

LÉVY–KHINTCHINE REPRESENTATION OF TOADER–QI MEAN

FENG QI AND BAI-NI GUO

(Communicated by I. Perić)

Abstract. In the paper, by virtue of a Lévy-Khintchine representation and an alternative integral representation for the weighted geometric mean, the authors establish a Lévy-Khintchine representation and an alternative integral representation for the Toader-Qi mean, verify that the Toader-Qi mean is a Bernstein function and that the divided difference of the Toader-Qi mean is a Stieltjes function, and collect a probabilistic interpretation and an application in engineering of the Toader-Qi mean.

1. Introduction and main results

In this paper, by virtue of a Lévy-Khintchine representation and an alternative integral representation for the principal branch of the weighted geometric mean

$$G_{a,b;\lambda}(z) = (a+z)^\lambda (b+z)^{1-\lambda}, \quad b > a > 0, \quad z \in \mathbb{C} \setminus [-b, -a], \quad \lambda \in (0, 1),$$

we will establish a Lévy-Khintchine representation and an alternative integral representation for the Toader-Qi mean $TQ(x+a, x+b)$, verify that the mean $TQ(x, x+b-a)$ is a Bernstein function and that the divided difference of $TQ(x, x+b-a)$ is a Stieltjes function on $(0, \infty)$, and collect a probabilistic interpretation and an application in engineering of $TQ(x+a, x+b)$. For stating our main results, we recall some known results and prepare some necessary knowledge.

1.1. Toader-Qi mean

For $a, b > 0$ and $q \neq 0$, denote

$$M_q(a, b) = \left(\frac{1}{2\pi} \int_0^{2\pi} a^{q \cos^2 \theta} b^{q \sin^2 \theta} d\theta \right)^{1/q}.$$

It is easy to see that

$$M_q(a, b) = \left[\frac{2}{\pi} \int_0^{\pi/2} (a^{\cos^2 \theta} b^{\sin^2 \theta})^q d\theta \right]^{1/q} = \left[\frac{2}{\pi} \int_0^{\pi/2} (a^q)^{\cos^2 \theta} (b^q)^{\sin^2 \theta} d\theta \right]^{1/q}.$$

Mathematics subject classification (2010): Primary 44A15, Secondary 26E60, 30E20, 33C10, 60G50.

Keywords and phrases: Lévy-Khintchine representation, integral representation, Bernstein function, Stieltjes function, Toader-Qi mean, weighted geometric mean, Bessel function of the first kind, probabilistic interpretation, application in engineering, inequality.

In [25], it was remarked that

$$M_0(a, b) = \lim_{q \rightarrow 0} M_q(a, b) = G(a, b) = \sqrt{ab},$$

but it was not known how to determine any mean M_q for $q \neq 0$. In [26, p. 382, Section 5], the connection

$$M_q(a, b) = G(a, b) \left[J_0 \left(-i \frac{q}{2} \ln \frac{a}{b} \right) \right]^{1/q} \tag{1.1}$$

was underlined, where $i = \sqrt{-1}$ is the imaginary unit, the Bessel function of the first kind $J_\nu(z)$ can be defined [10, p. 217, 10.2.2] by

$$J_\nu = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2} \right)^{2k+\nu},$$

and the classical Euler gamma function $\Gamma(z)$ can be defined [3] by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z + k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

In [18, Lemma 2.1], the relation

$$M_q(a, b) = G(a, b) \left[I_0 \left(\frac{q}{2} \ln \frac{a}{b} \right) \right]^{1/q} \tag{1.2}$$

was established, where the modified Bessel functions of the first kind $I_\nu(z)$ can be defined by

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2} \right)^{2k+\nu}, \quad |\arg z| < \pi, \quad \nu \in \mathbb{C}.$$

Since

$$J_0(-iz) = I_0(z) = \sum_{k=0}^{\infty} \left(\frac{z}{2k!} \right)^{2k},$$

the formulas (1.1) and (1.2) are essentially the same one.

With the help of (1.2), the mean $M_q(a, b)$, the modified Bessel function of the first kind I_0 , and the arithmetic-geometric mean $M(a, b)$, which can be defined [5] by

$$\frac{1}{M(a, b)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} d\theta,$$

were bounded in [18, 28, 29, 30] successfully.

We remark that the mean $M_q(a, b)$ can be traced back to [5, 6, 25, 26] and the closely related references therein. See also [2, pp. 401–403].

Due to the paper [18], the notation $TQ(a, b)$ and the terminology ‘‘Toader-Qi mean’’ for the unnamed mean $M_1(a, b)$ were used in the papers [23, 28, 29, 30]. For the sake of consistency, we continue to adopt these notation and terminology in this paper.

In Section 4 of this paper, we will collect a probabilistic interpretation and an application in engineering of the Toader-Qi mean $TQ(a, b)$.

1.2. Lévy-Khintchine representation

For stating our main results, we need to recall the following notion and conclusions.

DEFINITION 1.1. ([27, Chapter IV]) An infinitely differentiable function f on an interval I is said to be completely monotonic on I if it satisfies $(-1)^{n-1} f^{(n-1)}(t) \geq 0$ for $x \in I$ and $n \in \mathbb{N}$, where \mathbb{N} stands for the set of all positive integers.

DEFINITION 1.2. ([4, Definition 1.2]) An infinitely differentiable function $f : I \rightarrow [0, \infty)$ is called a Bernstein function on an interval I if $f'(t)$ is completely monotonic on I .

PROPOSITION 1.1. ([24, Theorem 3.2]) A function $f : (0, \infty) \rightarrow [0, \infty)$ is a Bernstein function if and only if it admits the representation

$$f(x) = \alpha + \beta x + \int_0^\infty (1 - e^{-xt}) d\mu(t), \quad (1.3)$$

where $\alpha, \beta \geq 0$ and μ is a positive measure satisfying $\int_0^\infty \min\{1, t\} d\mu(t) < \infty$.

In particular, the triplet (α, β, μ) determines f uniquely and vice versa.

The integral representation (1.3) for $f(x)$ is called the Lévy-Khintchine representation of $f(x)$. The representing measure μ and the characteristic triplet (α, β, μ) are often respectively called the Lévy measure and the Lévy triplet of the Bernstein function $f(x)$.

It was pointed out in [24, Chapter 3] that the notion of the Bernstein functions originated from the potential theory and is useful to harmonic analysis, probability, statistics, and integral transforms.

DEFINITION 1.3. ([24, Chapter 2, Definition 2.1]) A Stieltjes function is a function $f : (0, \infty) \rightarrow [0, \infty)$ which can be written in the form

$$f(x) = \frac{a}{x} + b + \int_0^\infty \frac{1}{s+x} d\mu(s), \quad (1.4)$$

where $a, b \geq 0$ are nonnegative constants and μ is a nonnegative measure on $(0, \infty)$ such that $\int_0^\infty \frac{1}{1+s} d\mu(s) < \infty$.

The integral appearing in (1.4) is also called the Stieltjes transform of the measure μ . For more information on this kind of functions (transforms), please refer to the monographs [24, Chapter 2], [27, Chapter VIII], and the closely related references therein.

1.3. Main results

Our main results can be stated as the following theorems.

THEOREM 1.1. *For $b > a > 0$, the Toader-Qi mean $TQ(x + a, x + b)$ for $x > -a$ has the integral representation*

$$TQ(x + a, x + b) = TQ(a, b) + x \left[1 + \frac{1}{\pi} \int_a^b \frac{h(a, b; t)}{t} \frac{1}{t + x} dt \right] \tag{1.5}$$

and the Lévy-Khintchine representation

$$TQ(x + a, x + b) = TQ(a, b) + x + \frac{b - a}{\pi} \int_0^\infty \frac{H(a, b; s)}{s} e^{-as} (1 - e^{-xs}) ds, \tag{1.6}$$

where

$$h(a, b; t) = \frac{2}{\pi} \int_0^{\pi/2} \sin(\pi \cos^2 \theta) (t - a)^{\cos^2 \theta} (b - t)^{\sin^2 \theta} d\theta, \quad t \in [a, b],$$

$$H(a, b; s) = \frac{2}{\pi} \int_0^{\pi/2} \sin(\pi \cos^2 \theta) F(\sin^2 \theta, (b - a)s) d\theta, \quad s \in (0, \infty),$$

and

$$F(\lambda, s) = \int_0^1 \left(\frac{1}{u} - 1 \right)^\lambda \left(1 - \frac{\lambda}{1 - u} \right) e^{-su} du, \quad (\lambda, s) \in (0, 1) \times (0, \infty) \tag{1.7}$$

are positive. Consequently,

1. the Toader-Qi mean $TQ(x, x + b - a)$ is a Bernstein function of x on $(0, \infty)$,
2. the divided difference

$$\frac{TQ(x, x + b - a) - TQ(a, b)}{x - a} \tag{1.8}$$

is a Stieltjes function of x on $(0, \infty)$.

COROLLARY 1.1. *For $b > a > 0$ and $x > -a^q$, we have*

$$TQ(x + a^q, x + b^q) = [M_q(a, b)]^q + x \left[1 + \frac{1}{\pi} \int_{a^q}^{b^q} \frac{h(a^q, b^q; t)}{t} \frac{1}{t + x} dt \right]$$

and

$$TQ(x + a^q, x + b^q) = [M_q(a, b)]^q + x + \frac{b^q - a^q}{\pi} \int_0^\infty \frac{H(a^q, b^q; s)}{s} e^{-a^q s} (1 - e^{-xs}) ds.$$

Consequently, the divided difference

$$\frac{TQ(x, x + b^q - a^q) - [M_q(a, b)]^q}{x - a^q}, \quad q \neq 0$$

is a Stieltjes function of x on $(0, \infty)$.

2. Lemmas

For proving our main results, we need the following lemmas.

LEMMA 2.1. *Let $\lambda \in (0, 1)$ and $b > a > 0$. The principal branch of the weighted geometric mean $G_{a,b;\lambda}(z)$ for $z \in \mathbb{C} \setminus [-b, -a]$ can be represented by*

$$\frac{G_{a,b;\lambda}(z) - a^\lambda b^{1-\lambda}}{z} = \begin{cases} 1 + \frac{\sin(\lambda\pi)}{\pi} \int_a^b \frac{(t-a)^\lambda (b-t)^{1-\lambda}}{t} \frac{1}{t+z} dt, & z \neq 0; \\ \frac{(1-\lambda)a + \lambda b}{a^{1-\lambda} b^\lambda}, & z = 0. \end{cases} \tag{2.1}$$

Proof. This follows from letting $n = 2$ in [19, Theorems 3.1 and 4.6], rearranging, and taking the limit $z \rightarrow 0$. See also [14, eq. (2.1)], [17, p. 728, Section 4, eq. (4.1)], and the closely related references therein. \square

LEMMA 2.2. *Let $\lambda \in (0, 1)$ and $b > a > 0$. The principal branch of the weighted geometric mean $G_{a,b;\lambda}(z)$ for $z \in \mathbb{C} \setminus [-b, -a]$ has the Lévy-Khintchine representation*

$$G_{a,b;\lambda}(z) = a^\lambda b^{1-\lambda} + z + \frac{\sin(\lambda\pi)}{\pi} (b-a) \int_0^\infty \frac{F(1-\lambda, (b-a)s)}{s} e^{-as} (1 - e^{-zs}) ds, \tag{2.2}$$

where $F(\lambda, s)$ is defined by (1.7).

Proof. This is a slight reformulation of [21, Theorem 1.1]. \square

3. Proofs of main results

Proof of Theorem 1.1. Applying (2.1) gives

$$\begin{aligned} & TQ(x+a, x+b) \\ &= \frac{2}{\pi} \int_0^{\pi/2} (x+a)^{\cos^2 \theta} (x+b)^{\sin^2 \theta} d\theta \\ &= \frac{2}{\pi} \int_0^{\pi/2} \left[a^{\cos^2 \theta} b^{\sin^2 \theta} + x + \frac{\sin(\pi \cos^2 \theta)}{\pi} x \int_a^b \frac{(t-a)^{\cos^2 \theta} (b-t)^{\sin^2 \theta}}{t} \frac{1}{t+x} dt \right] d\theta \\ &= TQ(a, b) + x + \frac{2}{\pi} \frac{x}{\pi} \int_0^{\pi/2} \sin(\pi \cos^2 \theta) \int_a^b \frac{(t-a)^{\cos^2 \theta} (b-t)^{\sin^2 \theta}}{t} \frac{1}{t+x} dt \Big] d\theta \\ &= TQ(a, b) + x + \frac{x}{\pi} \int_a^b \frac{1}{t} \left[\frac{2}{\pi} \int_0^{\pi/2} \sin(\pi \cos^2 \theta) (t-a)^{\cos^2 \theta} (b-t)^{\sin^2 \theta} d\theta \right] \frac{1}{t+x} dt. \end{aligned}$$

The integral representation (1.5) is thus proved.

Applying (2.2) yields

$$\begin{aligned}
 & TQ(x+a, x+b) \\
 &= \frac{2}{\pi} \int_0^{\pi/2} (x+a)^{\cos^2 \theta} (x+b)^{\sin^2 \theta} d\theta \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left[a^{\cos^2 \theta} b^{\sin^2 \theta} + x \right. \\
 &\quad \left. + \frac{\sin(\pi \cos^2 \theta)}{\pi} (b-a) \int_0^\infty \frac{F(\sin^2 \theta, (b-a)s)}{s} e^{-as} (1-e^{-xs}) ds \right] d\theta \\
 &= TQ(a, b) + x \\
 &\quad + \frac{2(b-a)}{\pi^2} \int_0^{\pi/2} \left[\sin(\pi \cos^2 \theta) \int_0^\infty \frac{F(\sin^2 \theta, (b-a)s)}{s} e^{-as} (1-e^{-xs}) ds \right] d\theta \\
 &= TQ(a, b) + x \\
 &\quad + \frac{b-a}{\pi} \int_0^\infty \frac{1}{s} \left[\frac{2}{\pi} \int_0^{\pi/2} \sin(\pi \cos^2 \theta) F(\sin^2 \theta, (b-a)s) d\theta \right] e^{-as} (1-e^{-xs}) ds.
 \end{aligned}$$

The Lévy-Khintchine representation (1.6) is thus proved.

Since $F(\lambda, s)$ is positive on $(0, \infty)$, comparing the representation (1.6) with (1.3) readily reveals that the Toader-Qi mean $TQ(x, x+b-a)$ is a Bernstein function of x on $(0, \infty)$.

Reformulating (1.5) as

$$\frac{TQ(x+a, x+b) - TQ(a, b)}{x} = 1 + \frac{1}{\pi} \int_a^b \frac{h(a, b; t)}{t} \frac{1}{t+x} dt$$

and comparing with (1.4) immediately show that the divided difference (1.8) is a Stieltjes function. The proof of Theorem 1.1 is complete. \square

Proof of Corollary 1.1. This follows from replacing a and b in Theorem 1.1 respectively by a^q and b^q for $q \neq 0$. \square

4. An interpretation and an application

In this section, we collect a probabilistic interpretation and an application in engineering of the Toader-Qi mean $TQ(a, b)$. For more detailed information, please refer to the closely related posts on the ResearchGate and to the paper [8, 9].

4.1. A probabilistic interpretation

According to [7, Example 8.6.3.10], we have $I_0(t) = e^t P\{X_t = 0\}$, where X_t for $t \geq 0$ is a compound Poisson process with intensity 1, with jumps equal 1 or -1 , and with probability 0.5. Thus, according to (1.2), it follows that

$$TQ(1, e^{-2t}) = M_1(1, e^{-2t}) = e^{-t} I_0(t) = P\{X_t = 0\}.$$

4.2. An application in engineering

In [8, Section 2] and [9, Section II], the mean and the mean square over one period T of a continuous (and periodic) signal $x(t)$ were defined by

$$\kappa(x) = \frac{1}{T} \int_T x(t) dt \quad \text{and} \quad \kappa(x^2) = \frac{1}{T} \int_T x^2(t) dt.$$

It is easy to see that

$$TQ(a, b) = \frac{1}{2\pi} \int_0^{2\pi} (a^{\cos\theta} b^{\sin\theta})^2 d\theta = \kappa\left((a^{\cos\theta} b^{\sin\theta})^2\right),$$

where $T = 2\pi$ and $x(t) = a^{\cos t} b^{\sin t}$. In [8], the root of the mean square values of sampled periodic signals was calculated by using non-integer number of samples per period. This implies that the Toader-Qi mean $TQ(a, b)$ can be applied to engineering.

5. Remarks

Finally we give several remarks on some related thing, including several inequalities for bounding the Toader-Qi mean $TQ(a, b)$.

REMARK 5.1. Under the assumptions of Lemmas 2.1 and 2.2, the formulas (2.1) and (2.2) are equivalent to each other. For the outline to prove this equivalence, please refer to [15, Remark 4.1] and [20, Theorem 1.1].

REMARK 5.2. By the arithmetic-geometric-harmonic mean inequality, we have

$$a \cos^2 \theta + b \sin^2 \theta \geq a^{\cos^2 \theta} b^{\sin^2 \theta} \geq \frac{1}{\frac{\cos^2 \theta}{a} + \frac{\sin^2 \theta}{b}}.$$

Integrating on all sides on $(0, \frac{\pi}{2})$ gives

$$A(a, b) = \frac{a+b}{2} > TQ(a, b) > \frac{a+b}{2ab} = H(a, b)$$

and

$$\sqrt[q]{A(a^q, b^q)} > M_q(a, b) > \sqrt[q]{H(a^q, b^q)},$$

where $A(a, b)$ and $H(a, b)$ denote respectively the arithmetic and harmonic means. Comparing the geometric mean $G(a, b)$ and the Toader-Qi mean $TQ(a, b)$, which mean is bigger? The answer is

$$TQ(a, b) < \frac{A(a, b) + G(a, b)}{2} < \frac{2A(a, b) + G(a, b)}{3}$$

did in [18, Remark 4.1]. Consequently, we have

$$M_q(a, b) < \left[\frac{A(a^q, b^q) + G(a^q, b^q)}{2} \right]^{1/q} < \left[\frac{2A(a^q, b^q) + G(a^q, b^q)}{3} \right]^{1/q}, \quad q \neq 0.$$

REMARK 5.3. In [11, Theorem 1.1], it was derived that

$$[\lambda a + (1 - \lambda)b] - a^\lambda b^{1-\lambda} < \frac{\sin(\lambda\pi)}{\pi} \left((2\lambda - 1)(b - a) + [(1 - \lambda)b - \lambda a] \ln \frac{b}{a} \right) \quad (5.1)$$

for $b > a > 0$ and $\lambda \in (0, 1)$. Replacing λ by $\cos^2 \theta$ and integrating over $(0, \frac{\pi}{2})$ on both sides of (5.1) arrive at

$$\begin{aligned} & \frac{2}{\pi} \int_0^{\pi/2} (a \cos^2 \theta + b \sin^2 \theta) \, d\theta - \frac{2}{\pi} \int_0^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} \, d\theta \\ & < \frac{2}{\pi} \int_0^{\pi/2} \frac{\sin(\pi \cos^2 \theta)}{\pi} \left[(2 \cos^2 \theta - 1)(b - a) + (b \sin^2 \theta - a \cos^2 \theta) \ln \frac{b}{a} \right] \, d\theta \end{aligned}$$

which can be simplified as

$$\begin{aligned} TQ(a, b) - A(a, b) &> -\frac{2}{\pi} \int_0^{\pi/2} \frac{\sin(\pi \cos^2 \theta)}{\pi} \left[(2 \cos^2 \theta - 1)(b - a) \right. \\ & \quad \left. + (b \sin^2 \theta - a \cos^2 \theta) \ln \frac{b}{a} \right] \, d\theta \\ &= -\frac{2(b - a)}{\pi^2} \int_0^{\pi/2} \sin(\pi \cos^2 \theta) \cos(2\theta) \, d\theta \\ & \quad - \frac{2}{\pi^2} \ln \frac{b}{a} \int_0^{\pi/2} \sin(\pi \cos^2 \theta) (b \sin^2 \theta - a \cos^2 \theta) \, d\theta \\ &= -\frac{2}{\pi^2} \ln \frac{b}{a} \int_0^{\pi/2} \sin(\pi \cos^2 \theta) (b \sin^2 \theta - a \cos^2 \theta) \, d\theta \\ &= -J_0\left(\frac{\pi}{2}\right) \frac{b - a}{2\pi} \ln \frac{b}{a}, \end{aligned}$$

where

$$\begin{aligned} & \int_0^{\pi/2} \sin(\pi \cos^2 \theta) \cos(2\theta) \, d\theta = \int_0^{\pi/2} \sin \left[\pi \frac{1 + \cos(2\theta)}{2} \right] \cos(2\theta) \, d\theta \\ &= \int_0^{\pi/2} \cos \left[\frac{\pi}{2} \cos(2\theta) \right] \cos(2\theta) \, d\theta = \frac{1}{2} \int_0^\pi \cos \left(\frac{\pi}{2} \cos \theta \right) \cos \theta \, d\theta \\ &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos \left[\frac{\pi}{2} \cos \left(\theta + \frac{\pi}{2} \right) \right] \cos \left(\theta + \frac{\pi}{2} \right) \, d\theta \\ &= -\frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos \left(\frac{\pi}{2} \sin \theta \right) \sin \theta \, d\theta = 0, \\ & \int_0^{\pi/2} \sin(\pi \cos^2 \theta) \sin^2 \theta \, d\theta = \int_0^{\pi/2} \sin \left[\pi \frac{1 + \cos(2\theta)}{2} \right] \frac{1 - \cos(2\theta)}{2} \, d\theta \\ &= \int_0^{\pi/2} \cos \left[\frac{\pi}{2} \cos(2\theta) \right] \frac{1 - \cos(2\theta)}{2} \, d\theta = \frac{1}{2} \int_0^{\pi/2} \cos \left[\frac{\pi}{2} \cos(2\theta) \right] \, d\theta \\ &= \frac{1}{4} \int_0^\pi \cos \left(\frac{\pi}{2} \cos \theta \right) \, d\theta = \frac{\pi}{4} J_0 \left(\frac{\pi}{2} \right), \end{aligned}$$

and

$$\begin{aligned} \int_0^{\pi/2} \sin(\pi \cos^2 \theta) \cos^2 \theta \, d\theta &= \int_0^{\pi/2} \sin \left[\pi \frac{1 + \cos(2\theta)}{2} \right] \frac{1 + \cos(2\theta)}{2} \, d\theta \\ &= \int_0^{\pi/2} \cos \left[\frac{\pi}{2} \cos(2\theta) \right] \frac{1 + \cos(2\theta)}{2} \, d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} \cos \left[\frac{\pi}{2} \cos(2\theta) \right] \, d\theta = \frac{\pi}{4} J_0 \left(\frac{\pi}{2} \right) \end{aligned}$$

by virtue of the formula

$$\int_0^\pi \cos(z \cos x) \cos(nx) \, dx = \pi \cos \frac{n\pi}{2} J_n(z)$$

in [3, p. 425, item 18]. In conclusion, we obtain

$$TQ(a, b) > A(a, b) - J_0 \left(\frac{\pi}{2} \right) \frac{b-a}{2\pi} \ln \frac{b}{a}, \quad b > a > 0.$$

Consequently, it follows that

$$M_q(a, b) > \left[A(a^q, b^q) - q J_0 \left(\frac{\pi}{2} \right) \frac{b^q - a^q}{2\pi} \ln \frac{b}{a} \right]^{1/q}, \quad b > a > 0, \quad q \neq 0.$$

REMARK 5.4. The idea and motivation of Theorem 1.1 come from the papers [1, 12, 13, 14, 15, 16, 19, 20, 21, 22] and closely related references therein. In these papers, the Lévy-Khintchine representations, complete monotonicity, the Bernstein function property for the arithmetic mean, harmonic mean, logarithmic mean, exponential mean, special cases of the Stolarsky mean, (weighted) geometric mean (of many positive numbers), some other special means, and their reciprocal were established and applied. Recently, these results were surveyed, reviewed, further extended to bivariate complex geometric mean and its reciprocal, and applied to Heronian mean of power 2 and its reciprocal in the paper [16].

Acknowledgements. The authors are grateful to Joachim Domsta (State University of Applied Sciences in Elbląg, Poland) for his posts on the ResearchGate where he provides the probabilistic interpretation of the Toader-Qi mean $TQ(a, b)$.

The authors are grateful to Viera Čerňanová (Slovak University of Technology) for her posts on the ResearchGate where she supplies an application in engineering of the Toader-Qi mean $TQ(a, b)$ by recommending the papers [8, 9].

The authors are thankful to Erika Andirkó (a librarian at the Library, Institute of Mathematics, University of Debrecen, Hungary) for her sending an electronic copy of the paper [5] by e-mail.

The authors are thankful to anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

REFERENCES

- [1] Á. BESENYEI, *On complete monotonicity of some functions related to means*, Math. Inequal. Appl. **16**, 1 (2013), 233–239, available online at <https://doi.org/10.7153/mia-16-17>.
- [2] P. S. BULLEN, *Handbook of Means and Their Inequalities*, Mathematics and its Applications, vol. 560, Kluwer Academic Publishers, Dordrecht-Boston-London, 2003.
- [3] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products*, Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Eighth edition, Revised from the seventh edition, Elsevier/Academic Press, Amsterdam, 2015, available online at <https://doi.org/10.1016/B978-0-12-384933-5.00013-8>.
- [4] B.-N. GUO AND F. QI, *On the degree of the weighted geometric mean as a complete Bernstein function*, Afr. Mat. **26**, 7 (2015), 1253–1262, available online at <http://dx.doi.org/10.1007/s13370-014-0279-2>.
- [5] H. HARUKI, *New characterizations of the arithmetic-geometric mean of Gauss and other well-known mean values*, Publ. Math. Debrecen **38**, 3–4 (1991), 323–332.
- [6] H. HARUKI AND T. M. RASSIAS, *New characterizations of some mean-values*, J. Math. Anal. Appl. **202**, 1 (1996), 333–348, available online at <https://doi.org/10.1006/jmaa.1996.0319>.
- [7] M. JEANBLANC, M. YOR, AND M. CHESNEY, *Mathematical Methods for Financial Markets*, Springer Finance. Springer-Verlag London, Ltd., London, 2009, available online at <https://doi.org/10.1007/978-1-84628-737-4>.
- [8] G. E. MOG AND E. P. RIBEIRO, *Mean and RMS calculations for sampled periodic signals with non-integer number of samples per period applied to AC energy systems*, In: Congresso Ibero-Latino-Americano de Métodos Computacionais em Engenharia—XXV Iberian Latin-American Congress on Computational Methods in Engineering, Recife, Pernambuco, Brazil, November 10–12, 2004.
- [9] G. E. MOG AND E. P. RIBEIRO, *One cycle AC RMS calculations for power quality monitoring under frequency deviation*, EEE Xplore (2004 11th International Conference on Harmonics and Quality of Power (IEEE Cat. No. 04EX951)) (2004), 700–705, available online at <https://doi.org/10.1109/ICHQP.2004.1409438>.
- [10] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT, AND C. W. CLARK (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010, available online at <http://dlmf.nist.gov/>.
- [11] F. QI, *Bounding the difference and ratio between the weighted arithmetic and geometric means*, Int. J. Anal. Appl. **13**, 2 (2017), 132–135.
- [12] F. QI, V. ČERNÁNOVÁ, X.-T. SHI, AND B.-N. GUO, *Some properties of central Delannoy numbers*, J. Comput. Appl. Math. **328** (2018), 101–115, available online at <https://doi.org/10.1016/j.cam.2017.07.013>.
- [13] F. QI AND B.-N. GUO, *The reciprocal of the geometric mean of many positive numbers is a Stieltjes transform*, J. Comput. Appl. Math. **311** (2017), 165–170, available online at <https://doi.org/10.1016/j.cam.2016.07.006>.
- [14] F. QI AND B.-N. GUO, *The reciprocal of the weighted geometric mean is a Stieltjes function*, Bol. Soc. Mat. Mex. (3) **23** (2017), in press, available online at <https://doi.org/10.1007/s40590-016-0151-5>.
- [15] F. QI AND B.-N. GUO, *The reciprocal of the weighted geometric mean of many positive numbers is a Stieltjes function*, Quaest. Math. **41** (2018), in press, available online at <https://doi.org/10.2989/16073606.2017.1396508>.
- [16] F. QI AND D. LIM, *Integral representations of bivariate complex geometric mean and their applications*, J. Comput. Appl. Math. **330** (2018), 41–58, available online at <https://doi.org/10.1016/j.cam.2017.08.005>.
- [17] F. QI, X.-T. SHI, AND B.-N. GUO, *Some properties of the Schröder numbers*, Indian J. Pure Appl. Math. **47**, 4 (2016), 717–732, available online at <https://doi.org/10.1007/s13226-016-0211-6>.
- [18] F. QI, X.-T. SHI, F.-F. LIU, AND Z.-H. YANG, *A double inequality for an integral mean in terms of the exponential and logarithmic means*, Period. Math. Hungar. **75**, 2 (2017), 180–189, available online at <https://doi.org/10.1007/s10998-016-0181-9>.

- [19] F. QI, X.-J. ZHANG, AND W.-H. LI, *An integral representation for the weighted geometric mean and its applications*, Acta Math. Sin. (Engl. Ser.) **30**, 1 (2014), 61–68, available online at <https://doi.org/10.1007/s10114-013-2547-8>.
- [20] F. QI, X.-J. ZHANG, AND W.-H. LI, *Lévy-Khintchine representation of the geometric mean of many positive numbers and applications*, Math. Inequal. Appl. **17**, 2 (2014), 719–729, available online at <https://doi.org/10.7153/mia-17-53>.
- [21] F. QI, X.-J. ZHANG, AND W.-H. LI, *Lévy-Khintchine representations of the weighted geometric mean and the logarithmic mean*, Mediterr. J. Math. **11**, 2 (2014), 315–327, available online at <https://doi.org/10.1007/s00009-013-0311-z>.
- [22] F. QI, X.-J. ZHANG, AND W.-H. LI, *The harmonic and geometric means are Bernstein functions*, Bol. Soc. Mat. Mex. (3) **23**, 2 (2017), 713–736, available online at <https://doi.org/10.1007/s40590-016-0085-y>.
- [23] W.-M. QIAN, X.-H. ZHANG, AND Y.-M. CHU, *Sharp bounds for the Toader-Qi mean in terms of harmonic and geometric means*, J. Math. Inequal. **11**, 1 (2017), 121–127, available online at <https://doi.org/10.7153/jmi-11-11>.
- [24] R. L. SCHILLING, R. SONG, AND Z. VONDRAČEK, *Bernstein Functions – Theory and Applications*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012.
- [25] G. TOADER, *Some mean values related to the arithmetic-geometric mean*, J. Math. Anal. Appl. **218**, 2 (1998), 358–368, available online at <https://doi.org/10.1006/jmaa.1997.5766>.
- [26] G. TOADER AND T. M. RASSIAS, *New properties of some mean values*, J. Math. Anal. Appl. **232**, 2 (1999), 376–383, available online at <https://doi.org/10.1006/jmaa.1999.6278>.
- [27] D. V. WIDDER, *The Laplace Transform*, Princeton Mathematical Series **6**, Princeton University Press, Princeton, N. J., 1941.
- [28] Z.-H. YANG AND Y.-M. CHU, *A sharp lower bound for Toader-Qi mean with applications*, J. Funct. Spaces **2016**, Art. ID 4165601, 5 pages, available online at <https://doi.org/10.1155/2016/4165601>.
- [29] Z.-H. YANG AND Y.-M. CHU, *On approximating the modified Bessel function of the first kind and Toader-Qi mean*, J. Inequal. Appl. **2016**, 40 (2016), 21 pages, available online at <https://doi.org/10.1186/s13660-016-0988-1>.
- [30] Z.-H. YANG, Y.-M. CHU, AND Y.-Q. SONG, *Sharp bounds for Toader-Qi mean in terms of logarithmic and identric means*, Math. Inequal. Appl. **19**, 2 (2016), 721–730, available online at <https://doi.org/10.7153/mia-19-52>.

(Received April 5, 2017)

Feng Qi
 Institute of Mathematics
 Henan Polytechnic University
 Jiaozuo, Henan, 454010, China
 and
 College of Mathematics
 Inner Mongolia University for Nationalities
 Tongliao, Inner Mongolia, 028043, China
 and
 Department of Mathematics, College of Science
 Tianjin Polytechnic University
 Tianjin, 300387, China
 e-mail: qifeng618@gmail.com,
 qifeng618@hotmail.com, qifeng618@qq.com
<https://qifeng618.wordpress.com>

Bai-Ni Guo
 School of Mathematics and Informatics
 Henan Polytechnic University
 Jiaozuo, Henan, 454010, China
 e-mail: bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com
<http://orcid.org/0000-0001-6156-2590>