B) \in \mathscr{B}^+(H) \times \mathscr{B}^+(H) \to A\sigma B \in \mathscr{B}^+(H)$ in the cone of positive operators on a Hilbert space H is called a connection if the following conditions are satisfied:

(i) monotonicity: $\quad A \leqslant C$ and $B \leqslant D \implies A\sigma B \leqslant C\sigma D$,

(ii) upper continuity: $\quad A_n \downarrow A$ and $B_n \downarrow B \implies A_n\sigma B_n \downarrow A\sigma B$,

(iii) transformer inequality: $\quad T^*(A\sigma B)T \leqslant (T^*AT)\sigma(T^*BT)$ for every T.

An operator mean is a connection with

(iv) normalized condition: $\quad 1_H\sigma 1_H = 1_H$.

In condition (ii) symbol $\downarrow$ denotes the convergence in the strong operator topology.

The key of Kubo-Ando theory is the one-to-one correspondence between connections and nonnegative operator monotone functions on $\mathbb{R}_+$. Following the Kubo-Ando theory, Fujii *et.al.* [6], gave extension of connections to solidarities establishing the one-to-one correspondence between solidarities and operator monotone functions on $\mathbb{R}_+$. A binary operation $(A,B) \in \mathscr{D}_s \subseteq \mathscr{B}^+(H) \times \mathscr{B}^+(H) \to AsB \in \mathscr{B}_h(H)$ is called a solidarity if it has the following properties:

(i) $B \leqslant C \implies AsB \leqslant AsC$,

(ii) $B_n \downarrow B \implies AsB_n \downarrow AsB$,

(iii) $A_n \to A$ strongly $\implies A_n s 1_H \to As1_H$ strongly,

(iv) $T^*(AsB)T \leqslant (T^*AT)s(T^*BT)$ for every T.

Although the solidarity s is defined for every ordered pair of positive invertible operators, it is not defined for every pair of positive operators. Hence, $\mathscr{D}_s$ denotes the maximal subset of $\mathscr{B}^+(H) \times \mathscr{B}^+(H)$ on which solidarity exists as a bounded self-adjoint operator.

Both connections and solidarities posses numerous common properties, one of them is the so called joint concavity. More precisely, the following relations hold:

$$((1-t)A_1+tB_1)\,\sigma\,((1-t)A_2+tB_2) \geqslant (1-t)A_1\sigma A_2+tB_1\sigma B_2, \tag{30}$$

$$((1-t)A_1+tB_1)\,s\,((1-t)A_2+tB_2) \geqslant (1-t)A_1 s A_2+tB_1 s B_2, \tag{31}$$

where $0 \leqslant t \leqslant 1$ and A_1,A_2,B_1,B_2 are positive operators provided that all the expressions with solidarities exist as bounded operators. In particular, if $t = \frac{1}{2}$, the joint concavity property reduces to the so-called subadditivity property of connections and solidarities, e.g.

$$(A_1+B_1)\sigma(A_2+B_2) \geqslant A_1\sigma A_2 + B_1\sigma B_2$$

in the case of connections. It should be noticed here that the joint concavity property of connections and solidarities corresponds to operator concavity in two variables, in the sense of definition from the previous section. Therefore, utilizing Corollary 6 we obtain relations which are more accurate than (30) and (31).

COROLLARY 7. *Let σ be connection and let N ba a nonnegative integer. If A_1, A_2, B_1, $B_2 \in \mathscr{B}^+(H)$, then the inequality*

$$\begin{aligned}(1-t)A_1\sigma A_2 &+ tB_1\sigma B_2 - ((1-t)A_1+tB_1)\sigma((1-t)A_2+tB_2)\\ &\leqslant \sum_{n=0}^{N-1}\frac{r_n(t)}{2^n}\sum_{k=1}^{2^n}\delta_\sigma(\mathbf{A},\mathbf{B},n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t),\end{aligned} \tag{32}$$

where

$$\begin{aligned}&\delta_\sigma(\mathbf{A},\mathbf{B},n,k)\\ &\quad=((2^n-k+1)A_1+(k-1)B_1)\,\sigma\,((2^n-k+1)A_2+(k-1)B_2)\\ &\qquad+((2^n-k)A_1+kB_1)\,\sigma\,((2^n-k)A_2+kB_2)\\ &\qquad-\left((2^{n+1}-2k+1)A_1+(2k-1)B_1\right)\sigma\left((2^{n+1}-2k+1)A_2+(2k-1)B_2\right),\end{aligned}$$

holds for all $t\in[0,1]$. In addition, the relation (32) also holds when connection σ is replaced by a solidarity s, provided that all expressions with solidarities exist as bounded operators.

Proof. We utilize Corollary 6 with $m=2$ and with connection σ instead of F. Moreover, taking into account homogeneity property of a connection (solidarity) i.e. $\alpha(X\sigma Y)=(\alpha X)\,\sigma\,(\alpha Y)$, $X,Y\in\mathscr{B}^+(H)$, $\alpha>0$ (see [9], p.140), it follows that

$$\Delta_\sigma(\mathbf{A},\mathbf{B},n,k)=\frac{1}{2^n}\delta_\sigma(\mathbf{A},\mathbf{B},n,k),\quad \mathbf{A}=(A_1,A_2),\mathbf{B}=(B_1,B_2),$$

so (32) holds. □

REMARK 9. The inequality (32) provides better estimate than the joint concavity relation (30). Note also that the inequality $\delta_\sigma(\mathbf{A},\mathbf{B},n,k)\leqslant 0$ represents the subadditivity property of connection σ. If $N=1$, the right-hand side of (32) reduces to

$$r_0(t)\delta_\sigma(\mathbf{A},\mathbf{B},0,1)=r_0(t)\left(A_1\sigma A_2+B_1\sigma B_2-(A_1+B_1)\sigma(A_2+B_2)\right),$$

providing the relation derived in [17].

REMARK 10. Our first application of Corollary 7 refers to a geometric mean. Let p and q be conjugate exponents, i.e. $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. By letting σ to be geometric mean $\sharp_{\frac{1}{p}}$ and replacing A_1, A_2, B_1, $B_2 \in \mathscr{B}^{++}(H)$ respectively by A_1^p, A_2^q, B_1^p, B_2^q, the inequality (32) reads

$$
\begin{aligned}
&(1-t)A_1^p\sharp_{\frac{1}{p}}A_2^q+tB_1^p\sharp_{\frac{1}{p}}B_2^q-\left((1-t)A_1^p+tB_1^p\right)\sharp_{\frac{1}{p}}\left((1-t)A_2^q+tB_2^q\right)\\
&\qquad\leqslant \sum_{n=0}^{N-1}\frac{r_n(t)}{2^n}\sum_{k=1}^{2^n}\delta_{\sharp_{\frac{1}{p}}}(\mathbf{A}^{p,q},\mathbf{B}^{p,q},n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t),
\end{aligned}
$$

where $\mathbf{A}^{p,q}=\left(A_1^p,A_2^q\right)$ and $\mathbf{B}^{p,q}=\left(B_1^p,B_2^q\right)$. This relation represents a refinement of the weighted operator Hölder's inequality in two dimensional case.

Another example of connection is a parallel sum. Recall that for $X,Y\in\mathscr{B}^{++}(H)$ the parallel sum $:$ is defined by $X:Y=\left(X^{-1}+Y^{-1}\right)^{-1}$. Now, considering (32) with A_1^{-1}, A_2^{-1}, B_1^{-1}, B_2^{-1} instead of A_1, A_2, B_1, $B_2\in\mathscr{B}^{++}(H)$, we obtain the inequality

$$
\begin{aligned}
&(1-t)\left(A_1+A_2\right)^{-1}+t\left(B_1+B_2\right)^{-1}\\
&\quad-\left((1-t)A_1^{-1}+tB_1^{-1}\right)^{-1}-\left((1-t)A_2^{-1}+tB_2^{-1}\right)^{-1}\\
&\leqslant \sum_{n=0}^{N-1}\frac{r_n(t)}{2^n}\sum_{k=1}^{2^n}\delta_{:}(\mathbf{A}^{-1},\mathbf{B}^{-1},n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t),
\end{aligned}
$$

where $\mathbf{A}^{-1}=\left(A_1^{-1},A_2^{-1}\right)$ and $\mathbf{B}^{-1}=\left(B_1^{-1},B_2^{-1}\right)$. This inequality yields a refinement of the weighted form of Minkowski's inequality in two dimensional case. Note also that nonweighted versions of operator Hölder's and Minkowski's inequalities were established in [20], so our relations may be regarded as more accurate weighted extensions in two dimensional case. Furthermore, scalar forms of these relations were obtained in [21] (see also [19], p.718).

REMARK 11. A common example of a solidarity is relative operator entropy defined by $S(X|Y)=X^{\frac{1}{2}}\left(\log X^{-\frac{1}{2}}YX^{-\frac{1}{2}}\right)X^{\frac{1}{2}}$, where $X,Y\in\mathscr{B}^{++}(H)$. In this setting, the inequality (32) becomes

$$
\begin{aligned}
&(1-t)S(A_1|A_2)+tS(B_1|B_2)-S\left((1-t)A_1+tB_1|(1-t)A_2+tB_2\right)\\
&\qquad\leqslant \sum_{n=0}^{N-1}\frac{r_n(t)}{2^n}\sum_{k=1}^{2^n}\delta_S(\mathbf{A},\mathbf{B},n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t),
\end{aligned}
$$

providing a more accurate joint concavity relation for relative operator entropy. The previous relation can also be extended to hold for a parametric extension of the relative operator entropy known as the Tsallis relative operator entropy $T_\lambda(X|Y)=X^{\frac{1}{2}}\left(\log_\lambda X^{-\frac{1}{2}}YX^{-\frac{1}{2}}\right)X^{\frac{1}{2}}$, $0<\lambda\leqslant 1$, $X,Y\in\mathscr{B}^{++}(H)$ (see [7]).

REMARK 12. It is interesting that the weighted geometric mean can be considered as a jointly concave mapping. Fujii *et.al.* [5], introduced the weighted geometric mean $G[m,\mu]$, $0 \leqslant \mu \leqslant 1$, for an m-tuple of positive invertible operators $A_1, A_2, \dots, A_m$ as follows: Let $G[2,\mu](A_1,A_2) = A_1 \sharp_\mu A_2 = A_1^{\frac{1}{2}}(A_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}})^\mu A_1^{\frac{1}{2}}$. For $m \geqslant 3$, $G[m,\mu]$ is defined inductively: Put $A_i^{(1)} = A_i$, $i = 1,2,\dots,m$, and

$$A_i^{(r)} = G[n-1,\mu]\big(A_1^{(r-1)},\dots,A_{i-1}^{(r-1)},A_{i+1}^{(r-1)},\dots,A_n^{(r-1)}\big).$$

Then, there exist $\lim_{r\to\infty} A_i^{(r)}$ in the Thompson metric which does not depend on i and the weighted geometric mean is defined as $G[m,\mu](A_1,A_2,\dots,A_m) = \lim_{r\to\infty} A_i^{(r)}$. It has been shown in [5] that the geometric mean $G[m,\mu]$ is jointly concave mapping acting on m-tuple of positive invertible operators. Consequently, Corollary 6 yields more accurate joint concavity relation, i.e.

$$\begin{aligned}&(1-t)G[m,\mu](\mathbf{A}) + tG[m,\mu](\mathbf{B}) - G[m,\mu]((1-t)\mathbf{A}+t\mathbf{B})\\ &\quad\leqslant \sum_{n=0}^{N-1} r_n(t) \sum_{k=1}^{2^n} \Delta_{G[m,\mu]}(\mathbf{A},\mathbf{B},n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(t),\end{aligned}$$

where $\mathbf{A} = (A_1,A_2,\dots,A_m)$, $\mathbf{B} = (B_1,B_2,\dots,B_m)$, and $\Delta_{G[m,\mu]}$ is defined in Corollary 6.

5. Some strengthened Young-type inequalities for unitarily invariant norms

Finally, in the last section we will improve some important Young-type inequalities for unitarily invariant norms. For that sake, we are going to exploit the scalar inequality (8) presented in the Introduction.

In this section, $M_m(\mathbb{C})$ is the algebra of all $m \times m$ complex matrices and $|||\cdot|||$ stands for any unitarily invariant norm on $M_m(\mathbb{C})$. So, $|||UAV||| = |||A|||$ for all $A \in M_m(\mathbb{C})$ and for all unitary matrices $U,V \in M_m(\mathbb{C})$. The Hilbert-Schmidt norm, the trace norm, and the spectral norm of $A \in M_n(\mathbb{C})$ are defined by $\|A\|_2 = (\sum_{j=1}^m s_j^2(A))^{\frac{1}{2}}$, $\|A\|_1 = \sum_{j=1}^m s_j(A)$, and $\|A\| = s_1(A)$, respectively, where $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_m(A)$ are the singular values of A, i.e. the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$. Clearly, these norms are unitarily invariant.

The Hilbert-Schmidt norm plays an important role in matrix analysis since $\|A\|_2 = (\sum_{i,j=1}^m |a_{ij}|^2)^{\frac{1}{2}}$ for $A = [a_{ij}] \in M_m(\mathbb{C})$. This fact makes this norm easily computable and geometrically tractable.

Bhatia and Parthasarathy [2] and Kosaki [16], proved the so-called X-version of the Young inequality for the case of the Hilbert-Schmidt norm: if $A,B,X \in M_m(\mathbb{C})$ are such that A and B are positive semidefinite and $0 \leqslant t \leqslant 1$, then

$$\|(1-t)AX + tXB\|_2 \geqslant \|A^{1-t}XB^t\|_2. \tag{33}$$

Now, the scalar Young inequality (8) yields the improved form of (33).

THEOREM 3. *Let $A,B,X \in M_m(\mathbb{C})$ be such that A and B are positive semidefinite, and let $0 \leqslant t \leqslant 1$. If N is a nonnegative integer, then holds the inequality*

$$\begin{aligned}\|(1-t)AX+tXB\|_2^2 \geqslant \|A^{1-t}XB^t\|_2^2 + r_0^2(t)\|AX-XB\|_2^2 \\ + \sum_{n=1}^{N-1} r_n(t) \sum_{k=1}^{2^n} \|A^{1-\frac{k-1}{2^n}}XB^{\frac{k-1}{2^n}} - A^{1-\frac{k}{2^n}}XB^{\frac{k}{2^n}}\|_2^2 \chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t).\end{aligned} \tag{34}$$

Proof. Utilizing the spectral theorem for positive semidefinite matrices A and B, it follows that there exist unitary matrices $U,V \in M_m(\mathbb{C})$ such that $A = U\Lambda U^*$ and $B = V\Gamma V^*$, where $\Lambda = \operatorname{diag}(\lambda_1,\lambda_2,\dots,\lambda_m)$ and $\Gamma = \operatorname{diag}(\gamma_1,\gamma_2,\dots,\gamma_m)$, with $\lambda_i,\gamma_i \geqslant 0$ for $i = 1,2,\dots,m$. If $Y = U^*XV = [y_{ij}]$, then

$$\begin{aligned}(1-t)AX+tXB =& U\left[((1-t)\lambda_i + t\gamma_j)y_{ij}\right]V^* \\ A^{1-t}XB^t =& U\left[\lambda_i^{1-t}\gamma_j^t y_{ij}\right]V^*\end{aligned}$$

and

$$A^{1-\frac{k-1}{2^n}}XB^{\frac{k-1}{2^n}} - A^{1-\frac{k}{2^n}}XB^{\frac{k}{2^n}} = U\left[\left(\lambda_i^{1-\frac{k-1}{2^n}}\gamma_j^{\frac{k-1}{2^n}} - \lambda_i^{1-\frac{k}{2^n}}\gamma_j^{\frac{k}{2^n}}\right)y_{ij}\right]V^*.$$

On the other hand, a straightforward computation shows that the relation

$$((1-t)a+tb)^2 - r_0^2(t)(a-b)^2 = (1-t)a^2 + tb^2 - r_0(t)(a-b)^2$$

holds for all nonnegative real numbers a and b. Now, taking into account this relation and the inequality (8), it follows that

$$\begin{aligned}\|(1-t)AX+tXB\|_2^2 &= \sum_{i,j=1}^m ((1-t)\lambda_i + t\gamma_j)^2 |y_{ij}|^2 \\ &= r_0^2(t)\sum_{i,j=1}^m (\lambda_i-\gamma_j)^2|y_{ij}|^2 \\ &\quad + \sum_{i,j=1}^m \left((1-t)\lambda_i^2 + t\gamma_j^2 - r_0(t)(\lambda_i-\gamma_j)^2\right)|y_{ij}|^2 \\ &\geqslant r_0^2(t)\sum_{i,j=1}^m (\lambda_i-\gamma_j)^2|y_{ij}|^2 + \sum_{i,j=1}^m \left(\lambda_i^{1-t}\gamma_j^t\right)^2|y_{ij}|^2 \\ &\quad + \sum_{n=1}^{N-1} r_n(t)\sum_{k=1}^{2^n}\sum_{i,j=1}^m \left(\lambda_i^{1-\frac{k-1}{2^n}}\gamma_j^{\frac{k-1}{2^n}} - \lambda_i^{1-\frac{k}{2^n}}\gamma_j^{\frac{k}{2^n}}\right)^2|y_{ij}|^2\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t) \\ &= r_0^2(t)\|AX-XB\|_2^2 + \|A^{1-t}XB^t\|_2^2 \\ &\quad + \sum_{n=1}^{N-1} r_n(t)\sum_{k=1}^{2^n}\|A^{1-\frac{k-1}{2^n}}XB^{\frac{k-1}{2^n}} - A^{1-\frac{k}{2^n}}XB^{\frac{k}{2^n}}\|_2^2\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t). \quad \square\end{aligned}$$

REMARK 13. If $N=1$, the inequality (34) becomes the refinement of (33) due to Hirzallah and Kittaneh [10]. In addition, if $N=2$ the inequality (34) becomes the relation established in [25]. Some related refinements of the Young inequality (33), with some extra conditions on matrices A and B can also be found in [12].

If X is a unit matrix, the inequality (33) holds for every unitarily invariant norm (see [1]). Otherwise, the inequality (33) may not hold for other unitarily invariant norms. However, it has been shown in [16] that the following weakened form of the Young inequality holds for every unitarily invariant norm:

$$(1-t)|||AX|||+t|||XB|||\geqslant |||A^{1-t}XB^t|||.$$

This weakened form of the Young inequality can also be refined via (8).

COROLLARY 8. *Let $A,B,X\in M_m(\mathbb{C})$ be such that A and B are positive semidefinite and $0\leqslant t\leqslant 1$. If N is a nonnegative integer then holds the inequality*

$$\begin{aligned}(1-t)&|||AX|||+t|||XB|||\\ &\geqslant |||A^{1-t}XB^t|||+\sum_{n=0}^{N-1} r_n(t)\sum_{k=1}^{2^n}\left(|||AX|||^{\frac12-\frac{k-1}{2^{n+1}}}|||XB|||^{\frac{k-1}{2^{n+1}}}\right.\\ &\qquad\left.-|||AX|||^{\frac12-\frac{k}{2^{n+1}}}|||XB|||^{\frac{k}{2^{n+1}}}\right)^2\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t).\end{aligned} \tag{35}$$

Proof. It follows from the inequality (8) with $a=|||AX|||$ and $b=|||XB|||$, since $|||A^{1-t}XB^t|||\leqslant |||AX|||^{1-t}|||XB|||^t$ (see [13]). $\square$

REMARK 14. Specializing the inequality (35) for the trace norm and letting X to be a unit matrix, it follows that

$$\begin{aligned}&\operatorname{tr}((1-t)A+tB)\\ &\geqslant \operatorname{tr}\left|A^{1-t}B^t\right|+\sum_{n=0}^{N-1} r_n(t)\sum_{k=1}^{2^n}\left((\operatorname{tr}A)^{\frac12-\frac{k-1}{2^{n+1}}}(\operatorname{tr}B)^{\frac{k-1}{2^{n+1}}}-(\operatorname{tr}A)^{\frac12-\frac{k}{2^{n+1}}}(\operatorname{tr}B)^{\frac{k}{2^{n+1}}}\right)^2\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(t),\end{aligned}$$

which represents improved trace version of the Young inequality. If $N=1$ the above inequality reduces to the relation established in [14].

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