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THE COMPLEX L_p LOOMIS–WHITNEY INEQUALITY

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Abstract. The complex L_p Loomis-Whitney inequality for complex isotropic measures is established, which extends the real version of the L_p Loomis-Whitney inequality for isotropic measures due to the first two authors.

1. Introduction

A convex body K is a compact convex set in $\mathbb{R}^n$ which is assumed to contain the origin in its interior. Denote by $V(K)$ the corresponding dimensional volume. Each convex body K is uniquely determined by its support function $h(K,\cdot)$ defined by, for $x \in \mathbb{R}^n$, $h(K,x) = \max\{\langle x,y\rangle : y \in K\}$, where $\langle x,y\rangle = \sum_{k=1}^n x_k y_k$ denotes the scalar product of x and y in $\mathbb{R}^n$. If $x,y \in \mathbb{C}^n$, we denote their complex scalar product by $\langle x,y\rangle_c = \sum_{k=1}^n x_k \overline{y_k}$ and the modulus of x by $\|x\| = \sqrt{\langle x,x\rangle_c}$.

The classical Loomis-Whitney inequality states that for a convex body K in $\mathbb{R}^n$,

$$V(K)^{n-1} \leqslant \prod_{k=1}^n V(K|e_k^\perp), \tag{1}$$

where $K|e_k^\perp$ denotes the orthogonal projection of K onto the 1-codimensional subspace $e_k^\perp$ perpendicular to e_k and $\{e_1,\ldots,e_n\}$ is the canonical basis of $\mathbb{R}^n$. Moreover, equality in (1) holds if and only if K is a coordinate box; i.e., up to translations, there are positive numbers $(\alpha_k)_{k=1}^n$ such that

$$K = \sum_{k=1}^n \alpha_k[-e_k,e_k],$$

where $[-e_k,e_k]$ is the segment jointing $-e_k$ to e_k and the sum is the Minkowski addition of convex sets. This inequality was first proved by Loomis and Whitney [27] in

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1949 and has been widely studied in recent years (see e.g., [8, 9, 11, 12, 15, 16, 17, 18, 20, 24, 25, 26, 33]). It is well-known (see e.g., [32, (5.77)]) that

$$V(K|e_k^{\perp}) = \frac{n}{2}V(K,[n-1];[-e_k,e_k]),$$

where $V(K,[n-1];[-e_k,e_k])$ is the mixed volume of $(n-1)$-copies of K and one copy of $[-e_k,e_k]$. Thus, the Loomis-Whitney inequality (1) can be rewritten as

$$V(K)^{n-1} \leqslant \frac{n^n}{2^n}\prod_{k=1}^{n} V(K,[n-1];[-e_k,e_k]). \tag{2}$$

In order to define the volume in $\mathbb{C}^n$, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the standard mapping from $x=(x_1,\dots,x_n)=(x_{11}+ix_{12},\dots,x_{n1}+ix_{n2})$ to $(x_{11},x_{12},\dots,x_{n1},x_{n2})$. A complex version of (2), as a special case of our main result, can be stated as follows: if K is a convex body in $\mathbb{R}^{2n}$, then

$$V(K)^{2n-1} \leqslant \frac{n^{2n}}{\pi^n}\prod_{k=1}^{n} V(K,[2n-1];D_k)^2, \tag{3}$$

where D_k is a unit disc in $\operatorname{span}\{e_{2k-1},e_{2k}\}$ and $\{e_1,\dots,e_{2n}\}$ denotes the canonical basis of $\mathbb{R}^{2n}$. Moreover, equality in (3) holds if and only if K is a polydisc; i.e., up to translations, there are positive numbers $(\alpha_k)_{k=1}^n$ such that

$$K=\sum_{k=1}^{n}\alpha_k D_k.$$

Motivated by the recent work of the first two authors [24] on the L_p Loomis-Whitney inequality for isotropic measures, this paper is devoted to the *complex L_p Loomis-Whitney inequality* for complex isotropic measures. The following two notions are essential to our main result.

The *complex isotropic measure*, recently introduced by the first author and He [19], is a Borel measure μ on the unit sphere S^{2n-1} of $\mathbb{C}^n$ satisfying

$$\int_{S^{2n-1}} |\langle x,v\rangle_c|^2 d\mu(v) = \|x\|^2, \tag{4}$$

for all $x\in\mathbb{C}^n$. Since we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, (4) can be written as

$$\int_{S^{2n-1}} [\langle x,v\rangle^2 + \langle x,v^{\dagger}\rangle^2] d\mu(v) = \|x\|^2, \tag{5}$$

where the operator $\dagger:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ is defined as

$$v=(v_{11},v_{12},\dots,v_{n1},v_{n2}) \mapsto v^{\dagger}=(-v_{12},v_{11},\dots,-v_{n2},v_{n1}).$$

An important example of complex isotropic measures on S^{2n-1} is the *complex cross measure* introduced in [19], which is the R_θ-invariant complex isotropic measure μ such that

$$\operatorname{supp}\mu = \{\operatorname{span}\{v_1,v_1^{\dagger}\}\cap S^{2n-1},\dots,\operatorname{span}\{v_n,v_n^{\dagger}\}\cap S^{2n-1}\},$$

where $\{v_1, v_1^\dagger \ldots, v_n, v_n^\dagger\}$ is an orthonormal basis of $\mathbb{R}^{2n}$. Furthermore, a *generalized $\ell_p(\mathbb{C}^n)$-ball* $B_{p,\alpha}(\mathbb{C}^n) := B_{p,\alpha}(\mathbb{C}^n)(\mu)$ formed by the complex cross measure μ (concentrated on $\{\mathrm{span}\{v_1, v_1^\dagger\} \cap S^{2n-1}, \ldots, \mathrm{span}\{v_n, v_n^\dagger\} \cap S^{2n-1}\}$) is defined as follows: there are positive numbers $(\alpha_k)_{k=1}^n$ such that

$$\begin{aligned} B_{p,\alpha}(\mathbb{C}^n) &= \Big\{x \in \mathbb{R}^{2n} : \Big(\sum_{k=1}^n \alpha_k |\langle x, v_k\rangle_c|^p\Big)^{\frac{1}{p}} \leqslant 1\Big\} \\ &= \Big\{x \in \mathbb{R}^{2n} : \Big(\sum_{k=1}^n \alpha_k [\langle x, v_k\rangle^2 + \langle x, v_k^\dagger\rangle^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}} \leqslant 1\Big\}, \quad 1 \leqslant p < \infty, \end{aligned} \tag{6}$$

and for $p = \infty$,

$$\begin{aligned} B_{\infty,\alpha}(\mathbb{C}^n) &= \Big\{x \in \mathbb{R}^{2n} : \alpha_k |\langle x, v_k\rangle_c| \leqslant 1 \text{ for all } k = 1, \ldots, n\Big\} \\ &= \Big\{x \in \mathbb{R}^{2n} : \alpha_k [\langle x, v_k\rangle^2 + \langle x, v_k^\dagger\rangle^2]^{\frac{1}{2}} \leqslant 1 \text{ for all } k = 1, \ldots, n\Big\}. \end{aligned} \tag{7}$$

We shall mention that $B_{\infty,\alpha}(\mathbb{C}^n) = \sum_{k=1}^n \alpha_k^{-1}(B_2^{2n} \cap \mathrm{span}\{v_k, v_k^\dagger\})$ is also called a polydisc formed by μ, where B_2^{2n} is the Euclidean unit ball in $\mathbb{R}^{2n}$.

For $p \geqslant 1$, we define the *R_θ-invariant L_p complex projection body* $\Pi_p^D K$ of a convex body K in $\mathbb{R}^{2n}$, in terms of its support function is given by, for $v \in S^{2n-1}$,

$$\begin{aligned} h(\Pi_p^D K, v) &= \Big(\frac{1}{2n}\int_{S^{2n-1}} |\langle v, u\rangle_c|^p dS_p(K, u)\Big)^{\frac{1}{p}} \\ &= \Big(\frac{1}{2n}\int_{S^{2n-1}} [\langle v, u\rangle^2 + \langle v, u^\dagger\rangle^2]^{\frac{p}{2}} dS_p(K, u)\Big)^{\frac{1}{p}}, \end{aligned}$$

where $dS_p(K, \cdot)$ is the L_p surface area measure of K. For $p = 1$, it reduces to the R_θ-invariant complex projection body introduced by Abardia and Bernig [3].

Thus, the complex Loomis-Whitney inequality for complex isotropic measures can be formulated as follows:

THEOREM 1. *Suppose $p \geqslant 1$ and K is a convex body in $\mathbb{R}^{2n}$. If μ is a complex isotropic measure on S^{2n-1}, then*

$$V(K)^{\frac{2n-p}{p}} \leqslant A_{n,p} \exp\Big\{\int_{S^{2n-1}} \log h(\Pi_p^D K, v)^2 d\mu(v)\Big\}, \tag{8}$$

where

$$A_{n,p} = \frac{n^{2n/p}\Gamma(2n+1-\frac{2n}{p})}{\pi^n \Gamma(3-\frac{2}{p})^n}.$$

In addition, if μ is a complex cross measure on S^{2n-1}, then equality in (8) *holds for $p > 1$ if and only if K is a generalized $\ell_{p^*}(\mathbb{C}^n)$-ball formed by μ and equality in* (8) *holds for $p = 1$ if and only if K is a polydisc formed by μ (up to translations).*

Here p^* is the Hölder conjugate of p; i.e., $1/p+1/p^*=1$.

When $p=1$, together with (22), inequality (8) reduces to

$$V(K)^{2n-1} \leqslant \frac{n^{2n}}{\pi^n} \exp\Big(\int_{S^{2n-1}} \log V(K,[2n-1];D\cdot v)^2 d\mu(v)\Big), \tag{9}$$

where $D\cdot v := \{cv : c \in D\}$ and D is the unit disk in $\mathbb{C}$. Inequality (3) now follows from (9) by taking the *basic* complex cross measure μ, which is a complex cross measure such that $\operatorname{supp}\mu = \{\operatorname{span}\{e_1,e_2\}\cap S^{2n-1},\ldots,\operatorname{span}\{e_{2n-1},e_{2n}\}\cap S^{2n-1}\}$. Note that a complex cross measure is just a rotation of the basic complex cross measure, since $\{v_1,v_1^\dagger,\ldots,v_n,v_n^\dagger\}$ is an orthonormal basis of $\mathbb{R}^{2n}$.

2. Background materials

2.1. Elements of the L_p Brunn-Minkowski theory

We collect in this section some elements of the L_p Brunn-Minkowski theory, which has its origins in the work of Firey from the 1960s and has expanded rapidly over the last two decade since the remarkable works of Lutwak [28, 29]. For further details we refer the reader to [32, Chapter 9] and the references therein.

The Minkowski functional $\|\cdot\|_K$ of a convex body K in $\mathbb{R}^n$ is defined by $\|x\|_K = \min\{\lambda \geqslant 0 : x \in \lambda K\}$. In this case,

$$h(K,\cdot) = \|\cdot\|_{K^*}, \tag{10}$$

where the polar body K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : \langle x,y\rangle \leqslant 1 \text{ for all } y \in K\}.$$

For $A \in \mathrm{GL}(\mathbb{R}^n)$, we have

$$(AK)^* = A^{-t}K^*, \tag{11}$$

where A^{-t} is the inverse of the transpose of A. Using the polar coordinate formula, it is easy to see that the volume of a convex body K in $\mathbb{R}^n$ is given by

$$V(K) = \frac{1}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n} \exp(-\|x\|_K^p)dx, \tag{12}$$

where the integral is with respect to Lebesgue measure on $\mathbb{R}^n$.

For $p \geqslant 1$ and $\varepsilon > 0$, the L_p Minkowski-Firey combination $K +_p \varepsilon\cdot L$ of convex bodied K,L is the convex body whose support function is given by

$$h(K +_p \varepsilon\cdot L,\cdot)^p = h(K,\cdot)^p + \varepsilon h(L,\cdot)^p.$$

The L_p mixed volume $V_p(K,L)$ of K,L was defined in [28] by

$$V_p(K,L) = \frac{p}{n}\lim_{\varepsilon\to 0^+} \frac{V(K +_p \varepsilon\cdot L) - V(K)}{\varepsilon}.$$

In particular, $V_p(K,K) = V(K)$. It was shown in [28] that for convex bodies K, L, there is a positive Borel measure, $S_p(K,\cdot)$, on S^{n-1} so that

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)^p dS_p(K,u), \tag{13}$$

where $dS_p(K,\cdot) = h(K,\cdot)^{1-p} dS(K,\cdot)$ is the L_p surface area measure of K and $dS(K,\cdot)$ is the classical surface area measure of K. Recall that for a Borel set $\omega \subset S^{n-1}$, $S(K,\omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K for which there exists a normal vector of K belonging to ω.

The L_p Minkowski inequality [28] states that for convex bodies K, L,

$$V_p(K,L)^n \geqslant V(K)^{n-p} V(L)^p, \tag{14}$$

with equality if and only if K and L are dilates when $p > 1$, and if and only if K and L are homothetic (i.e., they coincide up to translations and dilatations) when $p = 1$.

2.2. Complex isotropic measures

The unit sphere $\{x \in \mathbb{C}^n : \|x\| = 1\}$ of $\mathbb{C}^n$ is denoted by S^{2n-1}. Since we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we can say that a convex body K is R_θ-invariant if for each $\theta \in [0, 2\pi]$ and each $x = (x_{11}, x_{12}, \dots x_{n1}, x_{n2}) \in \mathbb{R}^{2n}$,

$$\|x\|_K = \|R_\theta(x_{11}, x_{12}), \dots, R_\theta(x_{n1}, x_{n2})\|_K,$$

where R_θ stands for the counterclockwise rotation of $\mathbb{R}^2$ by the angle θ with respect to the origin. We say a measure (or a function) on S^{2n-1} is R_θ-invariant if it assumes the same value on a set (or a point) and its R_θ image for each $\theta \in [0, 2\pi]$. For $\xi \in \mathbb{C}^n$ such that $\|\xi\| = 1$, denote by

$$H_\xi = \left\{ x \in \mathbb{C}^n : \langle x, \xi \rangle_c = \sum_{k=1}^n x_k \overline{\xi_k} = 0 \right\}$$

the complex hyperplane through the origin perpendicular to ξ. Under the mapping from $\mathbb{C}^n$ to $\mathbb{R}^{2n}$ the hyperplane H_ξ is a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors

$$\xi = (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}) \quad \text{and} \quad \xi^\dagger = (-\xi_{12}, \xi_{11}, \dots, -\xi_{n2}, \xi_{n1}).$$

The complex isotropic measure μ defined in (4) has the following properties (see [19]):

- the complex isotropic measure μ is not concentrated on $H_\xi \cap S^{2n-1}$ for any $\xi \in S^{2n-1}$.
- $\mu(S^{2n-1}) = n$.

2.3. Generalized $\ell_p(\mathbb{C}^n)$-balls

Let $B_p(\mathbb{C}^n)$ denote the unit ball of $\ell_p(\mathbb{C}^n)$-space, understood as

$$B_p(\mathbb{C}^n)=\Big\{x\in\mathbb{R}^{2n}:\Big(\sum_{k=1}^n[x_{k1}^2+x_{k2}^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}}\leqslant 1\Big\},\quad 1\leqslant p<\infty,$$

and for $p=\infty$,

$$B_\infty(\mathbb{C}^n)=\{x\in\mathbb{R}^{2n}:[x_{k1}^2+x_{k2}^2]^{\frac{1}{2}}\leqslant 1,\ \text{for all}\ k=1,\dots,n\}.$$

For $n,p\in[1,\infty)$, denote by $\kappa_{2n}(p)$ the volume of the unit ball of $\ell_p(\mathbb{C}^n)$ (see [19, Proposition 6.1]), which equals to

$$\kappa_{2n}(p)=\frac{\pi^n(\Gamma(1+\frac{2}{p}))^n}{\Gamma(1+\frac{2n}{p})}.$$

Recall that a generalized $\ell_p(\mathbb{C}^n)$-ball $B_{p,\alpha}(\mathbb{C}^n)$ formed by the complex cross measure μ is defined in (6) and (7). Let $A=\operatorname{diag}\{\alpha_1^{1/p},\alpha_1^{1/p},\dots,\alpha_n^{1/p},\alpha_n^{1/p}\}$. Since there exists $U\in\mathrm{O}(\mathbb{R}^{2n})$ such that $v_k=Ue_{2k-1}$ and $v_k^\dagger=Ue_{2k}$ for $k=1,\dots,n$, we have

$$\begin{aligned}B_{p,\alpha}(\mathbb{C}^n)&=\Big\{x\in\mathbb{R}^{2n}:\Big(\sum_{k=1}^n\alpha_k[\langle x,v_k\rangle^2+\langle x,v_k^\dagger\rangle^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}}\leqslant 1\Big\}\\&=\Big\{x\in\mathbb{R}^{2n}:\Big(\sum_{k=1}^n\alpha_k[\langle x,Ue_{2k-1}\rangle^2+\langle x,Ue_{2k}\rangle^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}}\leqslant 1\Big\}\\&=\Big\{x\in\mathbb{R}^{2n}:\Big(\sum_{k=1}^n\alpha_k[\langle U^{-1}x,e_{2k-1}\rangle^2+\langle U^{-1}x,e_{2k}\rangle^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}}\leqslant 1\Big\}\\&=\Big\{x\in\mathbb{R}^{2n}:\Big(\sum_{k=1}^n[\langle AU^{-1}x,e_{2k-1}\rangle^2+\langle AU^{-1}x,e_{2k}\rangle^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}}\leqslant 1\Big\}\\&=\Big\{UA^{-1}x\in\mathbb{R}^{2n}:\Big(\sum_{k=1}^n[\langle x,e_{2k-1}\rangle^2+\langle x,e_{2k}\rangle^2]^{\frac{p}{2}}\Big)^{\frac{1}{p}}\leqslant 1\Big\}\\&=UA^{-1}B_p(\mathbb{C}^n).\end{aligned}\tag{15}$$

Then we immediately get

$$V(B_{p,\alpha}(\mathbb{C}^n))=V(UA^{-1}B_p(\mathbb{C}^n))=V(B_p(\mathbb{C}^n))\Big(\prod_{k=1}^n\alpha_k\Big)^{-\frac{2}{p}}.\tag{16}$$

It follows from (10) and (6) that

$$h((B_{p,\alpha}(\mathbb{C}^n))^*,x)=\Big(\sum_{k=1}^n\alpha_k|\langle x,v_k\rangle_c|^p\Big)^{\frac{1}{p}}.\tag{17}$$

Moreover, for $p>1$, by (15), (11) and [19, Proposition 2.1], we have

$$
\begin{aligned}
(B_{p,\alpha}(\mathbb{C}^n))^* &= (UA^{-1}B_p(\mathbb{C}^n))^* = UA^t(B_p(\mathbb{C}^n))^* = UA^tB_{p^*}(\mathbb{C}^n)\\
&= \Big\{UA^tx\in\mathbb{R}^{2n} : \Big(\sum_{k=1}^n[\langle x,e_{2k-1}\rangle^2+\langle x,e_{2k}\rangle^2]^{\frac{p^*}{2}}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}\\
&= \Big\{x\in\mathbb{R}^{2n} : \Big(\sum_{k=1}^n[\langle A^{-t}U^{-1}x,e_{2k-1}\rangle^2+\langle A^{-t}U^{-1}x,e_{2k}\rangle^2]^{\frac{p^*}{2}}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}\\
&= \Big\{x\in\mathbb{R}^{2n} : \Big(\sum_{k=1}^n[\langle x,UA^{-1}e_{2k-1}\rangle^2+\langle x,UA^{-1}e_{2k-1}\rangle^2]^{\frac{p^*}{2}}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}\\
&= \Big\{x\in\mathbb{R}^{2n} : \Big(\sum_{k=1}^n\alpha_k^{-p^*/p}[\langle x,v_k\rangle^2+\langle x,v_k^\dagger\rangle^2]^{\frac{p^*}{2}}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}\\
&= B_{p^*,\alpha^{-p^*/p}}(\mathbb{C}^n). \qquad (18)
\end{aligned}
$$

For $p=1$, by the same way, we have

$$(B_{1,\alpha}(\mathbb{C}^n))^* = B_{\infty,\alpha^{-1}}(\mathbb{C}^n). \qquad (19)$$

Then, from (16) we obtain, for $p>1$,

$$V((B_{p,\alpha}(\mathbb{C}^n))^*) = V(B_{p^*,\alpha^{-p^*/p}}(\mathbb{C}^n)) = V(B_{p^*}(\mathbb{C}^n))\Big(\prod_{k=1}^n\alpha_k\Big)^{\frac{2}{p}}, \qquad (20)$$

and for $p=1$,

$$V((B_{1,\alpha}(\mathbb{C}^n))^*) = V(B_{\infty,\alpha^{-1}}(\mathbb{C}^n)) = V(B_\infty(\mathbb{C}^n))\Big(\prod_{k=1}^n\alpha_k\Big)^2. \qquad (21)$$

2.4. Complex L_p projection bodies

In recent years, the study of varieties of convex bodies in $\mathbb{C}^n$ has received considerable attention; see, e.g., [1, 2, 3, 4, 5, 6, 7, 13, 14, 21, 22, 23, 31]. For example, the notion of the complex projection body was introduced by Abardia and Bernig [3] in 2011: for a convex body $K\subset\mathbb{C}^n$ and a convex body $C\subset\mathbb{C}$, the complex projection body Π^CK is the convex body whose support function is defined by

$$h(\Pi^CK,v) = V(K,[2n-1];C\cdot v),\quad v\in\mathbb{C}^n,$$

where $C\cdot v:=\{cv : c\in C\}\subset\mathbb{C}^n$. Obviously, the R_θ-invariant complex projection body Π^DK can be defined by letting C be a unit disk D in $\mathbb{C}$; i.e.,

$$
\begin{aligned}
h(\Pi^D K, v) &= V(K, [2n-1]; D\cdot v) \\
&= \frac{1}{2n}\int_{S^{2n-1}} h(D\cdot v, u)\, dS(K,u) \\
&= \frac{1}{2n}\int_{S^{2n-1}} \sup_{\theta\in[0,2\pi]} \{Re\langle e^{i\theta}v, u\rangle_c\}\, dS(K,u) \\
&= \frac{1}{2n}\int_{S^{2n-1}} \sup_{\theta\in[0,2\pi]} \{Re(e^{i\theta}\langle v, u\rangle_c)\}\, dS(K,u) \\
&= \frac{1}{2n}\int_{S^{2n-1}} |\langle v, u\rangle_c|\, dS(K,u), \qquad (22)
\end{aligned}
$$

for every $v\in\mathbb{C}^n$. For $p\geqslant 1$, the R_θ-invariant L_p complex projection body $\Pi_p^D K$ of a convex body K in $\mathbb{C}^n$ can be defined by

$$
h(\Pi_p^D K, v) = \Big(\frac{1}{2n}\int_{S^{2n-1}} |\langle v, u\rangle_c|^p\, dS_p(K,u)\Big)^{\frac{1}{p}}, \quad v\in\mathbb{C}^n. \qquad (23)
$$

The fact that $h(\Pi_p^D K, v)$ is the support function of a convex body in $\mathbb{R}^{2n}$ can be verified as in [19, Theorem 4.3].

3. Proof of the main result

Assume that the measure μ is not concentrated on $H_\xi\cap S^{2n-1}$ for any $\xi\in S^{2n-1}$. Let $\alpha: S^{2n-1}\to(0,+\infty)$ be a R_θ-invariant positive continuous function. For $p\geqslant 1$, we define the *complex L_p zonoid* $Z_{p,\alpha}(\mu)$ with generating measure $\alpha d\mu$ as the R_θ-invariant convex body in $\mathbb{C}^n$, in terms of its support function, for $u\in S^{2n-1}$,

$$
\begin{aligned}
h(Z_{p,\alpha}(\mu), u) &= \Big(\int_{S^{2n-1}} |\langle u, v\rangle_c|^p \alpha(v)\, d\mu(v)\Big)^{\frac{1}{p}} \\
&= \Big(\int_{S^{2n-1}} \big\|u|\mathrm{span}\{v, v^\dagger\}\big\|^p \alpha(v)\, d\mu(v)\Big)^{\frac{1}{p}}. \qquad (24)
\end{aligned}
$$

Here $\big\|u|\mathrm{span}\{v,v^\dagger\}\big\|$ is the length of the orthogonal projection of u onto the 2-dimensional subspace $\mathrm{span}\{v,v^\dagger\}$. The fact that $h(Z_{p,\alpha}(\mu),u)$ is the support function of a convex body in $\mathbb{R}^{2n}$ can be verified as in [19, Theorem 4.3].

In particular, if μ is a complex cross measure, then we may assume that $\mathrm{supp}\mu = \{\mathrm{span}\{v_1, v_1^\dagger\}\cap S^{2n-1}, \ldots, \mathrm{span}\{v_n, v_n^\dagger\}\cap S^{2n-1}\}$. It was shown in [19, Lemma 4.1] that $\mu(\mathrm{span}\{v_k, v_k^\dagger\}\cap S^{2n-1}) = 1$ for $1\leqslant k\leqslant n$. Denote $\alpha(v_k) =: \alpha_k > 0$. By (24), (17), (18) and (19), we have, for $p>1$,

$$
\begin{aligned}
h(Z_{p,\alpha}(\mu), x) &= \Big(\sum_{k=1}^n \alpha(v_k)|\langle x, v_k\rangle_c|^p\Big)^{\frac{1}{p}} \\
&= h((B_{p,\alpha}(\mathbb{C}^n))^*, x) = h(B_{p^*,\alpha^{-p^*/p}}(\mathbb{C}^n), x), \qquad (25)
\end{aligned}
$$

and

$$h(Z_{1,\alpha}(\mu),x) = h((B_{1,\alpha}(\mathbb{C}^n))^*,x) = h(B_{\infty,\alpha^{-1}}(\mathbb{C}^n),x), \tag{26}$$

for each $x \in \mathbb{R}^{2n}$. From (20) and (21), we get, for $p > 1$,

$$V(Z_{p,\alpha}(\mu)) = V(B_{p^*,\alpha^{-p^*/p}}(\mathbb{C}^n)) = V(B_{p^*}(\mathbb{C}^n))\Big(\prod_{k=1}^{n}\alpha_k\Big)^{\frac{2}{p}}, \tag{27}$$

and

$$V(Z_{1,\alpha}(\mu)) = V(B_{\infty,\alpha^{-1}}(\mathbb{C}^n)) = V(B_\infty(\mathbb{C}^n))\Big(\prod_{k=1}^{n}\alpha_k\Big)^2. \tag{28}$$

The following particular case of multidimensional reverse Brascamp-Lieb inequality [10] is needed.

LEMMA 1. *Suppose $v_1,\dots,v_m \in S^{2n-1}$ and $c_1,\dots,c_m > 0$ such that*

$$\sum_{k=1}^{m} c_k\big\|x|\mathrm{span}\{v_k,v_k^\dagger\}\big\|^2 = \|x\|^2 \quad \textit{for every } x \in \mathbb{R}^{2n}. \tag{29}$$

Then for all integrable functions $f_i : \mathrm{span}\{v_k,v_k^\dagger\} \to [0,\infty)$, $1 \leqslant k \leqslant m$,

$$\int_{\mathbb{R}^{2n}}^{*} \sup\Big\{\prod_{k=1}^{m} f_k(y_k)^{c_k} : x = \sum_{k=1}^{m} c_k y_k, y_k \in \mathrm{span}\{v_k,v_k^\dagger\}\Big\}dx \geqslant \prod_{k=1}^{m}\Big(\int_{\mathrm{span}\{v_k,v_k^\dagger\}} f_k\Big)^{c_k}.$$

The following lemma extends Theorem 6.5 in [19].

LEMMA 2. *Suppose $p \geqslant 1$ and α is a R_θ-invariant continuous positive function on S^{2n-1}. If μ is a complex isotropic measure on S^{2n-1}, then*

$$V(Z_{p,\alpha}(\mu)) \geqslant V(B_{p^*}(\mathbb{C}^n))\Big(\exp\int_{S^{2n-1}}\log\alpha(v)d\mu(v)\Big)^{\frac{2}{p}}, \tag{30}$$

with equality if μ is a complex cross measure on S^{2n-1}.

Proof. Suppose the measure $\mu = \sum_{k=1}^{m} c_k\delta_{v_k}$ is a discrete complex isotropic measure on S^{2n-1}. Then the complex isotropic condition (5) is just the condition (29). Write $\alpha(v_k) =: \alpha_k > 0$.

Case $p = 1$: By (24) and the fact that $\big\|x|\mathrm{span}\{v_k,v_k^\dagger\}\big\| = h(B_2^{2n}|\mathrm{span}\{v_k,v_k^\dagger\},x)$, we have, for every $x \in \mathbb{R}^{2n}$,

$$\begin{aligned} h(Z_{1,\alpha}(\mu),x) &= \sum_{k=1}^{m}\alpha_k c_k\big\|x|\mathrm{span}\{v_k,v_k^\dagger\}\big\| \\ &= \sum_{k=1}^{m}\alpha_k c_k h(B_2^{2n}|\mathrm{span}\{v_k,v_k^\dagger\},x) \\ &= h\Big(\sum_{k=1}^{m} c_k\alpha_k B_2^{2n}|\mathrm{span}\{v_k,v_k^\dagger\},x\Big). \end{aligned}$$

Hence

$$Z_{1,\alpha}(\mu) = \left\{x \in \mathbb{R}^{2n} : x = \sum_{k=1}^{m} c_k y_k, y_k \in \alpha_k B_2(\mathbb{R}^{2n})|\text{span}\{v_k, v_k^{\dagger}\}\right\}. \tag{31}$$

Define functions $f_k : \text{span}\{v_k, v_k^{\dagger}\} \to [0,\infty), 1 \leqslant k \leqslant m$, by

$$f_k(y) = \mathbf{1}_{[0,\alpha_k]}(\|y\|).$$

From (31), Lemma 1 and the fact that $\mu(S^{2n-1}) = \sum_{k=1}^{m} c_k = n$, we obtain

$$\begin{aligned}
V(Z_{1,\alpha}(\mu)) &= \int_{\mathbb{R}^{2n}} \sup\left\{\prod_{k=1}^{m} \mathbf{1}_{[0,\alpha_k]}(\|y_k\|)^{c_k} : x = \sum_{k=1}^{m} c_k y_k, y_k \in \text{span}\{v_k, v_k^{\dagger}\}\right\} dx \\
&= \int_{\mathbb{R}^{2n}} \sup\left\{\prod_{k=1}^{m} f_k(y_k)^{c_k} : x = \sum_{k=1}^{m} c_k y_k, y_k \in \text{span}\{v_k, v_k^{\dagger}\}\right\} dx \\
&\geqslant \prod_{k=1}^{m}\left(\int_{\text{span}\{v_k, v_k^{\dagger}\}} f_k\right)^{c_k} = \prod_{k=1}^{m}\left(\int_{\text{span}\{v_k, v_k^{\dagger}\}} \mathbf{1}_{[0,\alpha_k]}(\|x\|) dx\right)^{c_k} \\
&= \pi^n \prod_{k=1}^{m} \alpha_k^{2c_k}.
\end{aligned}$$

Case $p > 1$: We claim that

$$\|x\|_{Z_{p,\alpha}(\mu)}^{p^*} \leqslant \inf\left\{\sum_{k=1}^{m} \alpha_k^{1-p^*} c_k (r_{k1}^2 + r_{k2}^2)^{p^*/2} : \sum_{k=1}^{m} c_k (r_{k1} v_k + r_{k2} v_k^{\dagger}) = x\right\}. \tag{32}$$

In fact, let $x = \sum_{k=1}^{m} c_k (r_{k1} v_k + r_{k2} v_k^{\dagger})$. By Hölder's inequality twice, (24) and (10), we have

$$\begin{aligned}
\langle x, y\rangle &= \sum_{k=1}^{m} c_k (r_{k1}\langle y, v_k\rangle + r_{k2}\langle y, v_k^{\dagger}\rangle) \\
&\leqslant \sum_{k=1}^{m} c_k (r_{k1}^2 + r_{k2}^2)^{\frac{1}{2}} (\langle y, v_k\rangle^2 + \langle y, v_k^{\dagger}\rangle^2)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{k=1}^{m} \alpha_k^{1-p^*} c_k (r_{k1}^2 + r_{k2}^2)^{\frac{p^*}{2}}\right)^{\frac{1}{p^*}} \left(\sum_{k=1}^{m} \alpha_k c_k (\langle y, v_k\rangle^2 + \langle y, v_k^{\dagger}\rangle^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
&= \left(\sum_{k=1}^{m} \alpha_k^{1-p^*} c_k (r_{k1}^2 + r_{k2}^2)^{\frac{p^*}{2}}\right)^{\frac{1}{p^*}} \|y\|_{(Z_{p,\alpha}(\mu))^*}.
\end{aligned}$$

Let $m_x = (\sum_{k=1}^{m} \alpha_k^{1-p^*} c_k (r_{k1}^2 + r_{k2}^2)^{p^*/2})^{1/p^*}$. Thus, the fact that $y/\|y\|_{(Z_{p,\alpha}(\mu))^*}$ lies on the boundary of the convex body $(Z_{p,\alpha}(\mu))^*$ implies

$$\frac{x}{m_x} \in Z_{p,\alpha}(\mu).$$

Hence

$$\Big\| \frac{x}{m_x} \Big\|_{Z_{p,\alpha}(\mu)} \leqslant 1.$$

That is,

$$\|x\|_{Z_{p,\alpha}(\mu)} \leqslant \Big(\sum_{k=1}^{m} \alpha_k^{1-p^*} c_k (r_{k1}^2 + r_{k2}^2)^{p^*/2} \Big)^{1/p^*},$$

for $x = \sum_{k=1}^{m} c_k (r_{k1} v_k + r_{k2} v_k^{\dagger})$. Taking the infimum yields the claim.

Define functions $f_k : \operatorname{span}(v_k, v_k^{\dagger}) \to [0,\infty), 1 \leqslant k \leqslant m$, by

$$f_k(y) = \exp(-\alpha_k^{1-p^*} \|y\|^{p^*}).$$

From (12), (32), Lemma 1 and the fact that $\mu(S^{2n-1}) = \sum_{k=1}^m c_k = n$, we have

$$\begin{aligned}
&\Gamma\Big(1+\frac{2n}{p^*}\Big) V(Z_{p,\alpha}(\mu)) \\
&= \int_{\mathbb{R}^{2n}} \exp(-\|x\|^{p^*}_{Z_{p,\alpha}(\mu)}) dx \\
&\geqslant \int_{\mathbb{R}^{2n}} \sup\Big\{ \prod_{k=1}^{m} \exp(-\alpha_k^{1-p^*} c_k (r_{k1}^2 + r_{k2}^2)^{\frac{p^*}{2}}) : \sum_{k=1}^{m} c_k (r_{k1} v_k + r_{k2} v_k^{\dagger}) = x \Big\} dx \\
&= \int_{\mathbb{R}^{2n}} \sup\Big\{ \prod_{k=1}^{m} f_k(y_k)^{c_k} : x = \sum_{k=1}^{m} c_k y_k, y_k \in \operatorname{span}\{v_k, v_k^{\dagger}\} \Big\} dx \\
&\geqslant \prod_{k=1}^{m} \Big(\int_{\operatorname{span}\{v_k, v_k^{\dagger}\}} f_i \Big)^{c_k} \\
&= \prod_{k=1}^{m} \Big(\int_{\operatorname{span}\{v_k, v_k^{\dagger}\}} e^{-\alpha_k^{1-p^*} \|x\|^{p^*}} dx \Big)^{c_k} = \Big(\pi \Gamma\Big(1+\frac{2}{p^*}\Big)\Big)^{n} \Big(\prod_{k=1}^{m} \alpha_k^{c_k} \Big)^{\frac{2}{p}}.
\end{aligned}$$

Therefore, $V(Z_{p,\alpha}(\mu)) \geqslant \kappa_{2n}(p^*) \Big(\prod_{k=1}^{m} \alpha_k^{c_k} \Big)^{\frac{2}{p}}$.

Now let μ be an arbitrary complex isotropic measure on S^{2n-1}. As shown in [19, Theorem 3.2], there exists a sequence $\mu_l, l \in \mathbb{N}$, of discrete complex isotropic measures such that μ_l converges weakly to μ as $l \to \infty$. Thus,

$$\lim_{l \to \infty} h(Z_{p,\alpha}(\mu_l), u) = h(Z_{p,\alpha}(\mu), u), \quad u \in S^{2n-1}.$$

Note that the pointwise convergence of support functions implies the convergence of the corresponding convex bodies in the Hausdorff metric (see e.g., [32]). Then the continuity of volume and the fact that

$$\Big(\prod_{k=1}^{m} \alpha_k^{c_k} \Big)^{\frac{2}{p}} = \Big(\exp\Big(\sum_{k=1}^{n} c_k \log \alpha_k \Big) \Big)^{\frac{2}{p}}$$

give inequality (30).

If μ is a complex cross measure on S^{2n-1} such that $\operatorname{supp}\mu = \{\operatorname{span}\{v_1, v_1^\dagger\} \cap S^{2n-1}, \dots, \operatorname{span}\{v_n, v_n^\dagger\} \cap S^{2n-1}\}$, then the equality of (30) follows from (27) and (28). □

Finally, we complete the proof of Theorem 1.

THEOREM 2. *Suppose $p \geqslant 1$ and K is a convex body in $\mathbb{R}^{2n}$. If μ is a complex isotropic measure on S^{2n-1}, then*

$$V(K)^{2n-p} \leqslant n^{2n}\kappa_{2n}(p^*)^{-p}\exp\Big(\int_{S^{2n-1}}\log h(\Pi_p^D K, v)^{2p}d\mu(v)\Big). \tag{33}$$

In addition, if μ is a complex cross measure on S^{2n-1}, then equality in (33) holds for $p>1$ if and only if K is a generalized $\ell_{p^*}(\mathbb{C}^n)$-ball formed by μ, and equality in (33) holds for $p=1$ if and only if K is a polydisc formed by μ (up to translations).

Proof. Let

$$\alpha(v) = h(\Pi_p^D K, v)^{-p} = \Big(\frac{1}{2n}\int_{S^{2n-1}}|\langle v,u\rangle_c|^p dS_p(K,u)\Big)^{-1}, \tag{34}$$

for $v \in \operatorname{supp}\mu$. From (14), (13), the definition of $Z_{p,\alpha}(\mu)$ (24), Fubini's theorem, (34) and the fact that $\mu(S^{2n-1}) = n$, we have

$$\begin{aligned}
V(K)^{2n-p} &\leqslant V(Z_{p,\alpha}(\mu))^{-p}V_p(K, Z_{p,\alpha}(\mu))^{2n}\\
&= V(Z_{p,\alpha}(\mu))^{-p}\Big(\frac{1}{2n}\int_{S^{2n-1}}h(Z_{p,\alpha}(\mu),u)^p dS_p(K,u)\Big)^{2n}\\
&= V(Z_{p,\alpha}(\mu))^{-p}\Big(\frac{1}{2n}\int_{S^{2n-1}}\Big(\int_{S^{2n-1}}|\langle u,v\rangle_c|^p\alpha(v)d\mu(v)\Big)dS_p(K,u)\Big)^{2n}\\
&= V(Z_{p,\alpha}(\mu))^{-p}\Big(\frac{1}{2n}\int_{S^{2n-1}}\int_{S^{2n-1}}|\langle u,v\rangle_c|^p dS_p(K,u)\alpha(v)d\mu(v)\Big)^{2n}\\
&= V(Z_{p,\alpha}(\mu))^{-p}\Big(\int_{S^{2n-1}}h(\Pi_p^D K, v)^p\alpha(v)d\mu(v)\Big)^{2n}\\
&= n^{2n}V(Z_{p,\alpha}(\mu))^{-p}.
\end{aligned}$$

By Lemma 2,

$$\begin{aligned}
V(K)^{2n-p} &\leqslant n^{2n}V(Z_{p,\alpha}(\mu))^{-p} \leqslant n^{2n}\kappa_{2n}(p^*)^{-p}\exp\Big(\int_{S^{2n-1}}\log\alpha^{-2}(v)d\mu(v)\Big)\\
&= n^{2n}\kappa_{2n}(p^*)^{-p}\exp\Big(\int_{S^{2n-1}}\log h(\Pi_p^D K, v)^{2p}d\mu(v)\Big),
\end{aligned} \tag{35}$$

which is the desired inequality.

For the equality conditions of (35), by the L_p Minkowski inequality (14), equality of the first inequality in (35) holds if and only if K and $Z_{p,\alpha}(\mu)$ are dilates when $p>1$ (K and $Z_{p,\alpha}(\mu)$ are homothetic when $p=1$). If μ is a complex cross measure on S^{n-1}, Lemma 2 implies that equality of the second inequality in (35) holds and $Z_{p,\alpha}(\mu)$ is the generalized $\ell_{p^*}(\mathbb{C}^n)$-ball $B_{p^*,\alpha^{-p^*/p}}(\mathbb{C}^n)$ formed by μ. Hence K is a dilation of the generalized $\ell^n_{p^*}$-ball formed by the cross measure μ, which is still the generalized $\ell^n_{p^*}$-ball formed by μ when $p>1$ (K is a polydisc formed by μ up to translations when $p=1$).

Conversely, we will show that, when $p>1$, equality in (35) holds if K is the generalized $\ell^n_{p^*}(\mathbb{C}^n)$-ball formed by μ; i.e., there are positive numbers $(\alpha_k)_{k=1}^n$ such that

$$K=\Big\{x\in\mathbb{R}^n:\Big(\sum_{k=1}^n \alpha_k|\langle x,v_k\rangle_c|^{p^*}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}, \tag{36}$$

where $\operatorname{supp}\mu=\{\operatorname{span}\{v_1,v_1^\dagger\}\cap S^{2n-1},\dots,\operatorname{span}\{v_n,v_n^\dagger\}\cap S^{2n-1}\}$ and $\{v_1,v_1^\dagger\dots,v_n,v_n^\dagger\}$ is an orthonormal basis of $\mathbb{R}^{2n}$. From (35), it is sufficient to verify that K and $Z_{p,\alpha}(\mu)$ are dilates. From (15), we have

$$K=B_{p^*,\alpha}(\mathbb{C}^n)=UA^{-1}B_{p^*}(\mathbb{C}^n),$$

where $A=\operatorname{diag}\{\alpha_1^{1/p^*},\alpha_1^{1/p^*},\dots,\alpha_n^{1/p^*},\alpha_n^{1/p^*}\}$ and $U\in \mathrm{O}(\mathbb{R}^{2n})$ such that $v_k=Ue_{2k-1}$, $v_k^\dagger=Ue_{2k}$ for $k=1,\dots,m$. From (34) and [30, Proposition 1.2], we get

$$\begin{aligned}
\alpha(v_k)&=h(\Pi_p^D K,v_k)^{-p}=h(\Pi_p^D(UA^{-1}B_{p^*}(\mathbb{C}^n)),v_k)^{-p}\\
&=\Big(\frac{1}{2n}\int_{S^{2n-1}}|\langle v_k,u\rangle_c|^p dS_p(UA^{-1}B_{p^*}(\mathbb{C}^n),u)\Big)^{-1}\\
&=\Big(\frac{1}{2n}\int_{S^{2n-1}}[\langle v_k,u\rangle^2+\langle v_k^\dagger,u\rangle^2]^{\frac{p}{2}} dS_p(UA^{-1}B_{p^*}(\mathbb{C}^n),u)\Big)^{-1}\\
&=\Big(\frac{1}{2n}\int_{S^{2n-1}}[\langle v_k,UA^tu\rangle^2+\langle v_k^\dagger,UA^tu\rangle^2]^{\frac{p}{2}} dS_p(B_{p^*}(\mathbb{C}^n)),u)\Big)^{-1}\\
&=\Big(\frac{1}{2n}\int_{S^{2n-1}}[\langle AU^tv_k,u\rangle^2+\langle AU^tv_k^\dagger,u\rangle^2]^{\frac{p}{2}} dS_p(B_{p^*}(\mathbb{C}^n)),u)\Big)^{-1}\\
&=\Big(\frac{1}{2n}\int_{S^{2n-1}}\alpha_k^{p/p^*}[\langle e_{2k-1},u\rangle^2+\langle e_{2k},u\rangle^2]^{\frac{p}{2}} dS_p(B_{p^*}(\mathbb{C}^n)),u)\Big)^{-1}\\
&=h(\Pi_p^D(B_{p^*}(\mathbb{C}^n)),e_{2k-1})^{-p}\alpha_k^{-p/p^*}
\end{aligned}$$

for every $k=1,\dots,n$. Notice that $h(\Pi_p^D(B_{p^*}(\mathbb{C}^n)),e_{2k-1})^{-p}$ is a constant for all $k=1,\dots,n$. Thus, there exists a constant $c>0$ such that $\alpha(v_k)=c\alpha_k^{-p/p^*}$ for every $k=1,\dots,n$. Now, it follows from (25) and (36) that

$$\begin{aligned}
Z_{p,\alpha}(\mu)&=B_{p^*,\alpha^{-p^*/p}}(\mathbb{C}^n)\\
&=\Big\{x\in\mathbb{R}^n:\Big(\sum_{k=1}^n \alpha(v_k)^{-p^*/p}|\langle x,v_k\rangle_c|^{p^*}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}\\
&=\Big\{x\in\mathbb{R}^n:\Big(\sum_{k=1}^n c^{-p^*/p}\alpha_k|\langle x,v_k\rangle_c|^{p^*}\Big)^{\frac{1}{p^*}}\leqslant 1\Big\}=c^{\frac{1}{p}}K.
\end{aligned}$$

That is, K and $Z_{p,\alpha}(\mu)$ are dilates when $p>1$. When $p=1$, the proof, together with the observation that $\Pi^D(K+v_0)=\Pi^D K$ for every $v_0\in\mathbb{R}^{2n}$, is the same. □

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REFERENCES

[1] J. ABARDIA, *Difference bodies in complex vector spaces*, J. Funct. Anal., **263** (2012), 3588–3603.

[2] J. ABARDIA, *Minkowski valuations in a 2-dimensional complex vector space*, Int. Math. Res. Not., **5** (2015), 1247–1262.

[3] J. ABARDIA AND A. BERNIG, *Projection bodies in complex vector spaces*, Adv. Math., **227** (2011), 830–846.

[4] J. ABARDIA, E. GALLEGO, AND G. SOLANES, *Gauss-Bonnet theorem and Crofton type formulas in complex space forms*, Israel J. Math., **187** (2012), 287–315.

[5] J. ABARDIA AND E. SAORÍN GÓMEZ, *How do difference bodies in complex vector spaces look like? A geometrical approach*, Commun. Contemp. Math., **17** (2015), 32 pp.

[6] J. ABARDIA AND T. WANNERER, *Aleksandrov-Fenchel inequalities for unitary valuations of degree* 2 *and* 3, Calc. Var. Partial Differential Equations, **54** (2015), 1767–1791.

[7] S. ALESKER, *Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations*, J. Differential Geom., **63** (2003), 63–95.

[8] P. BALISTER AND B. BOLLOBÁS, *Projections, entropy and sumsets*, Combinatorica, **32** (2012), 125–141.

[9] K. BALL, *Shadows of convex bodies*, Trans. Amer. Math. Soc., **327** (1991), 891–901.

[10] F. BARTHE, *On a reverse form of the Brascamp-Lieb inequality*, Invent. Math., **134** (1998), 685–693.

[11] J. BENNETT, A. CARBERY, AND T. TAO, *On the multilinear restriction and Kakeya conjectures*, Acta Math., **196** (2006), 261–302.

[12] J. BENNETT, A. CARBERY, AND J. WRIGHT, *A non-linear generalization of the Loomis-Whitney inequality and applications*, Math. Res. Lett., **12** (2005), 443–457.

[13] A. BERNIG AND J.H.G. FU, *Hermitian integral geometry*, Ann. of Math., **173** (2011), 907–945.

[14] A. BERNIG, J.H.G. FU, AND G. SOLANES, *Integral geometry of complex space forms*, Geom. Funct. Anal., **24** (2014), 403–492.

[15] U. BETKE AND P. MCMULLEN, *Estimating the sizes of convex bodies from projections*, J. London Math. Soc., **27** (1983), 525–538.

[16] B. BOLLOBAS AND A. THOMASON, *Projections bodies and hereditary properties of hypergraphs*, Bull. Lond. Math. Soc., **27** (1995), 417–424.

[17] S. CAMPI, R.J. GARDNER, AND P. GRONCHI, *Reverse and dual Loomis-Whitney-type inequalities*, Trans. Amer. Math. Soc., **368** (2016), 5093–5124.

[18] S. CAMPI AND P. GRONCHI, *Estimates of Loomis-Whitney type for intrinsic volumes*, Adv. in Appl. Math., **47** (2011), 545–561.

[19] Q. HUANG AND B. HE, *Volume inequalities for complex isotropic measures*, Geom. Dedicata., **177** (2015), 401–428.

[20] Q. HUANG AND A.-J. LI, *On the Loomis-Whitney inequality for isotropic measures*, Int. Math. Res. Not., **6** (2017), 1641–1652.

[21] A.KOLDOBSKY, *Stability of volume comparison for complex convex bodies*, Arch. Math., **97** (2011), 91–98.

[22] A.KOLDOBSKY, H. KÖNIG AND M. ZYMONOPOULOU, *The complex Busemann-Petty problem on sections of convex bodies*, Adv. Math., **218** (2008), 352–367.

[23] A. KOLDOBSKY, G. PAOURIS, AND M. ZYMONOPOULOU, *Complex Intersection Bodies*, J. London Math. Soc., **88** (2013), 538–562.

[24] A.-J. LI AND Q. HUANG, *The L_p Loomis-Whitney inequality*, Adv. in Appl. Math., **75** (2016), 94–115.

[25] A.-J. LI AND Q. HUANG, *The dual Loomis-Whitney inequality*, Bull. London Math. Soc., **48** (2016), 676–690.

[26] A.-J. LI, G. WANG, AND G. LENG, *An extended Loomis-Whitney inequality for positive double John bases*, Glasg. Math. J., **53** (2011), 451–462.

[27] L. H. LOOMIS AND H. WHITENY, *An inequality related to the isoperimetric inequality*, Bull. Amer. Math. Soc., **55** (1949), 961–962.

[28] E. LUTWAK, *The Brunn-Minkowski-Firey Theory I: Mixed volumes and the Minkowski Problem*, J. Differential Geom., **38** (1993), 131–150.

[29] E. LUTWAK, *The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas*, Adv. Math., **118** (1996), 244–294.

[30] E. LUTWAK, D. YANG, AND G. ZHANG, *L_p John ellipsoids*, Proc. London Math. Soc., **90** (2005), 497–520.
[31] P. NAYAR AND T. TKOCZ, *The unconditional case of the complex S-inequality*, Israel J. Math., **197** (2013), 99–106.
[32] R. SCHNEIDER, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, Vol. **44**, Cambridge University Press, Cambridge, 2014.
[33] G. ZHANG, *The affine Sobolev inequality*, J. Differential Geom., **53** (1999), 183–202.

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