Mathematical
Inequalities
& Applications
Volume 21, Number 2 (2018), 433–453 doi:10.7153/mia-2018-21-30

SOBOLEV'S INEQUALITIES FOR HERZ–MORREY–ORLICZ SPACES ON THE HALF SPACE

YOSHIHIRO MIZUTA, TAKAO OHNO AND TETSU SHIMOMURA

Dedicated to Professor Eiichi Nakai
on the occasion of his sixtieth birthday

(*Communicated by J. Soria*)

Abstract. We introduce Herz-Morrey-Orlicz spaces on the half space, and study the boundedness of the Hardy-Littlewood maximal operator. As an application, we establish Sobolev's inequality for Riesz potentials of functions in such spaces, which is one of mixed norm type inequalities.

1. Introduction

Let $\mathbf{R}^n$ be the Euclidean space. In harmonic analysis, the maximal operator is a classical tool when studying Sobolev functions and partial differential equations. This also plays a central role in the study of differentiation, singular integrals, smoothness of functions and so on (see [4, 9, 18], etc.).

It is well known that the maximal operator is bounded on the Lebesgue space $L^p(\mathbf{R}^n)$ if $p > 1$ (see [18]). The boundedness of the maximal operator was studied on Morrey spaces in [7, 14], on Orlicz-Morrey spaces in [16], and also on non-homogeneous Herz spaces in [10]. For Morrey spaces, which were introduced to estimate solutions of partial differential equations, we refer to [13, 17].

Recently, the boundedness of the maximal operator was studied for non-homogeneous central Herz-Morrey-Orlicz spaces on the whole space $\mathbf{R}^n$ (see [12]). Let $\mathbf{H}_+$ denote the half plane :

$$\mathbf{H}_+ = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} : x_n > 0\}.$$

Our first aim in this paper is to study the boundedness of the Hardy-Littlewood maximal operator on $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$ defined by a general function Φ and a weight ω satisfying certain conditions and $0 < q \leqslant \infty$ (see Theorems 3.4 and 3.5 below). The space $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$ is referred to as a Herz-Morrey-Orlicz space on the half space. See Section 2 for the definitions of Φ, ω and $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$. To prove Theorem 3.4, we need

Mathematics subject classification (2010): 31B15, 46E35.

Keywords and phrases: Herz-Morrey-Orlicz spaces, maximal functions, Riesz potentials, Sobolev's inequality.

the boundedness of maximal operator on L^{Φ} (see Lemma 3.1), and treat only the case $1 < q < \infty$, because the remaining case is easily settled.

One of the important applications of the boundedness of the maximal operator is Sobolev's inequality; in classical Lebesgue spaces, we know Sobolev's inequality:

$$\|I_\alpha f\|_{L^{p^*}(\mathbf{R}^n)} \leqslant C\|f\|_{L^p(\mathbf{R}^n)}$$

for $f \in L^p(\mathbf{R}^n)$, $0 < \alpha < n$ and $1 < p < n/\alpha$, where I_α is the Riesz kernel of order α and $1/p^* = 1/p - \alpha/n$ (see, e.g. [2, Theorem 3.1.4]). Sobolev's inequality for Morrey spaces was given by D. R. Adams [1] (also [7, 14]), and then the result was extended to Orlicz-Morrey spaces in [15]. For local Morrey-type spaces, we refer the reader to [5, 6], for example. See also [10] for non-homogeneous Herz spaces and [8] for non-homogeneous central Morrey spaces.

In [12], the boundedness of Riesz potential operators was studied for non-homogeneous central Herz-Morrey-Orlicz spaces on $\mathbf{R}^n$. Our second aim in this paper is to give a general version of Sobolev's inequality for Riesz potentials of functions in $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$ as an application of the boundedness of the maximal operator on $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$ (see Theorem 4.7 below). This seems to be new even for the case $\Phi(r) = r^p$ and $\omega(r) = r^{-\nu}$ (see Corollary 4.8 and Remark 4.9 below). The key lemma for our main Theorem 4.7 is Lemma 4.5 below.

2. Preliminaries

Throughout this paper, let C denote various positive constants independent of the variables in question.

Let Φ be a convex function on $[0,\infty)$ such that

(Φ1) $\Phi(0) = 0$ and $\Phi(r) > 0$ for $r > 0$;

(Φ2) for some $p_2 > 1$, $r^{-p_2}\Phi(r)$ is almost decreasing, that is, there exists a constant $A_1 > 0$ such that

$$\Phi(rt) \leqslant A_1 r^{p_2}\Phi(t) \qquad \text{when } r > 1 \text{ and } t > 0;$$

(Φ3) for some $1 < p_1 \leqslant p_2$, $r^{-p_1}\Phi(r)$ is almost increasing, that is, there exists a constant $A_2 > 0$ such that

$$\Phi(rt) \geqslant A_2 r^{p_1}\Phi(t) \qquad \text{when } r > 1 \text{ and } t > 0.$$

Here note from (Φ2) that Φ is doubling, that is, there exists a constant $A > 0$ such that

$$\Phi(2r) \leqslant A\Phi(r) \qquad \text{for } r > 0.$$

Further consider a weight ω such that

(ω1) $\omega(r) > 0$ for $r > 0$;

$(\omega 2)$ ω is almost decreasing in $(0,\infty)$;

$(\omega 3)$ ω is doubling.

We see that $\omega(r) = r^{-\nu}(\log(e+r))^{\theta}$ is almost decreasing when $\nu > 0$ and $\theta \in \mathbf{R}$.

Note here that $(\Phi 2)$ holds if and only if

$(\Phi 2')$ $\Phi(rt) \geqslant {A_1}^{-1} r^{p_2} \Phi(t)$ when $0 < r < 1$ and $t > 0$;

and $(\Phi 3)$ holds if and only if

$(\Phi 3')$ $\Phi(rt) \leqslant {A_2}^{-1} r^{p_1} \Phi(t)$ when $0 < r < 1$ and $t > 0$.

Moreover, if Φ is of the form $r^p(\log(e+r))^{\theta}$ for $p > 1$ and $\theta \in \mathbf{R}$, then $(\Phi 2)$ and $(\Phi 3)$ hold when $1 < p_1 < p < p_2$. In particular, $p_1 = p = p_2$ if and only if $\theta = 0$.

For $r > 0$, set

$$H(r) = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} : 0 < x_n < r\}$$

and

$$A(r) = H(2r) \setminus H(r).$$

For $0 < q \leqslant \infty$, we denote by $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$ the class of measurable functions f on the half space $\mathbf{H}_+$ satisfying

$$\|f\|_{\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)} = \inf\left\{\lambda > 0 : \int_{H(2)} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy + \left(\int_1^{\infty} \left(\omega(r) \int_{A(r)} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy\right)^q \frac{dr}{r}\right)^{1/q} \leqslant 1\right\} < \infty$$

when $0 < q < \infty$, $\mathscr{H}^{\Phi,\omega,\infty}(\mathbf{H}_+)$ is the space of all measurable functions f on $\mathbf{H}_+$ such that

$$\|f\|_{\mathscr{H}^{\Phi,\omega,\infty}(\mathbf{H}_+)} = \inf\left\{\lambda > 0 : \int_{H(2)} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy + \sup_{r \geqslant 1} \left(\omega(r) \int_{A(r)} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy\right) \leqslant 1\right\} < \infty.$$

The space $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$ is referred to as a non-homogeneous central Herz-Morrey-Orlicz space on the half space $\mathbf{H}_+$, which is one of spaces with mixed norm. In connection with $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$, we consider the space $\underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)$ of all measurable functions f on $\mathbf{H}_+$ satisfying

$$\|f\|_{\underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)} = \inf\left\{\lambda > 0 : \int_{H(2)} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy + \left(\int_1^{\infty} \left(\omega(r) \int_{H(r)} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy\right)^q \frac{dr}{r}\right)^{1/q} \leqslant 1\right\} < \infty$$

when $0 < q < \infty$; if $q = \infty$, then we need necessary modifications. For these spaces, see e.g. [3, 5, 6, 13].

LEMMA 2.1. *Suppose*

(ω4) $r^{\delta_0}\omega(r)$ *is almost decreasing on* $[1,\infty)$ *for some* $\delta_0 > 0$.

Then $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+) = \underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)$.

Proof. Let $0 < \delta < \delta_0$. First, we show the case $0 < q < \infty$. Let f be a measurable function on $\mathbf{H}_+$ with

$$\int_{H(2)} \Phi(|f(y)|)\,dy + \left(\int_1^\infty \left(\omega(r)\int_{A(r)} \Phi(|f(y)|)\,dy\right)^q \frac{dr}{r}\right)^{1/q} \leqslant 1.$$

In case $0 < q < \infty$, by Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{H(r)\setminus H(2)} \Phi(|f(y)|)\,dy \\
&\leqslant C\int_1^r \left(\int_{A(t)} \Phi(|f(y)|)\,dy\right)\frac{dt}{t} \\
&\leqslant C\left(\int_1^r \left(t^{-\delta}\omega(t)^{-1}\right)^{q'}\frac{dt}{t}\right)^{1/q'}\left(\int_1^r \left(t^{\delta}\omega(t)\int_{A(t)} \Phi(|f(y)|)\,dy\right)^q \frac{dt}{t}\right)^{1/q} \\
&\leqslant Cr^{-\delta}\omega(r)^{-1}\left(\int_1^r \left(t^{\delta}\omega(t)\int_{A(t)} \Phi(|f(y)|)\,dy\right)^q \frac{dt}{t}\right)^{1/q}
\end{aligned}$$

for $r \geqslant 1$ since

$$\begin{aligned}
\int_1^r \left(t^{-\delta}\omega(t)^{-1}\right)^{q'}\frac{dt}{t} &\leqslant C\left(r^{-\delta_0}\omega(r)^{-1}\right)^{q'}\int_1^r t^{(\delta_0-\delta)q'}\frac{dt}{t} \\
&= C\left(r^{-\delta_0}\omega(r)^{-1}\right)^{q'} r^{(\delta_0-\delta)q'} = Cr^{-\delta q'}\omega(r)^{-q'}
\end{aligned}$$

by (ω4). For the case $0 < q \leqslant 1$, we use the fact that $(a+b)^q \leqslant a^q + b^q$ for all $a, b \geqslant 0$ instead of Hölder's inequality and use

$$t^{-\delta}\omega(t)^{-1} \leqslant Cr^{-\delta}\omega(r)^{-1}$$

for all $1 \leqslant t \leqslant r$ by (ω4). Hence we have by (ω4)

$$\begin{aligned}
&\int_1^\infty \left(\omega(r)\int_{H(r)} \Phi(|f(y)|)\,dy\right)^q \frac{dr}{r} \\
&\leqslant C\left\{\int_1^\infty \left(\omega(r)\int_{H(r)\setminus H(2)} \Phi(|f(y)|)\,dy\right)^q \frac{dr}{r} + \int_1^\infty \left(\omega(r)\int_{H(2)} \Phi(|f(y)|)\,dy\right)^q \frac{dr}{r}\right\} \\
&\leqslant C\left\{\int_1^\infty \left(t^{\delta}\omega(t)\int_{A(t)} \Phi(|f(y)|)\,dy\right)^q \left(\int_t^\infty r^{-\delta q}\frac{dr}{r}\right)\frac{dt}{t} + \left(\int_{H(2)} \Phi(|f(y)|)\,dy\right)^q\right\} \\
&\leqslant C\left\{\int_1^\infty \left(\omega(t)\int_{A(t)} \Phi(|f(y)|)\,dy\right)^q \frac{dr}{r} + \left(\int_{H(2)} \Phi(|f(y)|)\,dy\right)^q\right\},
\end{aligned}$$

so that

$$\int_{H(2)} \Phi(|f(y)|)\,dy + \left(\int_1^\infty \left(\omega(r) \int_{H(r)} \Phi(|f(y)|)\,dy \right)^q \frac{dr}{r} \right)^{1/q} \leqslant C.$$

Next, we deal with the case $q = \infty$. Let f be a measurable function on $\mathbf{H}_+$ with

$$\int_{H(2)} \Phi(|f(y)|)\,dy + \sup_{r \geqslant 1} \left(\omega(r) \int_{A(r)} \Phi(|f(y)|)\,dy \right) \leqslant 1.$$

By $(\omega 4)$, we obtain

$$\begin{aligned}
\int_{H(r)\setminus H(2)} \Phi(|f(y)|)\,dy &\leqslant C \int_1^r \left(\int_{A(t)} \Phi(|f(y)|)\,dy \right) \frac{dt}{t} \\
&\leqslant C \left(\int_1^r t^{-\delta} \omega(t)^{-1} \frac{dt}{t} \right) \sup_{1 \leqslant t \leqslant r} \left(t^\delta \omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right) \\
&\leqslant C r^{-\delta} \omega(r)^{-1} \sup_{1 \leqslant t \leqslant r} \left(t^\delta \omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right) \\
&\leqslant C \omega(r)^{-1} \sup_{1 \leqslant t \leqslant r} \left(\omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right)
\end{aligned}$$

for $r \geqslant 1$. Hence we have by $(\omega 2)$

$$\begin{aligned}
&\sup_{r \geqslant 1} \left(\omega(r) \int_{H(r)} \Phi(|f(y)|)\,dy \right) \\
&\leqslant \sup_{r \geqslant 1} \left(\omega(r) \int_{H(r)\setminus H(2)} \Phi(|f(y)|)\,dy \right) + \sup_{r \geqslant 1} \left(\omega(r) \int_{H(2)} \Phi(|f(y)|)\,dy \right) \\
&\leqslant C \left\{ \sup_{r \geqslant 1} \left(\sup_{1 \leqslant t \leqslant r} \left(\omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right) \right) + \int_{H(2)} \Phi(|f(y)|)\,dy \right\} \\
&\leqslant C \left\{ \sup_{t \geqslant 1} \left(\omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right) + \int_{H(2)} \Phi(|f(y)|)\,dy \right\},
\end{aligned}$$

so that

$$\int_{H(2)} \Phi(|f(y)|)\,dy + \sup_{r \geqslant 1} \left(\omega(r) \int_{H(r)} \Phi(|f(y)|)\,dy \right) \leqslant C. \quad \square$$

3. The boundedness of the maximal operator

We define the maximal function of a locally integrable function f on $\mathbf{H}_+$ by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap \mathbf{H}_+} |f(y)|\,dy,$$

where $B(x,r)$ denotes the open ball centered at $x \in \mathbf{R}^n$ with radius $r > 0$.

In view of $(\Phi 3)$, we have the boundedness of maximal operator in $L^\Phi(\mathbf{H}_+) = \{ f \in L^1_{loc}(\mathbf{H}_+) : \int_{\mathbf{H}_+} \Phi(|f(y)|)\,dy < \infty \}$.

LEMMA 3.1. ([11, Corollary 4.4]) *There exists a constant $C>0$ such that*

$$\int_{\mathbf{H}_+} \Phi(Mf(x))\,dx \leqslant C\int_{\mathbf{H}_+} \Phi(|f(y)|)\,dy.$$

Let us start with the following lemma which will be useful in the sequel.

LEMMA 3.2. *Let $a<1$. Then there exists a constant $C>0$ such that*

$$\int_{\mathbf{R}^{n-1}} |x-y|^{a-n}\,dx' \leqslant C|x_n-y_n|^{a-1},$$

where $x=(x',x_n)$ and $y=(y',y_n)$.

Proof. For $x=(x',x_n)$ and $y=(y',y_n)$, set $r=|x'-y'|$ and $b=|x_n-y_n|$. Then

$$\int_{\mathbf{R}^{n-1}} |x-y|^{a-n}\,dx' = C\int_0^\infty (r^2+b^2)^{(a-n)/2} r^{n-2}\,dr \leqslant Cb^{a-1}. \quad \square$$

For a real number κ, $x\in H(r)$ and $r>1$, we define

$$J_{\kappa,r}(x) = \int_{\mathbf{H}_+\setminus(H(2r)\cup B(x,r))} |x-y|^{\kappa-n} f(y)\,dy$$

and set

$$f_{1,r} = f_{1,r;x} = f\chi_{\mathbf{H}_+\setminus(H(2r)\cup B(x,r))}.$$

Next result is instrumental for the boundedness of the maximal operator.

LEMMA 3.3. *Suppose*

(ω5) $r^{\kappa p_2-1+\beta_0}\omega(r)^{-1}$ *is almost decreasing on $[1,\infty)$ for some $\beta_0>0$.*

Let $\varepsilon>0$ and $\beta>0$ such that

$$\kappa p_2+\varepsilon(p_2-1)-1<0 \qquad \textit{and} \qquad \varepsilon(p_2-1)+\beta<\beta_0.$$

Then there exists a constant $C>0$ such that

$$r^{-\varepsilon p_2}\int_{H(r)} \Phi(r^{\varepsilon}J_{\kappa,r}(x))\,dx \leqslant Cr^{\kappa p_2+\beta}\omega(r)^{-1}\left(\int_r^\infty \left(t^{-\beta}\omega(t)\int_{A(t)}\Phi(f(y))\,dy\right)^q \frac{dt}{t}\right)^{1/q}$$

when $0<q<\infty$ and

$$r^{-\varepsilon p_2}\int_{H(r)} \Phi(r^{\varepsilon}J_{\kappa,r}(x))\,dx \leqslant Cr^{\kappa p_2}\omega(r)^{-1}\sup_{t\geqslant r}\left(\omega(t)\int_{A(t)}\Phi(f(y))\,dy\right)$$

when $q=\infty$, for all nonnegative measurable functions f on $\mathbf{H}_+$.

Proof. We show only the case $1 < q < \infty$. Let ε be given as in the present lemma. For a nonnegative measurable function f on $\mathbf{H}_+$ and $x \in H(r)$, we have

$$J_{\kappa,r}(x) = \int_{\mathbf{H}_+ \setminus (H(2r) \cup B(x,r))} |x-y|^{\kappa-n} f(y)\,dy \leqslant C \int_r^\infty \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} f_{1,r}(y)\,dy \right) t^{\kappa-1}\,dt.$$

We obtain by Jensen's inequality and $(\Phi 2)$

$$\begin{aligned}\Phi(r^\varepsilon J_{\kappa,r}(x)) &\leqslant C\Phi\left(r^\varepsilon \int_r^\infty \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} t^{\kappa+\varepsilon} f_{1,r}(y)\,dy \right) t^{-\varepsilon-1}\,dt \right) \\ &\leqslant Cr^\varepsilon \int_r^\infty \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} \Phi(t^{\kappa+\varepsilon} f_{1,r}(y))\,dy \right) t^{-\varepsilon-1}\,dt \\ &\leqslant Cr^\varepsilon \int_r^\infty t^{(\kappa+\varepsilon)p_2-n} \left(\int_{B(x,t)} \Phi(f_{1,r}(y))\,dy \right) t^{-\varepsilon-1}\,dt \\ &\leqslant Cr^\varepsilon \int_{\mathbf{H}_+ \setminus (H(2r) \cup B(x,r))} |x-y|^{\kappa p_2+\varepsilon(p_2-1)-n} \Phi(f(y))\,dy\end{aligned}$$

since $r > 1$ and

$$\kappa p_2 + \varepsilon(p_2-1) - n < \kappa p_2 + \varepsilon(p_2-1) - 1 < 0.$$

If $x \in H(r)$ and $y \in \mathbf{H}_+ \setminus B(x,r)$, then for $\bar{x} = (x', -x_n)$

$$|\bar{x}-y| \leqslant 2x_n + |x-y| < 2r + |x-y| < 3|x-y|.$$

Therefore, Lemma 3.2 gives

$$\begin{aligned}\int_{H(r)} |x-y|^{\kappa p_2+\varepsilon(p_2-1)-n}\,dx &\leqslant \int_{H(r)} (|\bar{x}-y|/3)^{\kappa p_2+\varepsilon(p_2-1)-n}\,dx \\ &\leqslant C \int_0^r (x_n+y_n)^{\kappa p_2+\varepsilon(p_2-1)-1}\,dx_n \\ &\leqslant Cry_n^{\kappa p_2+\varepsilon(p_2-1)-1}\end{aligned}$$

for $y \in \mathbf{H}_+ \setminus (H(2r) \cup B(x,r))$ since $\kappa p_2 + \varepsilon(p_2-1) < 1$. Hence we obtain by Hölder's inequality and $(\omega 5)$

$$\begin{aligned}\int_{H(r)} \Phi(r^\varepsilon J_{\kappa,r}(x))\,dx &\leqslant Cr^\varepsilon \int_{\mathbf{H}_+ \setminus H(2r)} \left(\int_{H(r)} |x-y|^{\kappa p_2+\varepsilon(p_2-1)-n}\,dx \right) \Phi(f(y))\,dy \\ &\leqslant Cr^{\varepsilon+1} \int_{\mathbf{H}_+ \setminus H(2r)} y_n^{\kappa p_2+\varepsilon(p_2-1)-1} \Phi(f(y))\,dy \\ &\leqslant Cr^{\varepsilon+1} \sum_{j=1}^\infty (2^j r)^{\kappa p_2+\varepsilon(p_2-1)-1} \int_{A(2^j r)} \Phi(f(y))\,dy\end{aligned}$$

$$\leqslant Cr^{\varepsilon+1}\left(\sum_{j=1}^{\infty}\left((2^jr)^{\kappa p_2+\varepsilon(p_2-1)-1+\beta}\omega(2^jr)^{-1}\right)^{q'}\right)^{1/q'}$$
$$\times\left(\sum_{j=1}^{\infty}\left((2^jr)^{-\beta}\omega(2^jr)\int_{A(2^jr)}\Phi(f(y))\,dy\right)^{q}\right)^{1/q}$$
$$\leqslant Cr^{(\kappa+\varepsilon)p_2+\beta}\omega(r)^{-1}$$
$$\times\left(\sum_{j=1}^{\infty}\left((2^jr)^{-\beta}\omega(2^jr)\int_{A(2^jr)}\Phi(f(y))\,dy\right)^{q}\right)^{1/q},$$

which gives

$$\int_{H(r)}\Phi(r^{\varepsilon}J_{\kappa,r}(x))\,dx$$
$$\leqslant Cr^{(\kappa+\varepsilon)p_2+\beta}\omega(r)^{-1}\left(\int_r^{\infty}\left(t^{-\beta}\omega(t)\int_{A(t)}\Phi(f(y))dy\right)^{q}\frac{dt}{t}\right)^{1/q},$$

which proves the assertion. $\square$

Now we are ready to prove the boundedness of maximal operator on $\underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)$.

THEOREM 3.4. *Suppose*

$(\omega 5')$ $r^{-1+\beta_0}\omega(r)^{-1}$ *is almost decreasing on* $[1,\infty)$ *for some* $\beta_0>0$.

Let $\varepsilon>0$ *be given such that*

$$\varepsilon(p_2-1)-1<0 \qquad \textit{and} \qquad \varepsilon(p_2-1)<\beta_0.$$

Then there exists a constant $C>0$ *such that*

$$\int_{H(2)}\Phi(Mf(x))\,dx+\left(\int_1^{\infty}\left(r^{-\varepsilon p_2}\omega(r)\int_{H(r)}\Phi(r^{\varepsilon}Mf(x))\,dx\right)^{q}\frac{dr}{r}\right)^{1/q}\leqslant C$$

when $0<q<\infty$ *and*

$$\int_{H(2)}\Phi(Mf(x))\,dx+\sup_{r\geqslant 1}\left(r^{-\varepsilon p_2}\omega(r)\int_{H(r)}\Phi(r^{\varepsilon}Mf(x))\,dx\right)\leqslant C$$

when $q=\infty$, *for all* $f\in\underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)$ *with* $\|f\|_{\underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)}\leqslant 1$.

Proof. We show only the case $1<q<\infty$. Let f be a nonnegative measurable function on $\mathbf{H}_+$ such that

$$\int_{H(2)}\Phi(f(y))\,dy+\left(\int_1^{\infty}\left(\omega(r)\int_{H(r)}\Phi(f(y))\,dy\right)^{q}\frac{dr}{r}\right)^{1/q}\leqslant 1.$$

For $r > 1$ write

$$f = f\chi_{\mathbf{H}_+ \setminus H(2r)} + f\chi_{H(2r)} = f_{2,r} + f_{3,r}.$$

First note by $(\Phi 2)$ and Lemma 3.1 that

$$\begin{aligned} r^{-\varepsilon p_2} \int_{H(r)} \Phi(r^{\varepsilon} M f_{3,r}(x))\,dx &\leqslant C \int_{H(r)} \Phi(M f_{3,r}(y))\,dy \\ &\leqslant C \int_{\mathbf{H}_+} \Phi(f_{3,r}(y))\,dy \\ &= C \int_{H(2r)} \Phi(f(y))\,dy, \end{aligned}$$

so that

$$\int_{H(2)} \Phi(M f_{3,2}(x))\,dx \leqslant C \int_{H(4)} \Phi(f(y))\,dy \leqslant C$$

and

$$\begin{aligned} \int_1^{\infty} \left(r^{-\varepsilon p_2} \omega(r) \int_{H(r)} \Phi(r^{\varepsilon} M f_{3,r}(x))\,dx \right)^q \frac{dr}{r} &\leqslant C \int_1^{\infty} \left(\omega(r) \int_{H(2r)} \Phi(f(y))\,dy \right)^q \frac{dr}{r} \\ &\leqslant C \int_1^{\infty} \left(\omega(r) \int_{H(r)} \Phi(f(y))\,dy \right)^q \frac{dr}{r} \\ &\leqslant C. \end{aligned}$$

For $x \in H(r)$, note that

$$M f_{2,r}(x) \leqslant C J_{0,r}(x).$$

Take $\beta > 0$ such that

$$\varepsilon(p_2 - 1) + \beta < \beta_0.$$

Then we obtain by Lemma 3.3 and $(\omega 5')$

$$r^{-\varepsilon p_2} \int_{H(2)} \Phi(r^{\varepsilon} M f_{2,r}(x))\,dx \leqslant C r^{\beta} \omega(r)^{-1} \left(\int_r^{\infty} \left(t^{-\beta} \omega(t) \int_{A(t)} \Phi(f(y))\,dy \right)^q \frac{dt}{t} \right)^{1/q},$$

so that

$$\begin{aligned} \int_{H(2)} \Phi(M f_{2,2}(x))\,dx &\leqslant C \left(\int_2^{\infty} \left(t^{-\beta} \omega(t) \int_{A(t)} \Phi(f(y))\,dy \right)^q \frac{dt}{t} \right)^{1/q} \\ &\leqslant C \left(\int_2^{\infty} \left(\omega(t) \int_{H(t)} \Phi(f(y))\,dy \right)^q \frac{dt}{t} \right)^{1/q} \\ &\leqslant C \end{aligned}$$

and again by Lemma 3.3

$$\begin{aligned}
&\int_1^\infty \left(r^{-\varepsilon p_2} \omega(r) \int_{H(r)} \Phi(r^\varepsilon M f_{2,r}(x))\, dx \right)^q \frac{dr}{r} \\
&\leqslant C \int_1^\infty r^{\beta q} \left(\int_r^\infty \left(t^{-\beta} \omega(t) \int_{A(t)} \Phi(f(y))\, dy \right)^q \frac{dt}{t} \right) \frac{dr}{r} \\
&\leqslant C \int_1^\infty \left(t^{-\beta} \omega(t) \int_{A(t)} \Phi(f(y))\, dy \right)^q \left(\int_1^t r^{\beta q} \frac{dr}{r} \right) \frac{dt}{t} \\
&\leqslant C \int_1^\infty \left(\omega(t) \int_{A(t)} \Phi(f(y))\, dy \right)^q \frac{dt}{t} \\
&\leqslant C \left\{ \int_{H(2)} \Phi(f(y))\, dy + \left(\int_1^\infty \left(\omega(t) \int_{H(t)} \Phi(f(y))\, dy \right)^q \frac{dt}{t} \right)^{1/q} \right\} \\
&\leqslant C.
\end{aligned}$$

Consequently,

$$\int_{H(2)} \Phi(Mf(x))\, dx \leqslant C$$

and

$$\int_1^\infty \left(r^{-\varepsilon p_2} \omega(r) \int_{H(r)} \Phi(r^\varepsilon M f(x))\, dx \right)^q \frac{dr}{r} \leqslant C,$$

and the proof of the theorem is completed. □

In view of Theorem 3.4 and Lemma 2.1, we present the boundedness of maximal operator on $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$.

THEOREM 3.5. *Suppose* $(\omega 4)$ *and* $(\omega 5')$ *hold. Let* $\varepsilon > 0$ *be given such that*

$$\varepsilon(p_2 - 1) - 1 < 0 \qquad \text{and} \qquad \varepsilon(p_2 - 1) < \beta_0.$$

Then there exists a constant $C > 0$ *such that*

$$\int_{H(2)} \Phi(Mf(x))\, dx + \left(\int_1^\infty \left(r^{-\varepsilon p_2} \omega(r) \int_{A(r)} \Phi(r^\varepsilon M f(x))\, dx \right)^q \frac{dr}{r} \right)^{1/q} \leqslant C$$

when $0 < q < \infty$ *and*

$$\int_{H(2)} \Phi(Mf(x))\, dx + \sup_{r \geqslant 1} \left(r^{-\varepsilon p_2} \omega(r) \int_{A(r)} \Phi(r^\varepsilon M f(x))\, dx \right) \leqslant C$$

when $q = \infty$, *for all* $f \in \mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$ *with* $\|f\|_{\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)} \leqslant 1$.

COROLLARY 3.6. *Let* $0 < \nu < 1$. *If* $p > 1$, *then there exists a constant* $C > 0$ *such that*

$$\left(\int_{\mathbf{H}_+} Mf(x)^p (1 + x_n)^{-\nu}\, dx \right)^{1/p} \leqslant C \left(\int_{\mathbf{H}_+} |f(y)|^p (1 + y_n)^{-\nu}\, dy \right)^{1/p}.$$

4. Sobolev inequality

The Riesz potential of order α $(0<\alpha<n)$ of f is defined by

$$I_\alpha f(x)=\int_{\mathbf{H}_+}|x-y|^{\alpha-n}f(y)\,dy$$

with $f\in L^1_{loc}(\mathbf{H}_+)$.

For $x=(x',x_n)\in\mathbf{H}_+$ and $r>0$, write

$$\begin{aligned}
I_{1,r}f(x)&=\int_{B(x,r)}|x-y|^{\alpha-n}f(y)\,dy,\\
I_{2,r}f(x)&=\int_{B(x,1)\setminus B(x,r)}|x-y|^{\alpha-n}f(y)\,dy,\\
I_{3,r}f(x)&=\int_{B(x,x_n/2)\setminus B(x,r)}|x-y|^{\alpha-n}f(y)\,dy,\\
I_{4,r}f(x)&=\int_{\mathbf{H}_+\setminus B(x,r)}|x-y|^{\alpha-n}f(y)\,dy.
\end{aligned}$$

In order to get our aims we shall need some lemmas.

LEMMA 4.1. *There exists a constant $C>0$ such that*

$$I_{1,r}f(x)\leqslant Cr^{\alpha}Mf(x)$$

for all nonnegative measurable function f on $\mathbf{H}_+$.

As in the proof of Lemma 3.3, we show the following result.

LEMMA 4.2. *Let $\varepsilon>0$ be a number such that $\alpha p_1+\varepsilon(p_1-1)-n<0$. Then there exists a constant $C>0$ such that*

$$\Phi(r^{\varepsilon}I_{2,r}f(x))\leqslant Cr^{\alpha p_1+\varepsilon p_1-n}\int_{B(x,1)\setminus B(x,r)}\Phi(f(y))\,dy$$

for all nonnegative measurable functions f on $\mathbf{H}_+$, $x\in\mathbf{H}_+$ and $0<r<1/2$.

Proof. Let f be a nonnegative measurable function on $\mathbf{H}_+$ and $x\in\mathbf{H}_+$. For $0<r<1/2$, we write

$$\tilde f_{2,r}=f\chi_{B(x,1)\setminus B(x,r)}.$$

Then we have by Jensen's inequality and $(\Phi3')$

$$\begin{aligned}
\Phi(r^{\varepsilon}I_{2,r}f(x))&\leqslant C\Phi\left(r^{\varepsilon}\int_r^2\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}t^{\alpha+\varepsilon}\tilde f_{2,r}(y)\,dy\right)t^{-\varepsilon-1}\,dt\right)\\
&\leqslant Cr^{\varepsilon}\int_r^2\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}\Phi(t^{\alpha+\varepsilon}\tilde f_{2,r}(y))\,dy\right)t^{-\varepsilon-1}\,dt
\end{aligned}$$

$$\leqslant Cr^{\varepsilon}\int_r^2 t^{\alpha p_1+\varepsilon p_1-n}\left(\int_{B(x,t)}\Phi(\tilde{f}_{2,r}(y))\,dy\right)t^{-\varepsilon-1}\,dt$$
$$\leqslant Cr^{\varepsilon}\int_{B(x,1)\setminus B(x,r)}|x-y|^{\alpha p_1+\varepsilon(p_1-1)-n}\Phi(f(y))\,dy$$
$$\leqslant Cr^{\alpha p_1+\varepsilon p_1-n}\int_{B(x,1)\setminus B(x,r)}\Phi(f(y))\,dy$$

since $\alpha p_1+\varepsilon(p_1-1)-n<0$. $\square$

LEMMA 4.3. *If $\varepsilon>0$ and $\alpha p_2+\varepsilon(p_2-1)-n<0$, then there exists a constant $C>0$ such that*

$$\Phi(r^{\varepsilon}I_{3,r}f(x))\leqslant Cr^{\alpha p_2+\varepsilon p_2-n}\int_{B(x,x_n/2)}\Phi(f(y))\,dy$$

for all nonnegative measurable functions f on $\mathbf{H}_+$, $x=(x',x_n)\in\mathbf{H}_+$ and $r\geqslant 1/2$.

Proof. Let f be a nonnegative measurable function on $\mathbf{H}_+$. It is sufficient to prove the result for $1/2\leqslant r<x_n/2$. We write

$$\tilde{f}_{3,r}=f\chi_{B(x,x_n/2)\setminus B(x,r)}.$$

Then

$$I_{3,r}f(x)=\int_{B(x,x_n/2)\setminus B(x,r)}|x-y|^{\alpha-n}f(y)\,dy$$
$$\leqslant C\int_r^{x_n}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}\tilde{f}_{3,r}(y)\,dy\right)t^{\alpha-1}\,dt.$$

We have by Jensen's inequality and $(\Phi 2)$

$$\Phi(r^{\varepsilon}I_{3,r}f(x))\leqslant C\Phi\left(r^{\varepsilon}\int_r^{x_n}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}t^{\alpha+\varepsilon}\tilde{f}_{3,r}(y)\,dy\right)t^{-\varepsilon-1}\,dt\right)$$
$$\leqslant Cr^{\varepsilon}\int_r^{x_n}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}\Phi(t^{\alpha+\varepsilon}\tilde{f}_{3,r}(y))\,dy\right)t^{-\varepsilon-1}\,dt$$
$$\leqslant Cr^{\varepsilon}\int_r^{\infty}t^{\alpha p_2+\varepsilon p_2-n}\left(\int_{B(x,t)}\Phi(\tilde{f}_{3,r}(y))\,dy\right)t^{-\varepsilon-1}\,dt$$
$$\leqslant Cr^{\varepsilon}\int_{B(x,x_n/2)\setminus B(x,r)}|x-y|^{\alpha p_2+\varepsilon(p_2-1)-n}\Phi(f(y))\,dy$$
$$\leqslant Cr^{\alpha p_2+\varepsilon p_2-n}\int_{B(x,x_n/2)}\Phi(f(y))\,dy,$$

which gives the assertion. $\square$

LEMMA 4.4. *Suppose*

(ω6) $r^{\alpha p_2+\varepsilon_0(p_2-1)-n}\omega(r)^{-1}$ *is almost decreasing on* $[1,\infty)$ *for some* $\varepsilon_0>0$.

If $0<\varepsilon<\varepsilon_0$, *then there exists a constant* $C>0$ *such that*

$$r^{\varepsilon}I_{4,r}f(x)\leqslant C\Phi^{-1}(r^{\alpha p_2+\varepsilon p_2-n}\omega(r)^{-1})$$

for all $r\geqslant 1/2$, $x=(x',x_n)\in\mathbf{H}_+$ *with* $r\geqslant x_n/2$ *and nonnegative measurable functions* f *on* $\mathbf{H}_+$ *with* $\|f\|_{\underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)}\leqslant 1$.

Proof. We show only the case $1<q<\infty$. Let f be a nonnegative measurable function on $\mathbf{H}_+$ such that

$$\int_{H(2)}\Phi(f(y))\,dy+\left(\int_1^{\infty}\left(\omega(r)\int_{H(r)}\Phi(f(y))\,dy\right)^q\frac{dr}{r}\right)^{1/q}\leqslant 1.$$

Let ε such that $0<\varepsilon<\varepsilon_0$. For $x=(x',x_n)\in\mathbf{H}_+$ with $r\geqslant x_n/2$ and $r\geqslant 1/2$, we write

$$\tilde{f}_{4,r}=f\chi_{\mathbf{H}_+\setminus B(x,r)}.$$

Then

$$I_{4,r}f(x)\leqslant C\int_r^{\infty}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}\tilde{f}_{4,r}(y)\,dy\right)t^{\alpha-1}\,dt.$$

Note from (ω6) that

$$\alpha p_2+\varepsilon_0(p_2-1)-n<0$$

since $\omega(r)$ is almost decreasing on $[1,\infty)$, so that

$$\alpha p_2+\varepsilon(p_2-1)-n=\alpha p_2+\varepsilon_0(p_2-1)-n+(\varepsilon-\varepsilon_0)(p_2-1)<0.$$

We have by Jensen's inequality and (Φ2)

$$\begin{aligned}\Phi(r^{\varepsilon}I_{4,r}f(x))&\leqslant C\Phi\left(r^{\varepsilon}\int_r^{\infty}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}t^{\alpha+\varepsilon}\tilde{f}_{4,r}(y)\,dy\right)t^{-\varepsilon-1}\,dt\right)\\&\leqslant Cr^{\varepsilon}\int_r^{\infty}\left(\frac{1}{|B(x,t)|}\int_{B(x,t)}\Phi(t^{\alpha+\varepsilon}\tilde{f}_{4,r}(y))\,dy\right)t^{-\varepsilon-1}\,dt\\&\leqslant Cr^{\varepsilon}\int_r^{\infty}t^{\alpha p_2+\varepsilon p_2-n}\left(\int_{B(x,t)}\Phi(\tilde{f}_{4,r}(y))\,dy\right)t^{-\varepsilon-1}\,dt\\&\leqslant Cr^{\varepsilon}\int_{\mathbf{H}_+\setminus B(x,r)}|x-y|^{\alpha p_2+\varepsilon(p_2-1)-n}\Phi(f(y))\,dy.\end{aligned}$$

Since $y_n\geqslant 4x_n/2=2x_n$ when $y\in\mathbf{H}_+\setminus(H(r)\cup B(x,r))$ and $y_n\geqslant 4r$, $|x-y|\geqslant|x_n-y_n|\geqslant y_n/2$. When $y\in\mathbf{H}_+\setminus(H(r)\cup B(x,r))$ and $r<y_n<4r$, $|x-y|>r>y_n/4$. Therefore,

$$\begin{aligned}&r^{\varepsilon}\int_{\mathbf{H}_+\setminus(H(r)\cup B(x,r))}|x-y|^{\alpha p_2+\varepsilon(p_2-1)-n}\Phi(f(y))\,dy\\&\leqslant Cr^{\varepsilon}\int_{\mathbf{H}_+\setminus H(r)}y_n^{\alpha p_2+\varepsilon(p_2-1)-n}\Phi(f(y))\,dy\end{aligned}$$

since $\alpha p_2+\varepsilon(p_2-1)-n<0$. Hence

$$\begin{aligned}&\Phi(r^{\varepsilon}I_{4,r}f(x))\\&\leqslant C\left\{r^{\varepsilon}\int_{\mathbf{H}_+\setminus H(r)}y_n^{\alpha p_2+\varepsilon(p_2-1)-n}\Phi(f(y))\,dy+r^{\alpha p_2+\varepsilon p_2-n}\int_{H(r)}\Phi(f(y))\,dy\right\}.\end{aligned}$$

Here we find by Hölder's inequality and $(\omega 6)$

$$\begin{aligned}&\int_{\mathbf{H}_+\setminus H(r)}y_n^{\alpha p_2+\varepsilon(p_2-1)-n}\Phi(f(y))\,dy\leqslant C\sum_{j=0}^{\infty}\int_{A(2^jr)}(2^jr)^{\alpha p_2+\varepsilon(p_2-1)-n}\Phi(f(y))\,dy\\&\leqslant C\left(\sum_{j=0}^{\infty}\left((2^jr)^{\alpha p_2+\varepsilon(p_2-1)-n}\omega(2^jr)^{-1}\right)^{q'}\right)^{1/q'}\left(\sum_{j=0}^{\infty}\left(\omega(2^jr)\int_{A(2^jr)}\Phi(f(y))\,dy\right)^{q}\right)^{1/q}\\&\leqslant Cr^{\alpha p_2+\varepsilon(p_2-1)-n}\omega(r)^{-1}\left(\sum_{j=0}^{\infty}\left(\omega(2^jr)\int_{A(2^jr)}\Phi(f(y))\,dy\right)^{q}\right)^{1/q}.\end{aligned}$$

Further,

$$r^{\alpha p_2+\varepsilon p_2-n}\int_{H(r)}\Phi(f(y))\,dy\leqslant Cr^{\alpha p_2+\varepsilon p_2-n}\omega(r)^{-1}.$$

Hence

$$\Phi(r^{\varepsilon}I_{4,r}f(x))\leqslant Cr^{\alpha p_2+\varepsilon p_2-n}\omega(r)^{-1},$$

which gives the assertion. □

LEMMA 4.5. *Suppose* $(\omega 6)$ *holds. Let* $0<\varepsilon<\varepsilon_0$. *Let* f *be a nonnegative measurable function on* $\mathbf{H}_+$ *such that* $\|f\|_{\underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)}\leqslant 1$.

(1) *Let* $N_{1,\varepsilon}=n+(\alpha+\varepsilon)(p_2-p_1)$. *If* $\alpha p_2+\varepsilon_0 p_2-n<0$, *then there exists a constant* $C>0$ *such that*

$$I_\alpha f(x)\leqslant CMf(x)\left(\omega(x_n)\Phi(Mf(x))\right)^{-\alpha/N_{1,\varepsilon}}$$

for all $x=(x',x_n)\in\mathbf{H}_+$ *with* $x_n\geqslant 1$ *and* $(\omega(x_n)\Phi(Mf(x)))^{-1/N_{1,\varepsilon}}<1$.

(2) *Let* $N_{2,\varepsilon}=n-(\alpha+\varepsilon)(p_2-p_1)$. *If* $\alpha p_2+\varepsilon_0 p_2-n<0$, *then there exists a constant* $C>0$ *such that*

$$I_\alpha f(x)\leqslant CMf(x)\left(\omega(x_n)\Phi(Mf(x))\right)^{-\alpha/N_{2,\varepsilon}}$$

for all $x=(x',x_n)\in\mathbf{H}_+$ *with* $x_n\geqslant 2$ *and* $1\leqslant(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\leqslant x_n/2$.

(3) *Suppose*

$(\omega 7)$ *there exist a real number* $N_{3,\varepsilon}$ *and a constant* $A>0$ *such that* $0<N_{3,\varepsilon}\leqslant N_{2,\varepsilon}$ *and*

$$t^{N_{3,\varepsilon}}\leqslant At^{N_{2,\varepsilon}}\omega(t)$$

for all $t\geqslant 1$.

Then there exists a constant $C>0$ such that

$$I_\alpha f(x) \leqslant CMf(x)\Phi(Mf(x))^{-\alpha/N_{3,\varepsilon}}$$

for all $x=(x',x_n)\in \mathbf{H}_+$ with $x_n \geqslant 1$ and $(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}} \geqslant x_n/2$.

Proof. First we show the case (1). By Lemmas 4.1–4.4, we have for $0<t<1$ and $x=(x',x_n)\in \mathbf{H}_+$ with $x_n \geqslant 1$

$$\begin{aligned}
I_\alpha f(x) &= \int_{B(x,t)} |x-y|^{\alpha-n} f(y)\,dy + \int_{B(x,1)\setminus B(x,t)} |x-y|^{\alpha-n} f(y)\,dy \\
&\quad + \int_{B(x,x_n/2)\setminus B(x,1)} |x-y|^{\alpha-n} f(y)\,dy + \int_{\mathbf{H}_+\setminus B(x,x_n/2)} |x-y|^{\alpha-n} f(y)\,dy \\
&\leqslant C\Big\{ t^\alpha Mf(x) + t^{-\varepsilon}\Phi^{-1}\Big(t^{\alpha p_1+\varepsilon p_1-n}\int_{H(2x_n)}\Phi(f(y))\,dy\Big) \\
&\quad + \Phi^{-1}\Big(\int_{H(3x_n/2)}\Phi(f(y))\,dy\Big) + x_n^{-\varepsilon}\Phi^{-1}(x_n^{\alpha p_2+\varepsilon p_2-n}\omega(x_n)^{-1})\Big\} \\
&\leqslant C\{t^\alpha Mf(x) + t^{-\varepsilon}\Phi^{-1}(t^{\alpha p_1+\varepsilon p_1-n}\omega(x_n)^{-1}) \\
&\quad + \Phi^{-1}(\omega(x_n)^{-1}) + x_n^{-\varepsilon}\Phi^{-1}(x_n^{\alpha p_2+\varepsilon p_2-n}\omega(x_n)^{-1})\} \\
&\leqslant C\{t^\alpha Mf(x) + t^{-\varepsilon}\Phi^{-1}(t^{\alpha p_1+\varepsilon p_1-n}\omega(x_n)^{-1})\}
\end{aligned}$$

since

$$\alpha p_1+\varepsilon p_1-n \leqslant \alpha p_2+\varepsilon p_2-n < \alpha p_2+\varepsilon_0 p_2-n<0$$

and

$$\alpha p_1+\varepsilon(p_1-1)-n \leqslant \alpha p_2+\varepsilon(p_2-1)-n<0.$$

If $t=(\omega(x_n)\Phi(Mf(x)))^{-1/N_{1,\varepsilon}}<1$, then

$$\begin{aligned}
\Phi(t^{-\alpha-\varepsilon}\Phi^{-1}(t^{\alpha p_1+\varepsilon p_1-n}\omega(x_n)^{-1})) &\leqslant A_1 t^{(-\alpha-\varepsilon)p_2} t^{\alpha p_1+\varepsilon p_1-n}\omega(x_n)^{-1} \\
&= A_1 t^{-N_{1,\varepsilon}}\omega(x_n)^{-1}
\end{aligned}$$

by $(\Phi2)$, so that

$$\begin{aligned}
I_\alpha f(x) &\leqslant C\left\{t^\alpha Mf(x) + t^\alpha\Phi^{-1}(A_1 t^{-N_{1,\varepsilon}}\omega(x_n)^{-1})\right\} \\
&\leqslant Ct^\alpha Mf(x) \\
&= CMf(x)\,(\omega(x_n)\Phi(Mf(x)))^{-\alpha/N_{1,\varepsilon}},
\end{aligned}$$

which gives Assertion (1).

Next we show the case (2). We have for $x=(x',x_n)\in\mathbf{H}_+$ and $1\leqslant t<x_n/2$

$$\begin{aligned}
I_\alpha f(x) &= \int_{B(x,t)} |x-y|^{\alpha-n} f(y)\,dy + \int_{B(x,x_n/2)\setminus B(x,t)} |x-y|^{\alpha-n} f(y)\,dy \\
&\quad + \int_{\mathbf{H}_+\setminus B(x,x_n/2)} |x-y|^{\alpha-n} f(y)\,dy
\end{aligned}$$

$$\leqslant C\left\{t^{\alpha}Mf(x)+t^{-\varepsilon}\Phi^{-1}(t^{\alpha p_2+\varepsilon p_2-n}\omega(x_n)^{-1})+x_n^{-\varepsilon}\Phi^{-1}(x_n^{\alpha p_2+\varepsilon p_2-n}\omega(x_n)^{-1})\right\}$$
$$\leqslant C\left\{t^{\alpha}Mf(x)+t^{-\varepsilon}\Phi^{-1}(t^{\alpha p_2+\varepsilon p_2-n}\omega(x_n)^{-1})\right\}$$

by Lemmas 4.1, 4.3 and 4.4. If $1<t=(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}<x_n/2$, then

$$\begin{aligned}\Phi(t^{-\alpha-\varepsilon}\Phi^{-1}(t^{\alpha p_2+\varepsilon p_2-n}\omega(x_n)^{-1})) &\leqslant A_2^{-1}t^{(-\alpha-\varepsilon)p_1}t^{\alpha p_2+\varepsilon p_2-n}\omega(x_n)^{-1}\\ &=A_2^{-1}t^{-N_{2,\varepsilon}}\omega(x_n)^{-1}\end{aligned}$$

by $(\Phi3')$, so that

$$\begin{aligned}I_\alpha f(x) &\leqslant C\left\{t^{\alpha}Mf(x)+t^{\alpha}\Phi^{-1}(A_2^{-1}t^{-N_{2,\varepsilon}}\omega(x_n)^{-1})\right\}\\ &\leqslant Ct^{\alpha}Mf(x)\\ &=CMf(x)\left(\omega(x_n)\Phi(Mf(x))\right)^{-\alpha/N_{2,\varepsilon}},\end{aligned}$$

which gives Assertion (2).

Finally we show the case (3). Let $x=(x',x_n)\in\mathbf{H}_+$ with $x_n\geqslant 1$.

If $(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\geqslant x_n/2\geqslant 1/2$, then we take $t=\Phi(Mf(x))^{-1/N_{3,\varepsilon}}$. Since $x_n\geqslant 1$, we see that

$$x_n^{N_{3,\varepsilon}}\leqslant Ax_n^{N_{2,\varepsilon}}\omega(x_n)\leqslant A2^{N_{2,\varepsilon}}\Phi(Mf(x))^{-1}=A2^{N_{2,\varepsilon}}t^{N_{3,\varepsilon}},$$

so that $t\geqslant A^{-1/N_{3,\varepsilon}}2^{-N_{2,\varepsilon}/N_{3,\varepsilon}}x_n\geqslant C$. We find

$$\begin{aligned}I_\alpha f(x) &=\int_{B(x,t)\cap\mathbf{H}_+}|x-y|^{\alpha-n}f(y)\,dy+\int_{\mathbf{H}_+\setminus B(x,t)}|x-y|^{\alpha-n}f(y)\,dy\\ &\leqslant C\left\{t^{\alpha}Mf(x)+t^{-\varepsilon}\Phi^{-1}(t^{\alpha p_2+\varepsilon p_2-n}\omega(t)^{-1})\right\}\end{aligned}$$

by Lemmas 4.1 and 4.4. Since

$$\begin{aligned}\Phi(t^{-\alpha-\varepsilon}\Phi^{-1}(t^{\alpha p_2+\varepsilon p_2-n}\omega(t)^{-1})) &\leqslant Ct^{(-\alpha-\varepsilon)p_1}t^{\alpha p_2+\varepsilon p_2-n}\omega(t)^{-1}\\ &=Ct^{-N_{2,\varepsilon}}\omega(t)^{-1}\leqslant Ct^{-N_{3,\varepsilon}}\end{aligned}$$

by $(\Phi3')$, we have

$$\begin{aligned}I_\alpha f(x) &\leqslant C\left\{t^{\alpha}Mf(x)+t^{-\varepsilon}\Phi^{-1}(t^{\alpha p_2+\varepsilon p_2-n}\omega(t)^{-1})\right\}\\ &\leqslant C\left\{t^{\alpha}Mf(x)+t^{\alpha}\Phi^{-1}(t^{-N_{3,\varepsilon}})\right\}\\ &\leqslant Ct^{\alpha}Mf(x)\\ &=CMf(x)\Phi(Mf(x))^{-\alpha/N_{3,\varepsilon}},\end{aligned}$$

which gives Assertion (3). □

Suppose there exist convex functions $\Psi_{i,\varepsilon}$ $(i=1,2,3)$ on $[0,\infty)$ such that

$(\Psi_{i,\varepsilon})$ there exists a constant $A_3>0$ such that

$$\Psi_{i,\varepsilon}\left(t\Phi(t)^{-\alpha/N_{i,\varepsilon}}\right)\leqslant A_3\Phi(t)$$

for $t>0$.

In view of previous results we have the following.

LEMMA 4.6. *Suppose* $(\omega 6)$ *and* $(\omega 7)$ *hold for* $\alpha p_2+\varepsilon_0 p_2-n<0$. *Let* $0<\varepsilon<\varepsilon_0$. *Then there exists a constant* $C>0$ *such that*

$$\int_{A(r)}\Big(\Psi_{1,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{1,\varepsilon}}x_n^{\varepsilon}I_\alpha f(x)\right)\chi_{\{x=(x',x_n)\in\mathbf{H}_+:(\omega(x_n)\Phi(Mf(x)))^{-1/N_{1,\varepsilon}}<1\}}$$
$$+\Psi_{2,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{2,\varepsilon}}x_n^{\varepsilon}I_\alpha f(x)\right)\chi_{\{x=(x',x_n)\in\mathbf{H}_+:1\leqslant(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\leqslant x_n/2\}}$$
$$+\Psi_{3,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\right)^{\alpha/N_{3,\varepsilon}}x_n^{\varepsilon}I_\alpha f(x)\right)\chi_{\{x=(x',x_n)\in\mathbf{H}_+:(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\geqslant x_n/2\}}\Big)\,dx$$
$$\leqslant C\int_{A(r)}\Phi(r^{\varepsilon}Mf(x))\,dx$$

for all $r\geqslant 1$ *and nonnegative measurable functions* f *on* $\mathbf{H}_+$ *such that* $\|f\|_{\underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)}\leqslant 1$.

Proof. Let $x=(x',x_n)\in A(r)$ with $r\geqslant 1$.

If $(\omega(x_n)\Phi(Mf(x)))^{-1/N_{1,\varepsilon}}<1$ with $N_{1,\varepsilon}=n+(\alpha+\varepsilon)(p_2-p_1)$, then we have by Lemma 4.5 (1) and $(\Phi 2)$

$$\begin{aligned}I_\alpha f(x)&\leqslant CMf(x)\left(\omega(x_n)\Phi(Mf(x))\right)^{-\alpha/N_{1,\varepsilon}}\\&\leqslant CMf(x)\left(\omega(x_n)x_n^{-\varepsilon p_2}\Phi(x_n^{\varepsilon}Mf(x))\right)^{-\alpha/N_{1,\varepsilon}},\end{aligned}$$

so that in view of $(\Psi_{1,\varepsilon})$

$$\begin{aligned}\Psi_{1,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{1,\varepsilon}}x_n^{\varepsilon}I_\alpha f(x)\right)&\leqslant C\Psi_{1,\varepsilon}\left(x_n^{\varepsilon}Mf(x)\Phi(x_n^{\varepsilon}Mf(x))^{-\alpha/N_{1,\varepsilon}}\right)\\&\leqslant C\Phi(x_n^{\varepsilon}Mf(x)).\end{aligned}$$

If $1\leqslant(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\leqslant x_n/2$ with $N_{2,\varepsilon}=n-(\alpha+\varepsilon)(p_2-p_1)$, then we have by Lemma 4.5 (2) and $(\Phi 2)$

$$\begin{aligned}I_\alpha f(x)&\leqslant CMf(x)\left(\omega(x_n)\Phi(Mf(x))\right)^{-\alpha/N_{2,\varepsilon}}\\&\leqslant CMf(x)\left(\omega(x_n)x_n^{-\varepsilon p_2}\Phi(x_n^{\varepsilon}Mf(x))\right)^{-\alpha/N_{2,\varepsilon}}\end{aligned}$$

and hence, in view of $(\Psi_{2,\varepsilon})$,

$$\Psi_{2,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{2,\varepsilon}}x_n^{\varepsilon}I_\alpha f(x)\right)\leqslant C\Phi(x_n^{\varepsilon}Mf(x)).$$

Finally, if $(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\geqslant x_n/2$, then we obtain by Lemma 4.5 (3) and $(\Phi 2)$

$$I_\alpha f(x)\leqslant CMf(x)\Phi(Mf(x))^{-\alpha/N_{3,\varepsilon}}\leqslant CMf(x)\left(x_n^{-\varepsilon p_2}\Phi(x_n^{\varepsilon}Mf(x))\right)^{-\alpha/N_{3,\varepsilon}},$$

so that in view of $(\Psi_{3,\varepsilon})$

$$\Psi_{3,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\right)^{\alpha/N_{3,\varepsilon}} x_n^{\varepsilon} I_\alpha f(x)\right) \leqslant C\Phi(x_n^{\varepsilon} Mf(x)),$$

as required. □

Now we establish our main result.

THEOREM 4.7. *Suppose* $(\omega 4)$, $(\omega 5')$, $(\omega 6)$ *and* $(\omega 7)$ *hold for* $\alpha p_2 + \varepsilon_0 p_2 - n < 0$. *If* $0 < \varepsilon < \varepsilon_0$, $\varepsilon(p_2-1)-1<0$ *and* $\varepsilon(p_2-1) < \beta_0$, *then there exists a constant* $C>0$ *such that*

$$\begin{aligned}\int_1^\infty \Bigg(& r^{-\varepsilon p_2}\omega(r) \\ & \times \int_{A(r)} \Bigg(\Psi_{1,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{1,\varepsilon}} x_n^{\varepsilon} |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+ : (\omega(x_n)\Phi(Mf(x)))^{-1/N_{1,\varepsilon}} <1\}} \\ & + \Psi_{2,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{2,\varepsilon}} x_n^{\varepsilon} |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+ : 1\leqslant (\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}} \leqslant x_n/2\}} \\ & + \Psi_{3,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\right)^{\alpha/N_{3,\varepsilon}} x_n^{\varepsilon} |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+ : (\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}} \geqslant x_n/2\}} \Bigg) dx \Bigg)^q \frac{dr}{r} \\ \leqslant C &\end{aligned}$$

when $0<q<\infty$ *and*

$$\begin{aligned}\sup_{r\geqslant 1} \Bigg(& r^{-\varepsilon p_2}\omega(r) \\ & \times \int_{A(r)} \Bigg(\Psi_{1,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{1,\varepsilon}} x_n^{\varepsilon} |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+ : (\omega(x_n)\Phi(Mf(x)))^{-1/N_{1,\varepsilon}} <1\}} \\ & + \Psi_{2,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{2,\varepsilon}} x_n^{\varepsilon} |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+ : 1\leqslant (\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}} \leqslant x_n/2\}} \\ & + \Psi_{3,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\right)^{\alpha/N_{3,\varepsilon}} x_n^{\varepsilon} |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+ : (\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}} \geqslant x_n/2\}} \Bigg) dx \Bigg) \\ \leqslant C &\end{aligned}$$

when $q=\infty$, *for all measurable functions* f *on* $\mathbf{H}_+$ *such that* $\|f\|_{\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)} \leqslant 1$.

Proof. We show only the case $1<q<\infty$. In view of Lemma 4.6 and Theorem 3.5, we obtain

$$\begin{aligned}\int_1^\infty \Bigg(& r^{-\varepsilon p_2}\omega(r) \\ & \times \int_{A(r)} \Bigg(\Psi_{1,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{1,\varepsilon}} x_n^{\varepsilon} |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+ : (\omega(x_n)\Phi(Mf(x)))^{-1/N_{1,\varepsilon}} <1\}} \\ & + \Psi_{2,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{2,\varepsilon}} x_n^{\varepsilon} |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+ : 1\leqslant (\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}} \leqslant x_n/2\}}\end{aligned}$$

$$+\Psi_{3,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\right)^{\alpha/N_{3,\varepsilon}} x_n^{\varepsilon}\,|I_\alpha f(x)|\right)\chi_{\{x=(x',x_n)\in\mathbf{H}_+:(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\geqslant x_n/2\}}\right)dx\Bigg)^q\frac{dr}{r}$$

$$\leqslant C\int_1^\infty\left(r^{-\varepsilon p_2}\omega(r)\int_{A(r)}\Phi(r^\varepsilon Mf(x))\,dx\right)^q\frac{dr}{r}$$

$$\leqslant C,$$

as required. □

When $\Phi(r)=r^p$ and $\omega(r)=r^{-\nu}$, we obtain the following corollary.

COROLLARY 4.8. *Let*

$$E_1=\{x=(x',x_n)\in\mathbf{H}_+ : Mf(x)\geqslant(2^n x_n^{\nu-n})^{1/p}\}$$

and

$$E_2=\{x=(x',x_n)\in\mathbf{H}_+ : Mf(x)<(2^n x_n^{\nu-n})^{1/p}\}.$$

Let $1/p^*=1/p-\alpha/n>0$ *and* $1/p_\nu=1/p-\alpha/(n-\nu)>0$. *If* $p>1$ *and* $0<\nu<1$, *then there exists a constant* $C>0$ *such that*

$$\int_{E_1\setminus H(1)}\left(|I_\alpha f(x)|(1+x_n)^{-\nu/p}\right)^{p^*}dx+\int_{E_2\setminus H(1)}|I_\alpha f(x)|^{p_\nu}x_n^{-\nu}\,dx\leqslant C$$

for all measurable functions f *on* $\mathbf{H}_+$ *such that*

$$\int_{\mathbf{H}_+}|f(y)|^p(1+y_n)^{-\nu}\,dy\leqslant 1.$$

In order to prove Corollary 4.8, it is enough to notice that for $\Phi(r)=r^p$ and $\omega(r)=r^{-\nu}$ we have $N_{1,\varepsilon}=N_{2,\varepsilon}=n$ and $N_{3,\varepsilon}=n-\nu$ and then we may take $\Psi_{1,\varepsilon}(r)=\Psi_{2,\varepsilon}(r)=r^{p^*}$ and $\Psi_{3,\varepsilon}(r)=r^{p_\nu}$ in view of $(\Psi_{i,\varepsilon})$ $(i=1,2,3)$.

REMARK 4.9. We shall show that the exponent p_ν in Corollary 4.8 is needed. To show this, consider the function

$$f(y)=|y|^{-(n-\nu)/p-\varepsilon}\chi_{\mathbf{H}_+\setminus H(1)}(y)$$

for $0<\nu<1$ and $0<\varepsilon<n-(n-\nu)/p$. Then, in view of Lemma 3.2, we have

$$\int_{\mathbf{H}_+}f(y)^p(1+y_n)^{-\nu}\,dy\leqslant\int_1^\infty y_n^{-\nu}\left(\int_{\mathbf{R}^{n-1}}|y|^{\nu-\varepsilon p-n}\,dy'\right)dy_n$$
$$\leqslant C\int_1^\infty y_n^{-\varepsilon p-1}\,dy_n<\infty.$$

We find

$$Mf(x)\leqslant C|x|^{-(n-\nu)/p-\varepsilon}\leqslant Cx_n^{-(n-\nu)/p-\varepsilon}<(2^n x_n^{\nu-n})^{1/p}$$

for $x \in \mathbf{H}_+ \setminus H(c)$ with some $c \geqslant 1$. Note that

$$\begin{aligned} I_\alpha f(x) &\geqslant \int_{\mathbf{H}_+ \cap B(x,|x|/2)} |x-y|^{\alpha-n} f(y)\,dy \\ &\geqslant C|x|^{-(n-\nu)/p-\varepsilon} \int_{\mathbf{H}_+ \cap B(x,|x|/2)} |x-y|^{\alpha-n}\,dy \\ &\geqslant C|x|^{-(n-\nu)/p-\varepsilon+\alpha} \end{aligned}$$

for $x \in \mathbf{H}_+ \setminus H(1)$. Let $s > 0$ such that

$$1/s < ((n-\nu)/p - \alpha + \varepsilon)/(n-1) = (1/p_\nu + \varepsilon/(n-\nu))(n-\nu)/(n-1).$$

Hence it follows from Lemma 3.2 that

$$\begin{aligned} \int_{E_2 \setminus H(1)} I_\alpha f(x)^s (1+x_n)^{-\nu}\,dx &\geqslant \int_{\mathbf{H}_+ \setminus H(c)} I_\alpha f(x)^s (1+x_n)^{-\nu}\,dx \\ &\geqslant C \int_c^\infty x_n^{-\nu} \left(\int_{\mathbf{R}^{n-1}} |x|^{(-(n-\nu)/p-\varepsilon+\alpha)s}\,dx' \right) dx_n \\ &\geqslant C \int_c^\infty x_n^{-\nu+(-(n-\nu)/p-\varepsilon+\alpha)s+n-1}\,dx_n \end{aligned}$$

since $(-(n-\nu)/p-\varepsilon+\alpha)s+n < 1$.

If

$$1/s \geqslant 1/p_\nu + \varepsilon/(n-\nu),$$

then

$$\int_{E_2 \setminus H(1)} I_\alpha f(x)^s (1+x_n)^{-\nu}\,dx = \infty.$$

This implies that $s \geqslant p_\nu$ by letting $\varepsilon \to 0$.

Acknowledgements. We would like to express our thanks to the referees for their kind comments and helpful suggestions.

REFERENCES

[1] D. R. ADAMS, *A note on Riesz potentials*, Duke Math. J. **42** (1975), 765–778.
[2] D. R. ADAMS AND L. I. HEDBERG, *Function spaces and potential theory*, Springer, 1996.
[3] D. R. ADAMS AND J. XIAO, *Morrey spaces in harmonic analysis*, Ark. Mat. **50**, 2 (2012), 201–230.
[4] B. BOJARSKI AND P. HAJŁASZ, *Pointwise inequalities for Sobolev functions and some applications*, Studia Math. **106**, 1 (1993), 77–92.
[5] V. I. BURENKOV, A. GOGATISHVILI, V. S. GULIYEV AND R. CH. MUSTAFAYEV, *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Complex Var. Elliptic Equ. **55**, 8–10 (2010), 739–758.
[6] V. I. BURENKOV, A. GOGATISHVILI, V. S. GULIYEV AND R. CH. MUSTAFAYEV, *Boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal. **35**, 1 (2011), 67–87.
[7] F. CHIARENZA AND M. FRASCA, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat. **7** (1987), 273–279.

[8] Z. FU, Y. LIN AND S. LU, *λ-central BMO estimates for commutators of singular integral operators with rough kernels*, Acta Math. Sin. (Engl. Ser.) **24**, 3 (2008), 373–386.
[9] J. L. LEWIS, *On very weak solutions of certain elliptic systems*, Comm. Partial Differential Equations **18** (9) (10) (1993), 1515–1537.
[10] X. LI AND D. YANG, *Boundedness of some sublinear operators on Herz spaces*, Illinois J. Math. **40** (1996), 484–501.
[11] F.-Y. MAEDA, Y. MIZUTA, T. OHNO AND T. SHIMOMURA, *Boundedness of maximal operators and Sobolev's inequality on Musielak-Orlicz-Morrey spaces*, Bull. Sci. math. **137** (2013), 76–96.
[12] Y. MIZUTA AND T. OHNO, *Boundedness of the maximal operator and Sobolev's inequality on non-homogeneous central Herz-Morrey-Orlicz spaces*, Nonlinear Anal. **128** (2015), 325–347.
[13] C. B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.
[14] E. NAKAI, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166** (1994), 95–103.
[15] E. NAKAI, *Calderón-Zygmund operators on Orlicz-Morrey spaces and modular inequalities*, Banach and Function Spaces II, 393–410, Yokohama Publ., Yokohama, 2008.
[16] E. NAKAI, *Orlicz-Morrey spaces and the Hardy-Littlewood maximal function*, Studia Math. **188** (2008), 193–221.
[17] J. PEETRE, *On the theory of $L_{p,\lambda}$ spaces*, J. Funct. Anal. **4** (1969), 71–87.
[18] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, 1970.

(Received April 18, 2017)

Yoshihiro Mizuta
4-13-11 Hachi-Hon-Matsu-Minami
Higashi-Hiroshima 739-0144, Japan
e-mail: yomizuta@hiroshima-u.ac.jp

Takao Ohno
Faculty of Education Oita University
Dannoharu Oita-city 870-1192, Japan
e-mail: t-ohno@oita-u.ac.jp

Tetsu Shimomura
Department of Mathematics
Graduate School of Education, Hiroshima University
Higashi-Hiroshima 739-8524, Japan
e-mail: tshimo@hiroshima-u.ac.jp