Mathematical
Inequalities
& Applications
Volume 21, Number 2 (2018), 481–487 doi:10.7153/mia-2018-21-33

NOTES ON SOME BOUNDS FOR THE ZEROS OF POLYNOMIALS

AMER ABU-OMAR

(*Communicated by I. Perić*)

Abstract. We apply some spectral radius and norm inequalities to the Frobenius companion matrix to present a simple proof of Kuniyeda's bound for the zeros of polynomials. We then use Kuniyeda's bound to derive a new bound for the zeros of polynomials. A partial comparison between the two bounds is given. Our new bound generalizes and refines classical bounds due to Guggenheimer and Walsh.

1. Introduction

Let M_n denote the algebra of all $n \times n$ complex matrices. A matrix norm $|||\cdot|||$ on M_n is a norm satisfying the submultiplicativity property

$$|||AB||| \leqslant |||A|||\,|||B|||$$

for all A and B. For $A \in M_n$, let $r(A)$ denote the spectral radius of A. It is well-known that

$$r(A) \leqslant |||A||| \tag{1}$$

holds for every matrix norm $|||\cdot|||$ on M_n.

Let

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \tag{2}$$

be a monic polynomial of degree $n > 1$, where $a_1, a_2, \ldots, a_n$ are complex numbers with $a_n \neq 0$. Then the Frobenius companion matrix of f is given by

$$C_f = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

It is well-known that the zeros of f coincide with the eigenvalues of C_f (see, e.g., [3, p. 316]). Hence, if z is any zero of f and if $|||\cdot|||$ is any matrix norm on M_n, then

$$|z| \leqslant r(C_f) \leqslant |||C_f|||. \tag{3}$$

Mathematics subject classification (2010): Primary 26C10, Secondary 15A60.

Keywords and phrases: Frobenius companion matrix, zeros of polynomials, matrix norms, spectral radius, inequality.

Throughout this paper, f will denote an arbitrary monic polynomial written in the form (2).

For $p \in [1,\infty]$, let $||\cdot||_p$ denote the p-norm defined for every $x = (x_1, x_2, \ldots, x_k)^t \in \mathbb{C}^k$ by

$$||x||_p = \left(\sum_{j=1}^{k} |x_j|^p \right)^{1/p}$$

and

$$||x||_\infty = \max_{1 \leqslant j \leqslant k} |x_j|.$$

Corresponding to each p-norm let $|||A|||_p$ denote the norm defined for every $A \in M_n$ by

$$|||A|||_p = \max\{||Ax||_p : x \in \mathbb{C}^n, ||x||_p = 1\}.$$

This norm is a matrix norm. It is sometimes called the (induced) p-norm. It can be shown that if $A = [a_{ij}] \in M_n$, then

$$|||A|||_1 = \max_{1 \leqslant j \leqslant n} \sum_{i=1}^{n} |a_{ij}|,$$

$$|||A|||_\infty = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |a_{ij}|$$

and

$$|||A|||_2 = \sqrt{r(A^*A)}, \text{ the largest singular value of } A.$$

Here A^* is the adjoint (conjugate transpose) of A. For proofs and more facts about the spectral radius and matrix norms we refer the reader to [3].

Matrix analysis methods have been successfully utilized to derive several bounds for the zeros of polynomials and to obtain new proofs of known bounds (see, e.g., [1], [2], [5], [6], [7], [8] and references therein). For instance, it is easily verified by computing $|||C_f|||_1$ that if z is any zero of f, then

$$|z| \leqslant |||C_f|||_1 = \max\{1 + |a_1|, \ldots, 1 + |a_{n-1}|, |a_n|\} \tag{4}$$

and hence

$$|z| \leqslant 1 + \max_{1 \leqslant j \leqslant n} |a_j|. \tag{5}$$

Similarly, by computing $|||C_f|||_\infty$, we have

$$|z| \leqslant |||C_f|||_\infty = \max\{1, \sum_{j=1}^{n} |a_j|\}. \tag{6}$$

Also, it can be shown (see, e.g., [3, p. 317]) that

$$|z| \leqslant |||C_f|||_2 \leqslant \left(1 + \sum_{j=1}^{n} |a_j|^2 \right)^{1/2}. \tag{7}$$

The bounds (5) and (6) are due to Cauchy and Montel, respectively, while the bound (7) is due to Carmichael and Mason.

In [4], the singular values of the Frobenius companion matrix C_f were computed and the following bound for the zeros of f (which is better than the bound (7)) was given

$$|z| \leqslant |||C_f|||_2 = \frac{1}{\sqrt{2}}\left(1+\alpha+\left((1+\alpha)^2-4|a_n|^2\right)^{1/2}\right)^{1/2}, \tag{8}$$

where $\alpha = \sum_{j=1}^{n} |a_j|^2$.

Kuniyeda [9] proved the following theorem which includes the bounds (5), (6) and (7) as special cases.

THEOREM 1. *If z is any zero of f and if $p,q \in [1,\infty]$ are such that $1/p+1/q=1$, then*

$$|z| \leqslant \left(1+\left(\sum_{j=1}^{n}|a_j|^q\right)^{p/q}\right)^{1/p}. \tag{9}$$

Note that the bounds (5), (6) and (7) follow from Kuniyeda's bound (9) by letting $p=1, p=\infty$ and $p=2$, respectively. Kuniyeda's bound and other bounds can be found in the comprehensive study of the zeros of polynomials [10].

In this paper, by estimating $|||C_f|||_p$, we give a simple proof of Kuniyeda's bound. We then use Kuniyeda's bound to derive a new bound for the zeros of polynomials, from which several bounds follow as special cases. Finally, a partial comparison between Kuniyeda's bound and the new bound is given.

2. Main results

Our first goal is to give a new proof of Kuniyeda's bound. The proof contains a simple application of Hölder's inequality.

Proof of Theorem 1. Let $x=(x_1,x_2,\ldots,x_n)^t \in \mathbb{C}^n$ be a unit vector in the p-norm such that $|||C_f|||_p = \|C_f x\|_p$. Then, by Hölder's inequality we have

$$\sum_{j=1}^{n}|a_j x_j| \leqslant \left(\sum_{j=1}^{n}|a_j|^q\right)^{1/q},$$

and hence

$$\left|\left|\left|C_f\right|\right|\right|_p^p = ||C_f x||_p^p = ||(\sum_{j=1}^{n} - a_j x_j, x_1, \ldots, x_{n-1})^t||_p^p$$
$$= |\sum_{j=1}^{n} a_j x_j|^p + |x_1|^p + \cdots + |x_{n-1}|^p$$
$$\leqslant 1 + \left(\sum_{j=1}^{n} \left|a_j x_j\right|\right)^p \leqslant 1 + \left(\sum_{j=1}^{n} \left|a_j\right|^q\right)^{p/q}.$$

The result now follows from the previous inequality by recalling that if z is any zero of f, then $|z| \leqslant \left|\left|\left|C_f\right|\right|\right|_p$. □

Let $B(a_1, a_2, \ldots, a_n)$ be a bound for the zeros of polynomials and let λ be a positive real number. Note that if z is a zero of f, then $\frac{z}{\lambda}$ is a zero of the polynomial g defined by

$$g(w) = w^n + \frac{a_1}{\lambda} w^{n-1} + \cdots + \frac{a_{n-1}}{\lambda^{n-1}} w + \frac{a_n}{\lambda^n}.$$

Thus, by applying the bound B to the zeros of g, we have

$$\frac{|z|}{\lambda} \leqslant B\left(\frac{a_1}{\lambda}, \frac{a_2}{\lambda^2}, \ldots, \frac{a_n}{\lambda^n}\right)$$

and hence

$$|z| \leqslant \lambda B\left(\frac{a_1}{\lambda}, \frac{a_2}{\lambda^2}, \ldots, \frac{a_n}{\lambda^n}\right).$$

Applying the above argument to Kuniyeda's bound, letting $\lambda = \max_{2 \leqslant j \leqslant n} \left|a_j\right|^{1/j}$ and noting that

$$\frac{|a_k|}{\max_{2 \leqslant j \leqslant n} \left|a_j\right|^{\frac{k-1}{j}}} \leqslant \frac{|a_k|}{|a_k|^{\frac{k-1}{k}}} = |a_k|^{1/k}$$

for $k = 2, 3, \ldots, n$, we have the following theorem.

THEOREM 2. *If z is any zero of f and if $p, q \in [1, \infty]$ are such that $1/p + 1/q = 1$, then*

$$|z| \leqslant \left(\max_{2 \leqslant j \leqslant n} \left|a_j\right|^{p/j} + \left(\sum_{j=1}^{n} \left|a_j\right|^{q/j}\right)^{p/q}\right)^{1/p}. \tag{10}$$

The following bounds follow from the bound (10) by letting $p = 1$, $p = \infty$ and $p = 2$, respectively.

COROLLARY 1. *If z is any zero of f, then*

$$|z| \leqslant \max_{2\leqslant j\leqslant n} \left|a_j\right|^{1/j} + \max_{1\leqslant j\leqslant n} \left|a_j\right|^{1/j}, \tag{11}$$

$$|z| \leqslant \sum_{j=1}^{n} \left|a_j\right|^{1/j}, \tag{12}$$

and

$$|z| \leqslant \left(\max_{2\leqslant j\leqslant n} \left|a_j\right|^{2/j} + \sum_{j=1}^{n} \left|a_j\right|^{2/j}\right)^{1/2}. \tag{13}$$

The bound (11) is better than the bound given by Guggenheimer [11], who proved that if z is any zero of f, then

$$|z| \leqslant 2 \max_{1\leqslant j\leqslant n} \left|a_j\right|^{1/j},$$

while the bound (12) was given by Walsh [12] using a completely different approach. The bound (13) is new.

In the following theorem, we show that if $p \in [2,\infty]$ then the bound (10) is better than Kuniyeda's for estimating the moduli of the zeros of any polynomial f with $\max\limits_{2\leqslant j\leqslant n} \left|a_j\right| \geqslant 1$.

THEOREM 3. *If $p \in [2,\infty]$ and if $\max\limits_{2\leqslant j\leqslant n} \left|a_j\right| \geqslant 1$, then*

$$\left(\max_{2\leqslant j\leqslant n} \left|a_j\right|^{p/j} + \left(\sum_{j=1}^{n} \left|a_j\right|^{q/j}\right)^{p/q}\right)^{1/p} \leqslant \left(1 + \left(\sum_{j=1}^{n} \left|a_j\right|^{q}\right)^{p/q}\right)^{1/p}. \tag{14}$$

Here, q is defined as in Theorem 2.

Proof. Assume that $\max\limits_{2\leqslant j\leqslant n} \left|a_j\right|^{1/j} = |a_k|^{1/k}$ for some $k \in \{2,1,\ldots,n\}$ and let $a = |a_k|^{q/2}$ and $s = \sum\limits_{\substack{j=1\\ j\neq k}}^{n} \left|a_j\right|^{q}$. Since $\max\limits_{2\leqslant j\leqslant n} \left|a_j\right| \geqslant 1$ we have $a \geqslant 1$ and hence

$$\begin{aligned}
\max_{2\leqslant j\leqslant n} \left|a_j\right|^{p/j} + \left(\sum_{j=1}^{n} \left|a_j\right|^{q/j}\right)^{p/q} &= |a_k|^{p/k} + \left(|a_k|^{q/k} + \sum_{\substack{j=1\\ j\neq k}}^{n} \left|a_j\right|^{q/j}\right)^{p/q} \\
&= \left(a^{2/k}\right)^{p/q} + \left(a^{2/k} + \sum_{\substack{j=1\\ j\neq k}}^{n} \left|a_j\right|^{q/j}\right)^{p/q} \\
&\leqslant a^{p/q} + (a+s)^{p/q}.
\end{aligned}$$

So, in order to prove the inequality (14), it is sufficient to show that

$$a^{p/q} + (a+s)^{p/q} \leqslant 1 + \left(a^2 + s\right)^{p/q},$$

which holds since the function

$$\varphi(s) = 1 + \left(a^2 + s\right)^{p/q} - a^{p/q} - (a+s)^{p/q}$$

is increasing on $[0,\infty)$ (recall that $p/q = p-1 \geqslant 1$) and

$$\varphi(0) = 1 + a^{2p/q} - 2a^{p/q} = (1 - a^{p/q})^2 \geqslant 0. \quad \square$$

Finally, we remark that a bound $B(a_1, a_2, \ldots, a_n)$ for the zeros of polynomials can be used to obtain a lower bound for the moduli of the zeros of f and hence determine an annulus in the complex plane containing all of its zeros. To see this, note that if z is a zero of f, then $\frac{1}{z}$ is a zero of the polynomial h defined by

$$h(w) = w^n + \frac{a_{n-1}}{a_n} w^{n-1} + \cdots + \frac{a_1}{a_n} w + \frac{1}{a_n}.$$

Now, by applying the bound B to the zeros of h, we have

$$\frac{1}{|z|} \leqslant B\left(\frac{a_{n-1}}{a_n}, \frac{a_{n-2}}{a_n}, \ldots, \frac{1}{a_n}\right)$$

and hence

$$1/B\left(\frac{a_{n-1}}{a_n}, \frac{a_{n-2}}{a_n}, \ldots, \frac{1}{a_n}\right) \leqslant |z| \leqslant B(a_1, a_2, \ldots, a_n).$$

REFERENCES

[1] A. ABU-OMAR AND F. KITTANEH, *Estimates for the numerical radius and the spectral radius of the Frobenius companion matrix and bounds for the zeros of polynomials*, Ann. Funct. Anal. **5**, 1 (2014), 56–62.

[2] M. FUJII AND F. KUBO, *Operator norms as bounds for roots of algebraic equations*, Proc. Japan Acad. Sci. **49**, 10 (1973), 805–808.

[3] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge Univ. Press, Cambridge, 1985.

[4] F. KITTANEH, *Singular values of companion matrices and bounds on zeros of polynomials*, SIAM J. Matrix Anal. Appl. **16**, 1 (1995), 333–340.

[5] F. KITTANEH, *Bounds for the zeros of polynomials from matrix inequalities*, Arch. Math. **81**, 5 (2003), 601–608.

[6] F. KITTANEH AND KHALID SHEBRAWI, *Bounds for the zeros of polynomials from matrix inequalities-II*, Linear and Multilinear Algebra **55**, 2 (2007), 147–158.

[7] H. LINDEN, *Bounds for the zeros of polynomials from eigenvalues and singular values of some companion matrices*, Linear Algebra Appl. **271**, 1 (1998), 41–82.

[8] H. LINDEN, *Numerical radii of some companion matrices and bounds for the zeros of polynomials*, Math. Appl. **478** (1999), 205–229.

[9] M. KUNIYEDA, *Note on the roots of algebraic equations*, Tôhoku Math. J. **9** (1916), 167–173.
[10] M. MARDEN, *Geometry of Polynomials*, American Mathematical Society, Mathematical Surveys and Monographs **3**, 2nd ed., Providence, 1969.
[11] H. GUGGENHEIMER, *Bounds for roots of algebraic equations*, Arch. Math. **31**, 1 (1978), 568–569.
[12] J. L. WALSH, *An inequality for the roots of an algebraic equation*, Ann. Math. **25**, 3 (1924), 285–286.

(Received June 23, 2017)

Amer Abu-Omar
College of Sciences and Humanities
Fahad Bin Sultan University
Tabuk, Saudi Arabia
e-mail: aabuomar@fbsu.edu.sa