Mathematical
Inequalities
& Applications
Volume 21, Number 2 (2018), 489–496 doi:10.7153/mia-2018-21-34

REMARKS ON SOME SINGULAR VALUE INEQUALITIES

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(*Communicated by J.-C. Bourin*)

Abstract. In this short note, we present new proofs for some singular value inequalities for matrices obtained by I. Garg and J. Aulja.

1. Introduction

Throughout this paper, let M_n be the space of $n\times n$ complex matrices. I_n denotes the identity matrix in M_n. For two Hermitian matrices A, $B\in M_n$, $A\geqslant B$ ($B\leqslant A$) means that $A-B$ is a positive semidefinite matrix. We shall always denote the singular values of $A\in M_n$ by $s_1(A)\geqslant s_2(A)\geqslant\cdots\geqslant s_n(A)\geqslant 0$, i.e., the eigenvalues of the positive semidefinite matrix $|A|=(A^*A)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. If $A\in M_n$ has real eigenvalues, we label them as $\lambda_1(A)\geqslant\lambda_2(A)\geqslant\cdots\geqslant\lambda_n(A)$. Let $s(A):=(s_1(A),s_2(A),\cdots,s_n(A))$ and $\lambda(A):=(\lambda_1(A),\lambda_2(A),\cdots,\lambda_n(A))$, where $s_i(A)$, $\lambda_i(A)\,(i=1,2,\cdots,n)$ are the singular values and the eigenvalues of $A\in M_n$, respectively. Let f be a real valued continuous function on an interval I and $A=Udiag(\lambda_1(A),\lambda_2(A),\cdots,\lambda_n(A))U^*$ be the spectral decomposition of an Hermitian matrix $A\in M_n$, if $\lambda(A)\subset I$, then $f(A)$ is defined as the matrix $f(A):=Udiag(f(\lambda_1(A)),f(\lambda_2(A)),\cdots,f(\lambda_n(A)))U^*$. Some of the important properties of singular values are that

$$s_j(A^*A)=s_j(AA^*) \tag{1}$$

and the singular values of matrices are unitarily invariance, that is

$$s_j(UAV)=s_j(A), \tag{2}$$

for $j=1,2,\cdots,n$, where A, U, $V\in M_n$ with U and V are unitary matrices.

A real valued continuous function f on an interval I is called matrix monotone of order n if for two Hermitian matrices A, B with spectrum in I, $A\geqslant B$ implies $f(A)\geqslant f(B)$. Further, f is called operator monotone if f is matrix monotone for all n. A function $f:I\to\mathbf{R}$ is called matrix convex on I of order n if $f(\alpha A+(1-\alpha)B)\leqslant \alpha f(A)+(1-\alpha)f(B)$ for all Hermitian matrices A, $B\in M_n$ with spectrum in I and $0<\alpha<1$. If $-f$ is matrix convex, then f is called matrix concave. It is well known

Mathematics subject classification (2010): 47A30, 15A60.
Keywords and phrases: Singular values, operator monotone functions, matrix inequalities.

that a function $f:[0,+\infty)\to[0,+\infty)$ is operator monotone if and only if it is operator concave.

Very recently, I. Garg and J. Aujla [5, Theorem 2.10] presented the following singular value inequality for matrices:

$$\prod_{j=1}^{k} s_j(I_n+f(|A+B|)) \leqslant \prod_{j=1}^{k} s_j(I_n+f(|A|))s_j(I_n+f(|B|)), \tag{3}$$

where $1\leqslant k\leqslant n$ and A, $B\in M_n$ and $f:[0,+\infty)\to[0,+\infty)$ is an operator concave function. Inequality (3) is a refinement of the Rotfel'd [6] result

$$\det(I_n+\mu|A+B|^p)\leqslant \det(I_n+\mu|A|^p)\det(I_n+\mu|B|^p),$$

where A, $B\in M_n$, $\mu>0$ and $0<p\leqslant 1$.

They also proved that for A, $B\in M_n$ and $1\leqslant r\leqslant 2$,

$$\prod_{j=1}^{k} s_j(|A+B|^r) \leqslant \prod_{j=1}^{k} s_j(I_n+|A|^r)s_j(I_n+|B|^r) \tag{4}$$

holds for $1\leqslant k\leqslant n$ [5, Theorem 2.8].

As a continuation, in this short note, we give new proofs for inequalities (3) and (4).

2. Main results

Let us recall some definitions of majorization. Given a real vector $x=(x_1,x_2,\cdots,x_n)\in\mathbf{R}^n$, we rearrange its components as $x_{[1]}\geqslant x_{[2]}\geqslant\cdots\geqslant x_{[n]}$. For two real vectors $x=(x_1,x_2,\cdots,x_n)$, $y=(y_1,y_2,\cdots,y_n)\in\mathbf{R}^n$, if

$$\sum_{i=1}^{k} x_{[i]}\leqslant\sum_{i=1}^{k} y_{[i]},\ \ k=1,2,\cdots,n,$$

then we say that x is weakly majorized by y and denotes by $x\prec_w y$. If $x\prec_w y$ and $\sum_{i=1}^{n} x_i=\sum_{i=1}^{n} y_i$, then we say that x is majorized by y and denotes by $x\prec y$. Further, if $x=(x_1,x_2,\cdots,x_n)$, $y=(y_1,y_2,\cdots,y_n)\in\mathbf{R}^n_+$ are two real vectors and

$$\prod_{i=1}^{k} x_{[i]}\leqslant\prod_{i=1}^{k} y_{[i]},\ \ k=1,2,\cdots,n,$$

then we say that x is weakly log-majorized by y and denotes by $x\prec_{w\log} y$. If $x\prec_{w\log} y$ and $\prod_{i=1}^{n} x_i=\prod_{i=1}^{n} y_i$, then we say that x is log-majorized by y and denotes by $x\prec_{\log} y$. It is well-known that if $x\prec_{w\log} y$, then $x\prec_w y$.

Now, we present some lemmas. The following well-known matrix inequality involving unitarily orbits is due to Thompson [7] (or [2, Theorem III.5.6]).

LEMMA 1. *Let $A, B \in M_n$. Then there exists two unitary matrices U and V such that*

$$|A+B| \leqslant U|A|U^* + V|B|V^*.$$

The next lemma was obtained by J. Aujla and J. Bourin [1, Theorem 2.1].

LEMMA 2. *Let f be monotone concave function on $[0,+\infty)$ with $f(0) \geqslant 0$ and $A, B \in M_n$ be two positive semidefinite matrices. Then there exists two unitary matrices U and V such that*

$$f(A+B) \leqslant Uf(A)U^* + Vf(B)V^*.$$

The next two lemmas are the Proposition 1.3.2 in [3] and Theorem IX 2.10 in [2], respectively.

LEMMA 3. *Let $A, B \in M_n$ be two positive semidefinite matrices. Then the matrix $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in M_{2n}$ is a positive semidefinite matrix if and only if $X = A^{\frac{1}{2}} W B^{\frac{1}{2}}$ for some contraction W.*

LEMMA 4. *Let $A, B \in M_n$ be two positive definite matrices. Then for every unitarily invariant norm $|||\cdot|||$, the following inequalities*

$$|||B^t A^t B^t||| \leqslant |||(BAB)^t|||, \quad for \quad 0 \leqslant t \leqslant 1,$$

and

$$|||(BAB)^t||| \leqslant |||B^t A^t B^t|||, \quad for \quad t \geqslant 1,$$

hold.

In fact, the Lemma 4's proof [2, Theorem IX 2.10] proved that

$$s(B^t A^t B^t) \prec_{w\log} s((BAB)^t), \quad 0 \leqslant t \leqslant 1, \tag{5}$$

and

$$s((BAB)^t) \prec_{w\log} s(B^t A^t B^t), \quad 1 \leqslant t, \tag{6}$$

where $A, B \in M_n$ are two positive definite matrices. By continuity, inequalities (5) and (6) hold for positive semidefinite matrices $A, B \in M_n$.

The following lemma is the famous Horn's inequality [8, Theorem 4.6].

LEMMA 5. *Let $A, B \in M_n$. Then*

$$s(AB) \prec_{\log} \{s_i(A)s_i(B)\}_{i=1}^n. \tag{7}$$

To prove inequality (3), the following result play an important role.

LEMMA 6. *Let A, $B \in M_n$ be two positive semidefinite matrices. Then*

$$s(I_n + A + B) \prec_{w\log} \{s_i(I_n + A)s_i(I_n + B)\}_{i=1}^{n},$$

equivalently,

$$\prod_{i=1}^{k} s_i(I_n + A + B) \leqslant \prod_{i=1}^{k} s_i(I_n + A)s_i(I_n + B), \tag{8}$$

for $k = 1, 2, \cdots, n$.

Proof. By (1) and noting that $(I_n + A)^{-1} \leqslant I_n$, we have

$$\begin{aligned}
\lambda_i((I_n + A)^{-\frac{1}{2}}(I_n\ + A + B)(I_n + A)^{-\frac{1}{2}}) \\
&= s_i((I_n + A)^{-\frac{1}{2}}(I_n + A + B)(I_n + A)^{-\frac{1}{2}}) \\
&= s_i(I_n + (I_n + A)^{-\frac{1}{2}}B(I_n + A)^{-\frac{1}{2}}) \\
&= 1 + s_i((I_n + A)^{-\frac{1}{2}}B(I_n + A)^{-\frac{1}{2}}) \\
&= 1 + s_i(B^{\frac{1}{2}}(I_n + A)^{-1}B^{\frac{1}{2}}) \\
&\leqslant 1 + s_i(B) \\
&= \lambda_i(I_n + B),
\end{aligned} \tag{9}$$

for $i = 1, 2, \cdots, n$.

According to inequality (9), there exists an unitary matrix U such that

$$(I_n + A)^{-\frac{1}{2}}(I_n + A + B)(I_n + A)^{-\frac{1}{2}} \leqslant U^*(I_n + B)U,$$

or equivalently,

$$I_n + A + B \leqslant (I_n + A)^{\frac{1}{2}}[U^*(I_n + B)U](I_n + A)^{\frac{1}{2}}. \tag{10}$$

By Lemma 5 and equality (2), inequality (10) gives

$$\begin{aligned}
\prod_{i=1}^{k} s_i(I_n + A + B) &\leqslant \prod_{i=1}^{k} s_i((I_n + A)^{\frac{1}{2}}[U^*(I_n + B)U](I_n + A)^{\frac{1}{2}}) \\
&\leqslant \prod_{i=1}^{k} s_i((I_n + A)^{\frac{1}{2}})s_i(U^*(I_n + B)U)s_i((I_n + A)^{\frac{1}{2}}) \\
&= \prod_{i=1}^{k} s_i((I_n + A))s_i(I_n + B),
\end{aligned}$$

for $k = 1, 2, \cdots, n$.

This completes the proof. □

REMARK 1. Using Lemma 2, we can obtain another different proof of Lemma 6. Indeed, taking $f(t) = \log(1+t)$, we get

$$\log(I_n + A + B) \leqslant U \log(I_n + A)U^* + V \log(I_n + B)V^*$$

for some unitary matrices U and V. so for each Ky Fan $k-$norm, the above inequality gives

$$\|\log(I_n + A + B)\|_{(k)} \leqslant \|\log(I_n + A)\|_{(k)} + \|\log(I_n + B)\|_{(k)},$$

which is exactly inequality (8).

Utilizing Lemma 6, we have the following theorem.

THEOREM 1. *Let $A,\ B \in M_n$ and $f : [0,+\infty) \to [0,+\infty)$ is a concave function. Then*

$$\prod_{j=1}^{k} s_j(I_n + f(|A+B|)) \leqslant \prod_{j=1}^{k} s_j(I_n + f(|A|))s_j(I_n + f(|B|)), \tag{11}$$

holds for $1 \leqslant k \leqslant n$.

Proof. By Lemma 1, there exists two unitary matrices U and V such that

$$|A+B| \leqslant U^*|A|U + V^*|B|V.$$

Since f is a nonnegative concave function, then it is a nondecreasing function. Hence by the above inequality, there exists an unitary matrix U_1 with

$$f(|A+B|) \leqslant U_1 f(U^*|A|U + V^*|B|V)U_1^*. \tag{12}$$

By Lemma 2, there exists two unitary matrices U_2 and V_1 such that

$$f(U^*|A|U + V^*|B|V) \leqslant U_2^* U^* f(|A|)UU_2 + V_1^* V f(|B|)VV_1. \tag{13}$$

Inequalities (12) and (13) imply

$$f(|A+B|) \leqslant U_1 U_2^* U^* f(|A|)UU_2 U_1^* + U_1 V_1^* V f(|B|)VV_1 U_1^*. \tag{14}$$

Noting that $UU_2U_1^*$ and $VV_1U_1^*$ are unitary matrices, the desired inequality (11) follows from the eigenvalue's monotone theorem for Hermitian matrices and inequalities (8) and (14).

This completes the proof. □

REMARK 2. It is not necessary to assume that $f(t)$ is operator concave in I. Garg and J. Aujla's result [5, Theorem 2.10].

We end this section by giving a new proof to inequality (4).

THEOREM 2. *Let $A,\ B \in M_n$. Then*

$$\prod_{j=1}^{k} s_j(|A+B|^r) \leqslant \prod_{j=1}^{k} s_j(I_n+|A|^r)s_j(I_n+|B|^r) \tag{15}$$

holds for $1 \leqslant k \leqslant n$ and $1 \leqslant r \leqslant 2$.

Proof. Since

$$\begin{pmatrix} I_n+AA^* & A+B \\ (A+B)^* & I_n+B^*B \end{pmatrix} = \begin{pmatrix} I_n & A \\ B^* & I_n \end{pmatrix} \begin{pmatrix} I_n & B \\ A^* & I_n \end{pmatrix} \geqslant 0,$$

by Lemma 3, there exists a contraction W with

$$\begin{aligned} A+B &= (I_n+AA^*)^{\frac{1}{2}} W (I_n+B^*B)^{\frac{1}{2}} \\ &= (I_n+|A^*|^2)^{\frac{1}{2}} W (I_n+|B|^2)^{\frac{1}{2}}. \end{aligned}$$

Thus,

$$|A+B|^{2r} = [(I_n+|B|^2)^{\frac{1}{2}} W^*(I_n+|A^*|^2)W(I_n+|B|^2)^{\frac{1}{2}}]^r, \tag{16}$$

for $1 \leqslant r \leqslant 2$. By equality (16) and inequalities (6) and (7), we obtain

$$\begin{aligned} &\prod_{i=1}^{k} s_j(|A\,+\,B|^{2r}) \\ &\qquad \leqslant \prod_{i=1}^{k} s_i((I_n+|B|^2)^{\frac{r}{2}} (W^*(I_n+|A^*|^2)W)^r (I_n+|B|^2)^{\frac{r}{2}}) \\ &\qquad \leqslant \prod_{i=1}^{k} s_i((I_n+|B|^2)^{\frac{r}{2}}) s_i((W^*(I_n+|A^*|^2)W)^r) s_i((I_n+|B|^2)^{\frac{r}{2}}) \\ &\qquad = \prod_{i=1}^{k} s_i(I_n+|B|^2)^r s_i(W^*(I_n+|A^*|^2)W)^r, \end{aligned} \tag{17}$$

for $1 \leqslant k \leqslant n$.

Since W is a contraction, then

$$s_i(W^*(I_n+|A^*|^2)W) \leqslant s_i(I_n+|A^*|^2), \tag{18}$$

$i = 1,2,\cdots,n$.

Inequalities (17) and (18) imply

$$\begin{aligned} \prod_{i=1}^{k} s_i(|A+B|^{2r}) &\leqslant \prod_{i=1}^{k} s_i(I_n+|B|^2)^r s_i(I_n+|A^*|^2)^r \\ &= \prod_{i=1}^{k} s_i(I_n+|B|^2)^r s_i(I_n+|A|^2)^r, \end{aligned} \tag{19}$$

for $1 \leqslant k \leqslant n$, the above equality holds due to the unitarily equivalent of $|A|^2$ and $|A^*|^2$.

When $r = 2$, by inequality (19), we have

$$\prod_{i=1}^{k} s_i(|A+B|^2) \leqslant \prod_{i=1}^{k} s_i(I_n+|B|^2)s_i(I_n+|A|^2). \tag{20}$$

for $1 \leqslant k \leqslant n$.

On the other hand, $f(x) = x^{\frac{r}{2}}$ $(1 \leqslant r < 2)$ is a concave function on $[0,+\infty)$, then $f(x)+f(y) \geqslant f(x+y)$, for $x,\ y \in [0,+\infty)$. It follows that

$$(1+s_i^2(|X|))^{\frac{r}{2}} \leqslant 1+s_i^r(|X|), \tag{21}$$

for $X \in M_n$ and $i = 1,2,\cdots,n$.

Combining inequalities (19) with (21) and noting that $s_i(I_n+|X|^2) = 1+s_i^2(|X|)$ $(i = 1,2,\cdots,n)$ for $X \in M_n$, we get

$$\prod_{i=1}^{k} s_i(|A+B|^{2r}) \leqslant \prod_{i=1}^{k} s_i(I_n+|B|^r)^2 s_i(I_n+|A|^r)^2,$$

or equivalently,

$$\prod_{i=1}^{k} s_i(|A+B|^r) \leqslant \prod_{i=1}^{k} s_i(I_n+|B|^r)s_i(I_n+|A|^r), \tag{22}$$

for $1 \leqslant k \leqslant n$ and $1 \leqslant r < 2$.

Thus, the desired inequality (15) follows from inequalities (20) and (22).

This completes the proof. $\square$

Acknowledgement. The author would like to thank the editor and anonymous referees for their helpful comments and suggestions. This work is supported by the National Natural Science Foundations of China (Grant Nos. 11161040).

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(Received July 20, 2017)

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