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ON APPROXIMATING THE ERROR FUNCTION

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Abstract. In the article, we find the best possible parameters p and q on the interval $(7/5,(7\pi-20)/(5\pi-15))$ such that the double inequality

$$\sqrt{1-\lambda(p)e^{-px^2}-(1-\lambda(p))e^{-\mu(p)x^2}}<\operatorname{erf}(x)<\sqrt{1-\lambda(q)e^{-qx^2}-(1-\lambda(q))e^{-\mu(q)x^2}}$$

holds for all $x>0$, where $\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ is the error function, $\lambda(p)=4[(7\pi-20)-5(\pi-3)p]/[\pi(15p^2-40p+28)]$, $\mu(p)=4(5p-7)/[5(3p-4)]$.

1. Introduction

Let $x>0$. Then the classical error function $\operatorname{erf}(x)$ is given by

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

It is well known that the error function $\operatorname{erf}(x)$ is one of the most important special functions, it has many important applications in probability, statistics and partial differential equations theory. Recently, the special functions have attracted the attention of many mathematicians [8, 10–16, 26–34, 38, 41–45]. In particular, many remarkable inequalities for the error function can be found in the literature [1, 5, 6, 9, 17–23, 35, 36].

Pólya [25] proved that the inequality

$$\operatorname{erf}(x)<\sqrt{1-e^{-4x^2/\pi}}$$

holds for all $x>0$.

In [7], Chu proved that the double inequality

$$\sqrt{1-e^{-px^2}}<\operatorname{erf}(x)<\sqrt{1-e^{-qx^2}}$$

holds for all $x>0$ if and only if $p\in(0,1]$ and $q\in[4/\pi,\infty)$.

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Alzer [2] presented the double inequality

$$\left(1-e^{-\beta(p)x^p}\right)^{1/p} < \frac{1}{\Gamma\left(1+\frac{1}{p}\right)} \int_0^x e^{-t^p} dt < \left(1-e^{-\alpha(p)x^p}\right)^{1/p}$$

for $x > 0$ and $p > 0$ with $p \neq 1$, where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the classical gamma function, and $\alpha(p)$ and $\beta(p)$ are respectively given by

$$\alpha(p) = \frac{1}{\Gamma^p\left(1+\frac{1}{p}\right)} \quad (p > 1), \qquad \alpha(p) = 1 \quad (0 < p < 1)$$

and

$$\beta(p) = \frac{1}{\Gamma^p\left(1+\frac{1}{p}\right)} \quad (0 < p < 1), \qquad \beta(p) = 1 \quad (p > 1).$$

Let $n \geqslant 2$, and α_n, β_n, α_n^*, β_n^* be respectively defined by

$$\alpha_2 = 0.90686\cdots, \quad \alpha_n = 1 \quad (n \geqslant 3), \quad \beta_n = n-1,$$

$$\alpha_n^* = n+1 \quad (n = 2k), \quad \alpha_n^* = n-1 \quad (n = 2k-1), \quad \beta_n^* = 1.$$

In [3, 4], Alzer proved that the double inequalities

$$\lambda_n \operatorname{erf}\left(\sum_{i=1}^n x_i\right) \leqslant \sum_{i=1}^n \operatorname{erf}(x_i) - \prod_{i=1}^n \operatorname{erf}(x_i) \leqslant \mu_n \operatorname{erf}\left(\sum_{i=1}^n x_i\right), \tag{1.1}$$

$$\lambda \operatorname{erf}(y+\operatorname{erf}(x)) < \operatorname{erf}(x+\operatorname{erf}(y)) < \mu \operatorname{erf}(y+\operatorname{erf}(x)),$$

$$\lambda^* \operatorname{erf}(y\operatorname{erf}(x)) < \operatorname{erf}(x\operatorname{erf}(y)) \leqslant \mu^* \operatorname{erf}(y\operatorname{erf}(x))$$

hold for all $x_i \geqslant 0$ and $y \geqslant x > 0$ if and only if $\lambda_n \leqslant \alpha_n$, $\mu_n \geqslant \beta_n$, $\lambda \leqslant \operatorname{erf}(1) = 0.8427\cdots$, $\mu \geqslant 2/\sqrt{\pi} = 1.1283\cdots$, $\lambda^* \leqslant 0$ and $\mu^* \geqslant 1$, and inequality (1.1) holds for all $x_i \leqslant 0$ if and only if $\lambda_n \geqslant \alpha_n^*$ and $\mu_n \leqslant \beta_n^*$.

Neuman [24] proved that the double inequality

$$\frac{2x}{\sqrt{\pi}} e^{-\frac{x^2}{3}} \leqslant \operatorname{erf}(x) \leqslant \frac{2x}{\sqrt{\pi}} \frac{e^{-x^2}+2}{3}$$

holds for all $x > 0$.

Let $\alpha \in (0,1)$, $\lambda(p)$, $\mu = \mu(p)$, $\eta(p)$ and $B(p,\alpha;x)$ be defined by

$$\lambda(p) = \frac{4[(7\pi-20)-5(\pi-3)p]}{\pi(15p^2-40p+28)}, \tag{1.2}$$

$$\mu(p) = \frac{4(5p-7)}{5(3p-4)}, \tag{1.3}$$

$$\eta(p) = \frac{16(5p-7)}{(15p^2-40p+28)(45p^2-60p-4)}, \tag{1.4}$$

$$B(p,\alpha;x)=\sqrt{1-\alpha e^{-px^2}-(1-\alpha)e^{-\mu(p)x^2}}, \tag{1.5}$$

respectively.

Very recently, Yang, Chu and Zhang [37] provide necessary and sufficient conditions for the parameters r and s on the interval $(7/5,\infty)$ such that the double inequality

$$B(r,\eta(r);x)<\operatorname{erf}(x)<B(s,\eta(s);x)$$

holds for all $x>0$.

From (1.2) we clearly see that $\lambda(p)\in(0,1)$ for $p\in(7/5,(7\pi-20)/(5\pi-15))$. The main purpose of this paper is to present the best possible parameters p and q on the interval $(7/5,(7\pi-20)/(5\pi-15))$ such that the double inequality

$$B(p,\lambda(p);x)<\operatorname{erf}(x)<B(q,\lambda(q);x)$$

holds for all $x>0$.

2. Lemmas

In order to prove our main results, we need to introduce an auxiliary function at first.

Let $-\infty\leqslant a<b\leqslant\infty$, f and g be differentiable on (a,b), and $g'\neq 0$ on (a,b). Then the function $H_{f,g}$ [40] is defined by

$$H_{f,g}\equiv\frac{f'}{g'}g-f. \tag{2.1}$$

LEMMA 2.1. (See [37, Lemma 2.1]) *Let $-\infty\leqslant a<b\leqslant\infty$, f and g be differentiable on (a,b) with $f(a^+)=g(a^+)=0$, $g'(x)\neq 0$ and $g'(x)H_{f,g}(b^-)<(>)0$. If there exists $\lambda_0\in(a,b)$ such that f'/g' is strictly increasing (decreasing) on (a,λ_0) and strictly decreasing (increasing) on (λ_0,b), then there exists $\mu_0\in(a,b)$ such that f/g is strictly increasing (decreasing) on (a,μ_0) and strictly decreasing (increasing) on (μ_0,b).*

LEMMA 2.2. (See [39, Lemma 7]) *Let $a_i\geqslant 0$ for all $i=0,1,2,\cdots$ with $\sum_{i=0}^{m}a_i>0$ and $\sum_{i=m+1}^{\infty}a_i>0$, and*

$$P(t)=\sum_{i=0}^{m}a_it^i-\sum_{i=m+1}^{\infty}a_it^i$$

be a convergent power series on the interval $(0,\infty)$. Then there exists $t_0\in(0,\infty)$ such that $P(t_0)=0$, $P(t)>0$ for $t\in(0,t_0)$ and $P(t)<0$ for $t\in(t_0,\infty)$.

LEMMA 2.3. (See [37, Lemma 2.2 (2)]) *Let $u_n=(5p-6)(5p-8)n-(15p^2-40p+28)$. Then $u_n<0$ for all $n\geqslant 2$ if $p\in(7/5,8/5]$.*

LEMMA 2.4. See [37, Theorem 3.2]) *Let* $\eta(p)$ *and* $B(p,\alpha;x)$ *be defined by (1.4) and (1.5), respectively. Then the inequality*

$$\mathrm{erf}(x) < B(p_0,\eta(p_0);x)$$

holds for $x>0$, *where* $p_0 = (21\pi - 60 + \sqrt{3(147\pi^2 - 920\pi + 1440)})/[30(\pi-3)] = 1.713\cdots$.

LEMMA 2.5. *Let* $x\in(0,\infty)$, $\lambda(p)$, $\mu(p)$, $B(p,\alpha;x)$ *and* $H_{f,g}(x)$ *be respectively defined by (1.2), (1.3), (1.5) and (2.1), and* $f_1(x)$, $g_1(x)$, $f_2(x)$ *and* $g_2(x)$ *be respectively defined by*

$$f_1(x) = B^2(p,\lambda(p);x) = 1-\lambda(p)e^{-px^2} - (1-\lambda(p))e^{-\mu(p)x^2}, \quad g_1(x) = \mathrm{erf}^2(x), \tag{2.2}$$

$$f_2(x) = \left[p\lambda(p)e^{(1-p)x^2} + \mu(p)(1-\lambda(p))e^{(1-\mu(p))x^2}\right]x, \quad g_2(x) = \frac{2}{\sqrt{\pi}}\mathrm{erf}(x). \tag{2.3}$$

Then

$$H_{f_1,g_1}(\infty) = \lim_{x\to\infty}\left(\frac{f_1'(x)}{g_1'(x)}g_1(x) - f_1(x)\right) = \infty,$$

$$H_{f_2,g_2}(\infty) = \lim_{x\to\infty}\left(\frac{f_2'(x)}{g_2'(x)}g_2(x) - f_2(x)\right) = \infty$$

for $p\in(7/5,8/5]$.

Proof. Let $t = (p-\mu(p))x^2$ and

$$\begin{aligned} h_1(t) = {} & p\lambda(p)(p-\mu(p)) - 2p\lambda(p)(p-1)t \\ & +\mu(p)(p-\mu(p))(1-\lambda(p))e^t - 2\mu(p)(\mu(p)-1)(1-\lambda(p))te^t. \end{aligned} \tag{2.4}$$

Then (2.1)–(2.4) lead to

$$\begin{aligned} H_{f_1,g_1}(x) = {} & \frac{\sqrt{\pi}}{2}x\,\mathrm{erf}(x)\left[p\lambda(p)e^{(1-p)x^2} + \mu(p)(1-\lambda(p))e^{(1-\mu(p))x^2}\right] \\ & -\left[1-\lambda(p)e^{-px^2} - (1-\lambda(p))e^{-\mu(p)x^2}\right], \end{aligned} \tag{2.5}$$

$$H_{f_2,g_2}(x) = te^{\frac{1-\mu(p)}{p-\mu(p)}t}\left[\frac{\sqrt{\pi}e^{\frac{2-p}{p-\mu(p)}t}}{2(p-\mu(p))}\mathrm{erf}(x)\frac{h_1(t)}{te^{\frac{1-\mu(p)}{p-\mu(p)}t}} - \frac{p\lambda(p)e^{-t} + \mu(p)(1-\lambda(p))}{\sqrt{(p-\mu(p))t}}\right]. \tag{2.6}$$

If $p\in(7/5,8/5]$, then it is not difficult to verify that

$$p > \mu(p), \quad 0 < \lambda(p) < 1, \quad 0 < \mu(p) \leqslant 1, \tag{2.7}$$

$$\lim_{t\to\infty}\frac{h_1(t)}{te^{\frac{1-\mu(p)}{p-\mu(p)}t}} = \lim_{t\to\infty}\left[\frac{p\lambda(p)(p-\mu(p))}{t}e^{\frac{\mu(p)-1}{p-\mu(p)}t} - 2p\lambda(p)(p-1)e^{\frac{\mu(p)-1}{p-\mu(p)}t}\right] \quad (2.8)$$
$$+\lim_{t\to\infty}\left[\frac{\mu(p)(1-\lambda(p))(p-\mu(p))}{t}e^{\frac{p-1}{p-\mu(p)}t} + 2\mu(p)(1-\lambda(p))(1-\mu(p))e^{\frac{p-1}{p-\mu(p)}t}\right]$$
$$=\infty.$$

Therefore, Lemma 2.5 follows easily from (2.4)–(2.8). □

LEMMA 2.6. *Let $x>0$, and $\lambda(p)$ and $B(p,\alpha;x)$ be respectively defined by (1.2) and (1.5). Then the function $p\mapsto B(p,\lambda(p);x)$ is strictly increasing on $(7/5,(7\pi-20)/(5\pi-15))$*

Proof. Let $\mu(p)$ be defined by (1.3), $t=(p-\mu(p))x^2$ and $f(t)$ be defined by

$$f(t) = -[p-\mu(p)]\lambda'(p)+\lambda(p)t+[p-\mu(p)]\lambda'(p)e^t+\mu'(p)[1-\lambda(p)]te^t.$$

Then elaborated computations lead to

$$2B(p,\lambda(p);x)\frac{\partial B(p,\lambda(p);x)}{\partial p} = \frac{e^{-px^2}}{p-\mu(p)}f(t), \quad (2.9)$$

$$f(0)=0, \quad (2.10)$$

$$f'(t)=\lambda(p)+[p-\mu(p)]\lambda'(p)e^t+\mu'(p)[1-\lambda(p)]e^t+\mu'(p)[1-\lambda(p)]te^t,$$
$$f'(0)=0, \quad (2.11)$$

$$f''(t)=\frac{4(\pi-3)}{(3p-4)\pi}e^t+\frac{4(\pi p-4)}{(3p-4)(15p^2-40p+28)}te^t>0 \quad (2.12)$$

for $t>0$ and $p\in(7/5,(7\pi-20)/(5\pi-15))$.

Therefore, Lemma 2.6 follows easily from (2.9)–(2.12) and $p>\mu(p)$ for $p\in(7/5,(7\pi-20)/(5\pi-15))$. □

3. Main results

THEOREM 3.1. *Let $\alpha\in(0,1)$, $p\in(7/5,8/5]$, and $\lambda(p)$ and $B(p,\alpha;x)$ be defined by (1.2) and (1.5), respectively. Then the inequality*

$$\operatorname{erf}(x)>B(p,\alpha;x) \quad (3.1)$$

holds for all $x>0$ if and only if $\alpha\leqslant\lambda(p)$.

Proof. If inequality (3.1) holds for all $x > 0$, then we clearly see that

$$\lim_{x\to 0^+} \frac{\operatorname{erf}(x) - B(p,\alpha;x)}{x} = \frac{2}{\sqrt{\pi}} - \sqrt{\alpha p + (1-\alpha)\mu(p)} \geqslant 0,$$

$$\alpha \leqslant \frac{4 - \pi\mu(p)}{\pi(p - \mu(p))} = \lambda(p).$$

Next, we prove that inequality (3.1) holds for all $x > 0$ if $\alpha \leqslant \lambda(p)$. We only need to prove that the inequality

$$\operatorname{erf}(x) > B(p,\lambda(p);x) \tag{3.2}$$

holds for all $x > 0$ due to the function $\alpha \mapsto B(p,\alpha;x)$ is strictly increasing on $(0,1)$.

Let $t = (p-\mu(p))x^2$, $\mu(p)$, $\eta(p)$, u_n, $f_1(x)$, $g_1(x)$, $f_2(x)$, $g_2(x)$ and $h_1(t)$ be respectively defined by (1.3), (1.4), Lemma 2.3, (2.2), (2.3) and (2.4), and $h_2(t)$ be defined by

$$\begin{aligned} h_2(t) = {} & 2\mu(p)(1-\lambda(p))(\mu(p)-1)(\mu(p)-2)te^t - \mu(p)(1-\lambda(p))(3\mu(p)-4) \\ & \times(p-\mu(p))e^t + 2p\lambda(p)(p-1)(p-2)t - p\lambda(p)(3p-4)(p-\mu(p)), \end{aligned}$$

Then elaborated computations lead to

$$\frac{f_1'(x)}{g_1'(x)} = \frac{f_2(x)}{g_2(x)}, \tag{3.3}$$

$$\frac{f_2'(x)}{g_2'(x)} = \frac{\pi}{4(p-\mu(p))} e^{\frac{2-p}{p-\mu(p)}t} h_1(t),$$

$$\left(\frac{f_2'(x)}{g_2'(x)}\right)' = \frac{\pi}{4(p-\mu(p))}\frac{d}{dt}\left[e^{\frac{2-p}{p-\mu(p)}t} h_1(t)\right]\frac{dt}{dx} = \frac{\pi x}{2(p-\mu(p))} e^{\frac{2-p}{p-\mu(p)}t} h_2(t), \tag{3.4}$$

$$\begin{aligned} h_2(t) = {} & 2\mu(p)(1-\lambda(p))(\mu(p)-1)(\mu(p)-2)\sum_{n=1}^{\infty}\frac{t^n}{(n-1)!} \\ & -\mu(p)(1-\lambda(p))(3\mu(p)-4)(p-\mu(p))\sum_{n=0}^{\infty}\frac{t^n}{n!} \\ & +2p\lambda(p)(p-1)(p-2)t - p\lambda(p)(3p-4)(p-\mu(p)) \\ = {} & -\frac{(15p^2-40p+28)^2(45p^2-60p-4)}{125(3p-4)^3}(\lambda(p)-\eta(p)) \\ & -\frac{2(p-1)(2-p)}{25(3p-4)^3}(\lambda(p)-\eta(p))t - \frac{4\mu(p)(1-\lambda(p))}{25(3p-4)^2}\sum_{n=2}^{\infty}\frac{u_n}{n!}t^n, \end{aligned} \tag{3.5}$$

$$\lambda(p) - \eta(p) = \frac{20(3p-4)(p_0-p)(p-p_1)}{\pi(15p^2-40p+28)(45p^2-60p-4)}, \tag{3.6}$$

where

$$p_0 = \frac{21\pi - 60 + \sqrt{3(147\pi^2 - 920\pi + 1440)}}{30(\pi-3)} = 1.713\cdots, \tag{3.7}$$

$$p_1 = \frac{21\pi - 60 - \sqrt{3(147\pi^2 - 920\pi + 1440)}}{30(\pi - 3)} = 1.099\cdots. \tag{3.8}$$

From (2.7) and (3.4)–(3.8) together with Lemmas 2.2 and 2.3 we clearly see that there exists $x_0 \in (0,\infty)$ such that $f_2'(x)/g_2'(x)$ is strictly decreasing on $(0,x_0)$ and strictly increasing on (x_0,∞). Then Lemmas 2.1 and 2.5 together with (3.3), $f_2(0) = g_2(0) = 0$ and $g_2'(x) > 0$ lead to the conclusion that there exists $x_1 \in (0,\infty)$ such that $f_1'(x)/g_1'(x)$ is strictly decreasing on $(0,x_1)$ and strictly increasing on (x_1,∞).

It follows from Lemma 2.1, $f_1(0) = g_1(0) = 0$ and $g_1'(x) > 0$ together with the piecewise monotonicity of $f_1'(x)/g_1'(x)$ that there exists $x_2 \in (0,\infty)$ such that $f_1(x)/g_1(x)$ is strictly decreasing on $(0,x_2)$ and strictly increasing on (x_2,∞).

Note that

$$\lim_{x\to 0^+} \frac{f_1(x)}{g_1(x)} = \lim_{x\to 0^+} \frac{f_2'(x)}{g_2'(x)} = \frac{\pi[p\lambda(p) + \mu(p)(1-\lambda(p))]}{4} = 1, \quad \lim_{x\to\infty} \frac{f_1(x)}{g_1(x)} = 1. \tag{3.9}$$

Therefore, inequality (3.2) follows easily from (2.2) and (3.9) together with the piecewise monotonicity of $f_1(x)/g_1(x)$. $\square$

THEOREM 3.2. *Let* $p_0 = (21\pi - 60 + \sqrt{3(147\pi^2 - 920\pi + 1440)})/[30(\pi-3)] = 1.713\cdots$, $\alpha \in (0,1)$, $p \in [p_0, (7\pi - 20)/(5\pi - 15))$, *and* $\lambda(p)$ *and* $B(p,\alpha;x)$ *be defined by (1.2) and (1.5), respectively. Then the inequality*

$$\operatorname{erf}(x) < B(p,\alpha;x) \tag{3.10}$$

holds for all $x > 0$ *if and only if* $\alpha \geqslant \lambda(p)$.

Proof. If inequality (3.10) holds for all $x > 0$, then we clearly see that

$$\lim_{x\to 0^+} \frac{\operatorname{erf}(x) - B(p,\alpha;x)}{x} = \frac{2}{\sqrt{\pi}} - \sqrt{\alpha p + (1-\alpha)\mu(p)} \leqslant 0,$$

$$\alpha \geqslant \frac{4 - \pi\mu(p)}{\pi(p - \mu(p))} = \lambda(p).$$

Next, we prove that inequality (3.10) holds for all $x > 0$ if $\alpha \geqslant \lambda(p)$. We only need to prove that the inequality

$$\operatorname{erf}(x) < B(p,\lambda(p);x) \tag{3.11}$$

holds for all $x > 0$ due to the function $\alpha \mapsto B(p,\alpha;x)$ is strictly increasing on $(0,1)$.

Inequality (3.11) follows easily from Lemma 2.4, Lemma 2.6, (3.6) and $p \geqslant p_0$. Indeed, we have

$$\operatorname{erf}(x) < B(p_0,\eta(p_0);x) = B(p_0,\lambda(p_0);x) \leqslant B(p,\lambda(p);x)$$

for all $x > 0$. $\square$

THEOREM 3.3. *Let* $p_0 = (21\pi - 60 + \sqrt{3(147\pi^2 - 920\pi + 1440)})/[30(\pi - 3)] = 1.713\cdots$, $\alpha \in (0,1)$, $p \in (7/5, (7\pi - 20)/(5\pi - 15))$, *and* $\lambda(p)$, $\mu(p)$ *and* $B(p,\alpha;x)$ *be respectively defined by (1.2), (1.3) and (1.5). Then the inequality*

$$\operatorname{erf}(x) > B(p,\lambda(p);x) \tag{3.12}$$

holds for all $x > 0$ *if and only if* $p \in (7/5, 8/5]$, *and the inequality*

$$\operatorname{erf}(x) < B(p,\lambda(p);x) \tag{3.13}$$

holds for all $x > 0$ *if and only if* $p \in [p_0, (7\pi - 20)/(5\pi - 15))$.

Proof. From Theorems 3.1 and 3.2 we clearly see that inequality (3.12) holds for all $x > 0$ if $p \in (7/5, 8/5]$, and inequality (3.13) holds for all $x > 0$ if $p \in [p_0, (7\pi - 20)/(5\pi - 15))$.

Next, we prove that $p \leqslant 8/5$ if inequality (3.12) holds for all $x > 0$. Indeed,

$$\begin{aligned}
&\lim_{x\to\infty} \frac{\operatorname{erf}^2(x) - B^2(p,\lambda(p);x)}{e^{-\mu(p)x^2}} \\
&= -\lim_{x\to\infty} \frac{\frac{4}{\sqrt{\pi}} e^{-x^2} \operatorname{erf}(x) - 2x\left(p\lambda(p)e^{-px^2} + \mu(p)(1-\lambda(p))e^{-\mu(p)x^2}\right)}{2\mu(p)xe^{-\mu(p)x^2}} \\
&= \lim_{x\to\infty}\left(-\frac{2\operatorname{erf}(x)e^{(\mu(p)-1)x^2}}{\sqrt{\pi}\mu(p)} + \frac{p\lambda(p)}{\mu(p)} e^{-(p-\mu(p))x^2} + 1 - \lambda(p)\right) = -\infty
\end{aligned}$$

due to $p > \mu(p)$ and $\mu(p) > 1$ if $p > 8/5$.

Finally, we prove that $p \geqslant p_0$ if inequality (3.13) holds for all $x > 0$. Let $x \to 0^+$, then making use of Taylor formula we get

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}x - \frac{2}{3\sqrt{\pi}}x^3 + o\left(x^3\right), \tag{3.14}$$

$$B(p,\lambda(p);x) = \sqrt{p\lambda(p) + \mu(p)(1-\lambda(p))}\,x - \frac{p^2\lambda(p) + \mu^2(p)(1-\lambda(p))}{4\sqrt{p\lambda(p) + \mu(p)(1-\lambda(p))}}x^3 + o\left(x^3\right). \tag{3.15}$$

Note that

$$\sqrt{p\lambda(p) + \mu(p)(1-\lambda(p))} = \frac{2}{\sqrt{\pi}}. \tag{3.16}$$

If inequality (3.13) holds for all $x > 0$, then (1.2), (1.3) and (3.14)–(3.16) lead to

$$\lim_{x\to 0^+} \frac{\operatorname{erf}(x) - B(p,\lambda(p);x)}{x^3} = -\frac{15(\pi-3)p^2 - 3(7\pi - 20)p + 4}{30\sqrt{\pi}(3p-4)} \leqslant 0. \tag{3.17}$$

Therefore, $p \geqslant p_0$ follows easily from (3.17) and $p > 7/5$. □

Let $\lambda(p)$, $\mu(p)$ and $B(p,\alpha;x)$ be respectively defined by (1.2), (1.3) and (1.5). Then simple computations lead to

$$\lambda\left(\frac{8}{5}\right)=\frac{5(4-\pi)}{3\pi},\quad \mu\left(\frac{8}{5}\right)=1, \tag{3.18}$$

$$\lambda\left(\frac{3}{2}\right)=\frac{8(5-\pi)}{7\pi},\quad \mu\left(\frac{3}{2}\right)=\frac{4}{5}, \tag{3.19}$$

$$\lambda\left(\frac{7}{5}\right)=\frac{20}{7\pi},\quad \mu\left(\frac{7}{5}\right)=0, \tag{3.20}$$

$$\lambda(2)=\frac{10-3\pi}{2\pi},\quad \mu(2)=\frac{6}{5}, \tag{3.21}$$

$$\lambda\left(\frac{7\pi-20}{5\pi-15}\right)=0,\quad \mu\left(\frac{7\pi-20}{5\pi-15}\right)=\frac{4}{\pi}. \tag{3.22}$$

From Lemma 2.6, Theorem 3.1, Theorem 3.2 and (3.18)–(3.22) together with $7/5<3/2<8/5$ and $(21\pi-60+\sqrt{3(147\pi^2-920\pi+1440)})/[30(\pi-3)]=1.713\cdots$ $<2<(7\pi-20)/(5\pi-15)=2.181\cdots$, we get Corollary 3.4 immediately.

COROLLARY 3.4. *Let* $p_0=(21\pi-60+\sqrt{3(147\pi^2-920\pi+1440)})/[30(\pi-3)]$, *and* $\lambda(p)$ *and* $\mu(p)$ *be respctviely defined by (1.2) and (1.3). Then the inequalities*

$$\begin{aligned}\sqrt{\frac{20}{7\pi}\left(1-e^{-7x^2/5}\right)} &< \sqrt{1-\frac{8(5-\pi)}{7\pi}e^{-3x^2/2}-\frac{5(3\pi-8)}{7\pi}e^{-4x^2/5}}\\ &< \sqrt{1-\frac{5(4-\pi)}{3\pi}e^{-8x^2/5}-\frac{4(2\pi-5)}{3\pi}e^{-x^2}}\\ &< \operatorname{erf}(x) < \sqrt{1-\lambda(p_0)e^{-p_0x^2}-(1-\lambda(p_0))e^{-\mu(p_0)x^2}}\\ &< \sqrt{1-\frac{10-3\pi}{2\pi}e^{-2x^2}-\frac{5(\pi-2)}{2\pi}e^{-6x^2/5}} < \sqrt{1-e^{-4x^2/\pi}}\end{aligned}$$

hold for all $x>0$.

REMARK 3.5. Let $\lambda(p)$, $\eta(p)$ and $B(p,\alpha;x)$ be respectively defined by (1.2), (1.4) and (1.5), and $p_0=(21\pi-60+\sqrt{3(147\pi^2-920\pi+1440)})/[30(\pi-3)]=1.713\cdots$. Then from (3.6) we clearly see that $\lambda(p)>\eta(p)$ for $p\in(7/5,8/5]$ and $\lambda(p)<\eta(p)$ for $p\in(p_0,(7\pi-20)/(5\pi-15))$. Therefore, Theorems 3.1 and 3.2 together with the monotonicity of the function $\alpha\to B(p,\alpha;x)$ lead to the conclusion that

$$\operatorname{erf}(x)>B(p,\lambda(p);x)>B(p,\eta(p);x)$$

for all $x>0$ and $p\in(7/5,8/5]$, and

$$\operatorname{erf}(x)<B(p,\lambda(p);x)<B(p,\eta(p);x)$$

for all $x>0$ and $p\in(p_0,(7\pi-20)/(5\pi-15))$.

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