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THE MULTI–PARAMETER HAUSDORFF OPERATORS ON H^1 AND L^p

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Abstract. In the present paper, we characterize the nonnegative functions φ for which the multi-parameter Hausdorff operator $\mathscr{H}_\varphi$ generated by φ is bounded on either the multi-parameter Hardy space $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$ or $L^p(\mathbb{R}^n)$, $p\in[1,\infty]$. The corresponding operator norms are also obtained. Our results improve some recent results in [4, 15, 16, 18] and give an answer to an open question posted by Liflyand [12].

1. Introduction and main result

Let φ be a locally integrable function on $(0,\infty)$. The classical one-parameter Hausdorff operator $\mathscr{H}_\varphi$ is defined for suitable functions f on $\mathbb{R}$ by

$$\mathscr{H}_\varphi f(x)=\int_0^\infty f\left(\frac{x}{t}\right)\frac{\varphi(t)}{t}dt.$$

The Hausdorff operator $\mathscr{H}_\varphi$ is an interesting operator in harmonic analysis. There are many classical operators in analysis which are special cases of the Hausdorff operator if one chooses suitable kernel functions φ, such as the classical Hardy operator, its adjoint operator, the Cesàro type operators, the Riemann-Liouville fractional integral operator. See the survey article [13] and the references therein. In the recent years, there is an increasing interest in the study of boundedness of the Hausdorff operator on some function spaces, see for example [1, 2, 4, 7, 8, 12, 13, 14, 15, 16, 17, 18, 19].

When φ is a locally integrable function on $(0,\infty)^n$, there are several high-dimensional extensions of $\mathscr{H}_\varphi$. One of them is the *multi-parameter Hausdorff operator* $\mathscr{H}_\varphi$ defined for suitable functions f on $\mathbb{R}^n$ by

$$\mathscr{H}_\varphi f(x_1,\dots,x_n)=\int_0^\infty\cdots\int_0^\infty f\left(\frac{x_1}{t_1},\dots,\frac{x_n}{t_n}\right)\frac{\varphi(t_1,\dots,t_n)}{t_1\dots t_n}dt_1\cdots dt_n.$$

Let $\Phi^{(1)},\dots,\Phi^{(n)}$ be C^∞-functions with compact support satisfying $\int_{\mathbb{R}}\Phi^{(1)}(x)dx=\cdots=\int_{\mathbb{R}}\Phi^{(n)}(x)dx=1$. Then, for any $(t_1,\dots,t_n)\in(0,\infty)^n$, we denote

$$\otimes_{j=1}^n\Phi_{t_j}^{(j)}(\mathbf{x}):=\prod_{j=1}^n\frac{1}{t_j}\Phi^{(j)}\left(\frac{x_j}{t_j}\right),\quad \mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}^n.$$

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Following Gundy and Stein [6], we define the *multi-parameter Hardy space* $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$ as the set of all functions $f\in L^1(\mathbb{R}^n)$ such that

$$\|f\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})} := \|M_\Phi f\|_{L^1(\mathbb{R})} < \infty,$$

where $M_\Phi f$ is the *multi-parameter smooth maximal function* of f defined by

$$M_\Phi f(\mathbf{x}) = \sup_{(t_1,\dots,t_n)\in(0,\infty)^n} |f*(\otimes_{j=1}^n \Phi_{t_j}^{(j)})(\mathbf{x})|, \quad \mathbf{x}\in\mathbb{R}^n.$$

REMARK 1.

(i) $\|\cdot\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})}$ defines a norm on $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$, whose size depends on the choice of $\{\Phi^{(j)}\}_{j=1}^n$, but the space $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$ does not depend on this choice.

(ii) If f is in $H^1(\mathbb{R})$, then the function

$$f\otimes\cdots\otimes f(\mathbf{x}) = \prod_{j=1}^n f(x_j), \quad \mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}^n,$$

is in $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$. Moreover, there exist two positive constants C_1, C_2 independent of f such that

$$C_1\|f\|^n_{H^1(\mathbb{R})} \leqslant \|f\otimes\cdots\otimes f\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})} \leqslant C_2\|f\|^n_{H^1(\mathbb{R})}.$$

In the setting of two-parameter, Liflyand and Móricz showed in [15] that $\mathscr{H}_\varphi$ is bounded on $H^1(\mathbb{R}\times\mathbb{R})$ provided $\varphi\in L^1((0,\infty)^2)$. In the setting of n-parameter, one of Weisz's important results (see [18, Theorem 7]) showed that $\mathscr{H}_\varphi$ is bounded on $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$ provided $\varphi(t_1,\dots,t_n)=\prod_{i=1}^n\varphi_i(t_i)$ with $\varphi_i\in L^1(\mathbb{R})$ for all $1\leqslant i\leqslant n$. Recently, in the setting of two-parameter, Fan and Zhao showed in [4] that the condition $\varphi\in L^1((0,\infty)^2)$ is also a necessary condition for $H^1(\mathbb{R}\times\mathbb{R})$-boundedness of $\mathscr{H}_\varphi$ if φ is nonnegative valued. However, it seems that Fan-Zhao's method can not be used to obtain the exact norm of $\mathscr{H}_\varphi$ on $H^1(\mathbb{R}\times\mathbb{R})$. So, in the setting of n-parameter, a natural question arises: Can one find the exact norm of $\mathscr{H}_\varphi$ on $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$? Very recently, in the setting of one-parameter, this question was solved by Hung, Ky and Quang [7].

Motivated by the above question and an open question posted by Liflyand [12, Problem 5], we characterize the nonnegative functions φ for which $\mathscr{H}_\varphi$ is bounded on $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$. More precisely, our main result is the following:

THEOREM 1. *Let φ be a nonnegative function in $L^1_{\rm loc}((0,\infty)^n)$. Then $\mathscr{H}_\varphi$ is bounded on $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$ if and only if*

$$\int_0^\infty\cdots\int_0^\infty \varphi(t_1,\dots,t_n)dt_1\cdots dt_n < \infty. \tag{1}$$

Moreover, in that case,

$$\|\mathscr{H}_\varphi\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})} = \int_0^\infty\cdots\int_0^\infty \varphi(t_1,\dots,t_n)dt_1\cdots dt_n.$$

Theorem 1 not only gives an affirmative answer to the above question, but also gives an answer to [12, Problem 5]. It should be pointed out that the norm of the Hausdorff operator $\mathscr{H}_\varphi$ ($\int_0^\infty \cdots \int_0^\infty \varphi(t_1,\dots,t_n)dt_1\dots dt_n$) does not depend on the choice of the above functions $\{\Phi^{(j)}\}_{j=1}^n$, moreover, it still holds when the above norm $\|\cdot\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})}$ is replaced by

$$\|f\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})} := \sum_{\mathbf{e}\in\{0,1\}^n} \|\mathbf{H_e} f\|_{L^1(\mathbb{R}^n)},$$

where $\mathbf{H_e} f$'s are the *multi-parameter Hilbert transforms* of f. See Theorem 8 for details.

Also we characterize the nonnegative functions φ for which $\mathscr{H}_\varphi$ is bounded on $L^p(\mathbb{R}^n)$, $p\in[1,\infty]$. Our next result can be stated as follows.

THEOREM 2. *Let $p\in[1,\infty]$ and let φ be a nonnegative function in $L^1_{\rm loc}((0,\infty)^n)$. Then $\mathscr{H}_\varphi$ is bounded on $L^p(\mathbb{R}^n)$ if and only if*

$$\int_0^\infty \cdots \int_0^\infty \frac{\varphi(t_1,\dots,t_n)}{t_1^{1-1/p}\dots t_n^{1-1/p}} dt_1\cdots dt_n < \infty. \tag{2}$$

Moreover, in that case,

$$\|\mathscr{H}_\varphi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \int_0^\infty \cdots \int_0^\infty \frac{\varphi(t_1,\dots,t_n)}{t_1^{1-1/p}\dots t_n^{1-1/p}} dt_1\cdots dt_n.$$

Throughout the whole article, *we always assume that φ is a nonnegative function in $L^1_{\rm loc}((0,\infty)^n)$* and denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $A\lesssim B$ means that $A\leqslant CB$. If $A\lesssim B$ and $B\lesssim A$, then we write $A\sim B$.

2. Norm of $\mathscr{H}_\varphi$ on $L^p(\mathbb{R}^n)$

The main purpose of this section is to give the proof of Theorem 2. Let us first consider the operator $\mathscr{H}_\varphi^*$ defined by

$$\mathscr{H}_\varphi^* f(x_1,\dots,x_n) = \int_0^\infty \cdots \int_0^\infty f(t_1x_1,\dots,t_nx_n)\varphi(t_1,\dots,t_n)dt_1\dots dt_n.$$

Studying this operator on the spaces $L^p(\mathbb{R}^n)$ is useful in proving the main theorem (Theorem 1) in the next section.

Remark that $\mathscr{H}_\varphi^* = \mathscr{H}_{\overline{\varphi}}$ with $\overline{\varphi}(\mathbf{t}) = \frac{\varphi(1/t_1,\dots,1/t_n)}{t_1\dots t_n}$ for all $\mathbf{t}=(t_1,\dots,t_n)\in(0,\infty)^n$. Hence, by Theorems 1 and 2, we obtain:

THEOREM 3. *$\mathscr{H}_\varphi^*$ is bounded on $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$ if and only if*

$$\int_0^\infty \cdots \int_0^\infty \frac{\varphi(t_1,\dots,t_n)}{t_1\dots t_n} dt_1\cdots dt_n < \infty. \tag{3}$$

Moreover, in that case,

$$\|\mathscr{H}_\varphi^*\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})} = \int_0^\infty \cdots \int_0^\infty \frac{\varphi(t_1,\dots,t_n)}{t_1\dots t_n} dt_1\cdots dt_n.$$

THEOREM 4. *Let $p \in [1,\infty]$. Then $\mathscr{H}_\varphi^*$ is bounded on $L^p(\mathbb{R}^n)$ if and only if*

$$\int_0^\infty \cdots \int_0^\infty \frac{\varphi(t_1,\dots,t_n)}{t_1^{1/p}\dots t_n^{1/p}} dt_1\cdots dt_n < \infty. \tag{4}$$

Moreover, in that case,

$$\|\mathscr{H}_\varphi^*\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \int_0^\infty \cdots \int_0^\infty \frac{\varphi(t_1,\dots,t_n)}{t_1^{1/p}\dots t_n^{1/p}} dt_1\cdots dt_n.$$

By Theorems 2, 4 and the Fubini theorem, $\mathscr{H}_\varphi^*$ can be viewed as the Banach space adjoint of $\mathscr{H}_\varphi$ and vice versa. More precisely, we have:

THEOREM 5. *Let $p \in [1,\infty]$ and $1/p' + 1/p = 1$.*

(i) *If (2) holds, then, for all $f \in L^p(\mathbb{R}^n)$ and all $g \in L^{p'}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \mathscr{H}_\varphi f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^n} f(\mathbf{x}) \mathscr{H}_\varphi^* g(\mathbf{x}) d\mathbf{x}.$$

(ii) *If (4) holds, then, for all $f \in L^p(\mathbb{R}^n)$ and all $g \in L^{p'}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \mathscr{H}_\varphi^* f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^n} f(\mathbf{x}) \mathscr{H}_\varphi g(\mathbf{x}) d\mathbf{x}.$$

As a consequence of the above theorem, we get the following.

COROLLARY 1. *Let $p \in [1,2]$.*

(i) *If (2) holds, then, for all $f \in L^p(\mathbb{R}^n)$,*

$$\widehat{\mathscr{H}_\varphi f} = \mathscr{H}_\varphi^* \hat{f}.$$

(ii) *If (4) holds, then, for all $f \in L^p(\mathbb{R}^n)$,*

$$\widehat{\mathscr{H}_\varphi^* f} = \mathscr{H}_\varphi \hat{f}.$$

Proof. We prove only (i) since the proof of (ii) is similar. Moreover, from the Hausdorff-Young theorem and the fact that $L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, we consider only the case $p = 1$. For all $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, by Theorem 5(i) and the Fubini theorem, we get

$$\begin{aligned}\widehat{\mathscr{H}_\varphi f}(\mathbf{y}) &= \int_{\mathbb{R}^n} \mathscr{H}_\varphi f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\mathbf{y}} d\mathbf{x} \\ &= \int_{\mathbb{R}^n} f(\mathbf{x}) d\mathbf{x} \int_0^\infty \cdots \int_0^\infty e^{-2\pi i \sum_{j=1}^n t_j x_j y_j} \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \\ &= \int_0^\infty \cdots \int_0^\infty \hat{f}(t_1 y_1, \dots, t_n y_n) \varphi(t_1, \dots, t_n) dt_1 \dots dt_n \\ &= \mathscr{H}_\varphi^* \hat{f}(\mathbf{y}).\end{aligned}$$

This completes the proof of Corollary 1. □

Proof of Theorem 2. Since the case $p = \infty$ is trivial, we consider only the case $p \in [1, \infty)$. Suppose that (2) holds. For any $f \in L^p(\mathbb{R}^n)$, by the Minkowski inequality, we obtain

$$\begin{aligned}\|\mathscr{H}_\varphi f\|_{L^p(\mathbb{R}^n)} &\leqslant \int_0^\infty \cdots \int_0^\infty \left\| f\left(\frac{\cdot}{t_1}, \dots, \frac{\cdot}{t_n}\right)\right\|_{L^p(\mathbb{R}^n)} \frac{\varphi(t_1, \dots, t_n)}{t_1 \dots t_n} dt_1 \cdots dt_n \\ &= \|f\|_{L^p(\mathbb{R}^n)} \int_0^\infty \cdots \int_0^\infty \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p} \dots t_n^{1-1/p}} dt_1 \cdots dt_n.\end{aligned}$$

This proves that $\mathscr{H}_\varphi$ is bounded on $L^p(\mathbb{R}^n)$, moreover,

$$\|\mathscr{H}_\varphi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \leqslant \int_0^\infty \cdots \int_0^\infty \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p} \dots t_n^{1-1/p}} dt_1 \cdots dt_n. \tag{5}$$

Conversely, suppose that $\mathscr{H}_\varphi$ is bounded on $L^p(\mathbb{R}^n)$. For any $\varepsilon > 0$, take

$$f_\varepsilon(\mathbf{x}) = \prod_{j=1}^n |x_j|^{-1/p-\varepsilon} \chi_{\{y_j \in \mathbb{R}: |y_j| \geqslant 1\}}(x_j)$$

for all $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then, it is easy to see that $f_\varepsilon \in L^p(\mathbb{R}^n)$ and

$$\mathscr{H}_\varphi f_\varepsilon(\mathbf{x}) = \prod_{j=1}^n |x_j|^{-1/p-\varepsilon} \int_0^{|x_1|} dt_1 \cdots \int_0^{|x_{n-1}|} dt_{n-1} \int_0^{|x_n|} \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p-\varepsilon} \dots t_n^{1-1/p-\varepsilon}} dt_n$$

for all $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Some simple computations give

$$\begin{aligned}\|\mathscr{H}_\varphi f_\varepsilon\|_{L^p(\mathbb{R}^n)} &\geqslant \int_0^{1/\varepsilon} \cdots \int_0^{1/\varepsilon} \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p-\varepsilon} \dots t_n^{1-1/p-\varepsilon}} dt_1 \cdots dt_n \times \\ &\quad \times \left(\prod_{j=1}^n \int_{\{x_j \in \mathbb{R}: |x_j| \geqslant 1/\varepsilon\}} |x_j|^{-1-p\varepsilon} dx_j \right)^{1/p} \\ &= \int_0^{1/\varepsilon} \cdots \int_0^{1/\varepsilon} \frac{\varphi(t_1, \dots, t_n)}{t_1^{1-1/p-\varepsilon} \dots t_n^{1-1/p-\varepsilon}} dt_1 \cdots dt_n \left(\varepsilon^{n\varepsilon} \|f_\varepsilon\|_{L^p(\mathbb{R}^n)}\right).\end{aligned}$$

Therefore,

$$\|\mathscr{H}_\varphi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \geqslant \frac{\|\mathscr{H}_\varphi f_\varepsilon\|_{L^p(\mathbb{R}^n)}}{\|f_\varepsilon\|_{L^p(\mathbb{R}^n)}}$$

$$\geqslant \varepsilon^{n\varepsilon}\int_0^{1/\varepsilon}\cdots\int_0^{1/\varepsilon}\frac{\varphi(t_1,\dots,t_n)}{t_1^{1-1/p-\varepsilon}\dots t_n^{1-1/p-\varepsilon}}dt_1\cdots dt_n.$$

Letting $\varepsilon\to 0$, we obtain

$$\|\mathscr{H}_\varphi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \geqslant \int_0^\infty\cdots\int_0^\infty\frac{\varphi(t_1,\dots,t_n)}{t_1^{1-1/p}\dots t_n^{1-1/p}}dt_1\cdots dt_n.$$

This, together (5), implies that

$$\|\mathscr{H}_\varphi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \int_0^\infty\cdots\int_0^\infty\frac{\varphi(t_1,\dots,t_n)}{t_1^{1-1/p}\dots t_n^{1-1/p}}dt_1\cdots dt_n,$$

and thus ends the proof of Theorem 2. □

3. Norm of $\mathscr{H}_\varphi$ on $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$

The main purpose of this section is to give the proof of Theorem 1 and to show that the norm of the Hausdorff operator $\mathscr{H}_\varphi$ in Theorem 1 still holds when one replaces the norm $\|\cdot\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})}$ by the norm $\|\cdot\|_*$ (see (12) and Theorem 8 below).

Let $\mathbb{C}^n_+$ be the upper half-plan in $\mathbb{C}^n$, that is,

$$\mathbb{C}^n_+ = \prod_{j=1}^n\{z_j = x_j + iy_j \in\mathbb{C} : y_j > 0\}.$$

Following Gundy-Stein [6] and Lacey [9], a function $F:\mathbb{C}^n_+\to\mathbb{C}$ is said to be in the Hardy space $\mathscr{H}^1_a(\mathbb{C}^n_+)$ if it is holomorphic in each variable separately and

$$\|F\|_{\mathscr{H}^1_a(\mathbb{C}^n_+)} := \sup_{(y_1,\dots,y_n)\in(0,\infty)^n}\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty |F(x_1+iy_1,\dots,x_n+iy_n)|dx_1\dots dx_n<\infty.$$

Let $j\in\{1,\dots,n\}$. For any $f\in L^1(\mathbb{R}^n)$, the Hilbert transform H_jf computed in the j^{th} variable is defined by

$$H_jf(\mathbf{x}) = \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^\infty\frac{f(x_1,\dots,x_j-y,\dots,x_n)}{y}dy.$$

For any $\mathbf{e}=(e_1,\dots,e_n)\in\mathbb{E}:=\{0,1\}^n$, denote

$$\mathbf{H_e} = \prod_{j=1}^n H_j^{e_j}$$

with $H_j^{e_j}=I$ for $e_j=0$ while $H_j^{e_j}=H_j$ for $e_j=1$.

The following two theorems are well-known, see for example [6, 9, 10, 18].

THEOREM 6. *A function f is in $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$ if and only if $\mathbf{H_e}f$ is in $L^1(\mathbb{R}^n)$ for all $\mathbf{e}\in\mathbb{E}$. Moreover, in that case,*

$$\|f\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})}\sim\sum_{\mathbf{e}\in\mathbb{E}}\|\mathbf{H_e}f\|_{L^1(\mathbb{R}^n)}.$$

THEOREM 7. *Let $F\in\mathscr{H}_a^1(\mathbb{C}_+^n)$. Then the boundary value function f of F, which is defined by*

$$f(x_1,\dots,x_n)=\lim_{(y_1,\dots,y_n)\to(0,\dots,0)}F(x_1+iy_1,\dots,x_n+iy_n),$$

a. e. $\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$, is in $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$. Moreover,

$$\|f\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})}\sim\|f\|_{L^1(\mathbb{R}^n)}=\|F\|_{\mathscr{H}_a^1(\mathbb{C}_+^n)}$$

and, for all $\mathbf{x}+i\mathbf{y}=(x_1+iy_1,\dots,x_n+iy_n)\in\mathbb{C}_+^n$,

$$\begin{aligned}F(\mathbf{x}+i\mathbf{y}) &= \int_{\mathbb{R}}\cdots\int_{\mathbb{R}}f(x_1-u_1,\dots,x_n-u_n)\prod_{j=1}^n\frac{1}{y_j}P\left(\frac{u_j}{y_j}\right)du_1\dots du_n\\ &=: f*(\otimes_{j=1}^nP_{y_j})(\mathbf{x}),\end{aligned}$$

where $P(u)=\frac{1}{1+u^2}, u\in\mathbb{R}$, is the Poisson kernel on $\mathbb{R}$.

In order to prove Theorem 1, we also need the following two lemmas.

LEMMA 1. *Let φ be such that $\mathscr{H}_\varphi$ is bounded from $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$ into $L^1(\mathbb{R}^n)$. Then (1) holds.*

LEMMA 2. *Let φ be such that (1) holds. Then:*

(i) *$\mathscr{H}_\varphi$ is bounded on $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$, moreover,*

$$\|\mathscr{H}_\varphi\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})}\leqslant\int_0^\infty\cdots\int_0^\infty\varphi(t_1,\dots,t_n)dt_1\dots dt_n.$$

(ii) *If supp $\varphi\subset[0,1]^n$, then*

$$\|\mathscr{H}_\varphi\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})}=\int_0^1\cdots\int_0^1\varphi(t_1,\dots,t_n)dt_1\dots dt_n.$$

Proof of Lemma 1. Since the function

$$f(x)=\frac{x}{(1+x^2)^2},\quad x\in\mathbb{R},$$

is in $H^1(\mathbb{R})$ (see [7, Theorem 3.3]), Remark 1(ii) yields that

$$f\otimes\cdots\otimes f(\mathbf{x})=\prod_{j=1}^n\frac{x_j}{(1+x_j^2)^2},\quad \mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}^n,$$

is in $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$. Hence, the function

$$\mathscr{H}_\varphi(f\otimes\cdots\otimes f)(\mathbf{x}) = \int_0^\infty\cdots\int_0^\infty \prod_{j=1}^n \frac{\frac{x_j}{t_j}}{\left[1+\left(\frac{x_j}{t_j}\right)^2\right]^2}\frac{\varphi(t_1,\ldots,t_n)}{t_1\ldots t_n}dt_1\cdots dt_n,$$

$\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n$, is in $L^1(\mathbb{R}^n)$ since $\mathscr{H}_\varphi$ is bounded from $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$ into $L^1(\mathbb{R}^n)$. As a consequence,

$$\begin{aligned}&\left[\int_0^\infty \frac{y}{(1+y^2)^2}dy\right]^n \int_0^\infty\cdots\int_0^\infty \varphi(t_1,\ldots,t_n)dt_1\cdots dt_n\\ &= \int_{[0,\infty)^n} d\mathbf{x}\int_0^\infty\cdots\int_0^\infty \prod_{j=1}^n \frac{\frac{x_j}{t_j}}{\left[1+\left(\frac{x_j}{t_j}\right)^2\right]^2}\frac{\varphi(t_1,\ldots,t_n)}{t_1\ldots t_n}dt_1\cdots dt_n\\ &\leqslant \|\mathscr{H}_\varphi(f\otimes\cdots\otimes f)\|_{L^1(\mathbb{R}^n)}<\infty\end{aligned}$$

which proves (1), and thus ends the proof of Lemma 1. □

Proof of Lemma 2. (i) For any $f\in H^1(\mathbb{R}\times\cdots\times\mathbb{R})$, by the Fubini theorem,

$$\begin{aligned}&M_\Phi(\mathscr{H}_\varphi f)(\mathbf{x})\\ &= \sup_{(r_1,\ldots,r_n)\in(0,\infty)^n}\left|\int_{\mathbb{R}^n} d\mathbf{y}\int_0^\infty\cdots\int_0^\infty(\otimes_{j=1}^n\Phi_{r_j}^{(j)})(\mathbf{x}-\mathbf{y})f\left(\frac{y_1}{t_1},\ldots,\frac{y_n}{t_n}\right)\frac{\varphi(t_1,\ldots,t_n)}{t_1\ldots t_n}dt_1\ldots dt_n\right|\\ &= \sup_{(r_1,\ldots,r_n)\in(0,\infty)^n}\left|\int_0^\infty\cdots\int_0^\infty\left(f*(\otimes_{j=1}^n\Phi_{r_j/t_j}^{(j)})\right)\left(\frac{x_1}{t_1},\ldots,\frac{x_n}{t_n}\right)\frac{\varphi(t_1,\ldots,t_n)}{t_1\ldots t_n}dt_1\ldots dt_n\right|\\ &\leqslant \mathscr{H}_\varphi(M_\Phi f)(\mathbf{x})\end{aligned}$$

for all $\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n$. Hence, by Theorem 2,

$$\begin{aligned}\|\mathscr{H}_\varphi f\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})} &= \|M_\Phi(\mathscr{H}_\varphi f)\|_{L^1(\mathbb{R}^n)}\\ &\leqslant \|\mathscr{H}_\varphi(M_\Phi f)\|_{L^1(\mathbb{R}^n)}\\ &\leqslant \int_0^\infty\cdots\int_0^\infty \varphi(t_1,\ldots,t_n)dt_1\ldots dt_n\|M_\Phi f\|_{L^1(\mathbb{R}^n)}\\ &= \int_0^\infty\cdots\int_0^\infty \varphi(t_1,\ldots,t_n)dt_1\ldots dt_n\|f\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})}.\end{aligned}$$

This proves that $\mathscr{H}_\varphi$ is bounded on $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$, moreover,

$$\|\mathscr{H}_\varphi\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})}\leqslant \int_0^\infty\cdots\int_0^\infty \varphi(t_1,\ldots,t_n)dt_1\ldots dt_n. \tag{6}$$

(ii) Let $\delta\in(0,1)$ be arbitrary. Set $\varphi_\delta(\mathbf{t}):=\varphi(\mathbf{t})\chi_{[\delta,1]^n}(\mathbf{t})$ for all $\mathbf{t}\in(0,\infty)^n$. Then, by (6), we see that

$$\begin{aligned}\|\mathscr{H}_{\varphi_\delta}\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})} &\leqslant \int_0^\infty\cdots\int_0^\infty \varphi_\delta(t_1,\ldots,t_n)dt_1\ldots dt_n\\ &= \int_\delta^1\cdots\int_\delta^1 \varphi(t_1,\ldots,t_n)dt_1\ldots dt_n<\infty\end{aligned}$$

and

$$\begin{aligned}
&\|\mathscr{H}_\varphi - \mathscr{H}_{\varphi_\delta}\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})} \\
&\leqslant \int_0^\infty\cdots\int_0^\infty [\varphi(t_1,\ldots,t_n)-\varphi_\delta(t_1,\ldots,t_n)]dt_1\ldots dt_n \\
&= \int_{(0,1]^n\setminus[\delta,1]^n} \varphi(\mathbf{t})d\mathbf{t} < \infty.
\end{aligned} \tag{7}$$

For any $\varepsilon > 0$, we define the function $F_\varepsilon : \mathbb{C}^n_+ \to \mathbb{C}$ by

$$F_\varepsilon(z_1,\ldots,z_n) = \prod_{j=1}^n \frac{1}{(z_j+i)^{1+\varepsilon}}$$

where $\zeta^{1+\varepsilon} = |\zeta|^{1+\varepsilon}e^{i(1+\varepsilon)\arg\zeta}$ for all $\zeta\in\mathbb{C}$. Denote by f_ε the boundary value function of F_ε, that is, $f_\varepsilon(\mathbf{x}) = \lim_{\mathbf{y}\to 0} F_\varepsilon(\mathbf{x}+i\mathbf{y})$. Then, by Theorem 7,

$$\|f_\varepsilon\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})} \sim \|F_\varepsilon\|_{\mathscr{H}^1_a(\mathbb{C}^n_+)} = \left[\int_{-\infty}^\infty \frac{1}{\sqrt{x^2+1}^{1+\varepsilon}}dx\right]^n < \infty, \tag{8}$$

where the constants are independent of ε.

For all $\mathbf{z} = \mathbf{x}+i\mathbf{y} = (x_1+iy_1,\ldots,x_n+iy_n) = (z_1,\ldots,z_n) \in \mathbb{C}^n_+$, by the Fubini theorem and Theorem 7, we get

$$\begin{aligned}
&\left(\mathscr{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon\int_{(0,\infty)^n}\varphi_\delta(\mathbf{t})d\mathbf{t}\right) * (\otimes_{j=1}^n P_{y_j})(\mathbf{x}) \\
&= \int_0^\infty\cdots\int_0^\infty \prod_{j=1}^n \frac{1}{(\frac{z_j}{t_j}+i)^{1+\varepsilon}}\frac{\varphi_\delta(t_1,\ldots,t_n)}{t_1\ldots t_n}dt_1\ldots dt_n \\
&\quad -\prod_{j=1}^n\frac{1}{(z_j+i)^{1+\varepsilon}}\int_0^\infty\cdots\int_0^\infty \varphi_\delta(t_1,\ldots,t_n)dt_1\ldots dt_n \\
&= \int_\delta^1\cdots\int_\delta^1 [\phi_{\varepsilon,\mathbf{z}}(t_1,\ldots,t_n)-\phi_{\varepsilon,\mathbf{z}}(1,\ldots,1)]\varphi(t_1,\ldots,t_n)dt_1\ldots dt_n,
\end{aligned}$$

where $\phi_{\varepsilon,\mathbf{z}}(t_1,\ldots,t_n) := \prod_{j=1}^n \frac{t_j^\varepsilon}{(z_j+it_j)^{1+\varepsilon}}$. For any $\mathbf{t} = (t_1,\ldots,t_n)\in[\delta,1]^n$, a simple calculus gives

$$\begin{aligned}
&|\phi_{\varepsilon,\mathbf{z}}(t_1,\ldots,t_n)-\phi_{\varepsilon,\mathbf{z}}(1,\ldots,1)| \\
&\leqslant \sup_{s\in[0,1]}\sum_{j=1}^n |t_j-1|\left|\frac{\partial\phi_{\varepsilon,\mathbf{z}}}{\partial t_j}(t_j+s(1-t_j))\right| \\
&\leqslant \sum_{j=1}^n\left(\frac{\varepsilon\delta^{-2}}{\sqrt{x_j^2+1}^{1+\varepsilon}}+\frac{(1+\varepsilon)\delta^{-2}}{\sqrt{x_j^2+1}^{2+\varepsilon}}\right)\prod_{\substack{k=1\\k\neq j}}^n\frac{\delta^{-1}}{\sqrt{x_k^2+1}^{1+\varepsilon}}.
\end{aligned}$$

Therefore, by Theorem 7 again,

$$\begin{aligned}
&\left\|\mathscr{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon \int_{(0,\infty)^n} \varphi_\delta(\mathbf{t})d\mathbf{t}\right\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})} \\
&\lesssim \left\| \sup_{(y_1,\ldots,y_n)\in(0,\infty)^n} \left(\mathscr{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon \int_{(0,\infty)^n} \varphi_\delta(\mathbf{t})d\mathbf{t}\right) * (\otimes_{j=1}^n P_{y_j})\right\|_{L^1(\mathbb{R}^n)} \\
&\leqslant \int_\delta^1 \cdots \int_\delta^1 \varphi(t_1,\ldots,t_n)dt_1\ldots dt_n \times \\
&\quad \times \sum_{j=1}^n \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \left(\frac{\varepsilon\delta^{-2}}{\sqrt{x_j^2+1}^{1+\varepsilon}} + \frac{(1+\varepsilon)\delta^{-2}}{\sqrt{x_j^2+1}^{2+\varepsilon}}\right) \prod_{\substack{k=1\\k\neq j}}^n \frac{\delta^{-1}}{\sqrt{x_k^2+1}^{1+\varepsilon}} dx_1\ldots dx_n.
\end{aligned}$$

This, together with (8), yields

$$\begin{aligned}
&\frac{\left\|\mathscr{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon \int_{(0,\infty)^n} \varphi_\delta(\mathbf{t})d\mathbf{t}\right\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})}}{\|f_\varepsilon\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})}} \qquad (9)\\
&\lesssim \int_\delta^1 \cdots \int_\delta^1 \varphi(t_1,\ldots,t_n)dt_1\ldots dt_n \times \\
&\quad \times \sum_{j=1}^n \frac{\delta^{1-n}\left[\varepsilon\delta^{-2}\int_{-\infty}^\infty \frac{1}{\sqrt{x_j^2+1}^{1+\varepsilon}}dx_j + (1+\varepsilon)\delta^{-2}\int_{-\infty}^\infty \frac{1}{\sqrt{x_j^2+1}^{2+\varepsilon}}dx_j\right]}{\int_{-\infty}^\infty \frac{1}{\sqrt{x_j^2+1}^{1+\varepsilon}}dx_j} \\
&\lesssim \int_\delta^1 \cdots \int_\delta^1 \varphi(t_1,\ldots,t_n)dt_1\ldots dt_n \times \\
&\quad \times \sum_{j=1}^n \left[\varepsilon\delta^{-1-n} + \frac{(1+\varepsilon)\delta^{-1-n}\int_{-\infty}^\infty \frac{1}{x_j^2+1}dx_j}{\int_{-\infty}^\infty \frac{1}{\sqrt{x_j^2+1}^{1+\varepsilon}}dx_j}\right] \to 0
\end{aligned}$$

as $\varepsilon \to 0$. As a consequence,

$$\begin{aligned}
\int_\delta^1 \cdots \int_\delta^1 \varphi(t_1,\ldots,t_n)dt_1\ldots dt_n &= \int_{(0,\infty)^n} \varphi_\delta(\mathbf{t})d\mathbf{t} \\
&\leqslant \|\mathscr{H}_{\varphi_\delta}\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})}.
\end{aligned}$$

This, together with (7), allows us to conclude that

$$\|\mathscr{H}_\varphi\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})} \geqslant \int_0^1 \cdots \int_0^1 \varphi(t_1,\ldots,t_n)dt_1\ldots dt_n$$

since $\lim_{\delta\to 0}\int_{(0,1]^n\setminus[\delta,1]^n}\varphi(\mathbf{t})d\mathbf{t} = 0$. Hence, by (6),

$$\|\mathscr{H}_\varphi\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})} = \int_0^1 \cdots \int_0^1 \varphi(t_1,\ldots,t_n)dt_1\ldots dt_n.$$

This completes the proof of Lemma 2. □

Now we are ready to give the proof of Theorem 1.

Proof of Theorem 1. By Lemma 2(i), it suffices to prove that

$$\int_0^\infty \cdots \int_0^\infty \varphi(t_1,\dots,t_n)dt_1\dots dt_n \leqslant \|\mathscr{H}_\varphi\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})} \tag{10}$$

provided $\mathscr{H}_\varphi$ is bounded on $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$. Indeed, by Lemma 1, we have

$$\int_0^\infty \cdots \int_0^\infty \varphi(t_1,\dots,t_n)dt_1\dots dt_n < \infty.$$

For any $m>0$, set $\varphi_m(\mathbf{t}) := \varphi(m\mathbf{t})\chi_{(0,1)^n}(\mathbf{t})$. Then, by Lemma 2(i), we see that

$$\begin{aligned} &\left\|\mathscr{H}_\varphi - \mathscr{H}_{\varphi_m(\frac{\cdot}{m})}\right\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})} \\ &= \left\|\mathscr{H}_{\varphi-\varphi_m(\frac{\cdot}{m})}\right\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})} \\ &\leqslant \int_0^\infty \cdots \int_0^\infty \left[\varphi(t_1,\dots,t_n) - \varphi_m\left(\frac{t_1}{m},\dots,\frac{t_n}{m}\right)\right] dt_1\dots dt_n \\ &= \int_{(0,\infty)^n\setminus(0,m)^n} \varphi(\mathbf{t})d\mathbf{t}. \end{aligned} \tag{11}$$

Noting that

$$\left\|f\left(\frac{\cdot}{m}\right)\right\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})} = m^n\|f(\cdot)\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})} \quad \text{and} \quad \mathscr{H}_{\varphi_m(\frac{\cdot}{m})} = \mathscr{H}_{\varphi_m} f\left(\frac{\cdot}{m}\right)$$

for all $f\in H^1(\mathbb{R}\times\cdots\times\mathbb{R})$, Lemma 2(ii) gives

$$\begin{aligned} \left\|\mathscr{H}_{\varphi_m(\frac{\cdot}{m})}\right\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})} &= m^n \left\|\mathscr{H}_{\varphi_m}\right\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})} \\ &= m^n \int_0^1 \cdots \int_0^1 \varphi_m(t_1,\dots,t_n)dt_1\dots dt_n \\ &= \int_0^m \cdots \int_0^m \varphi(t_1,\dots,t_n)dt_1\dots dt_n. \end{aligned}$$

Combining this with (11) allow us to conclude that

$$\|\mathscr{H}_\varphi\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})\to H^1(\mathbb{R}\times\cdots\times\mathbb{R})} \geqslant \int_0^\infty \cdots \int_0^\infty \varphi(t_1,\dots,t_n)dt_1\dots dt_n$$

since $\lim_{m\to\infty}\int_{(0,\infty)^n\setminus(0,m)^n}\varphi(\mathbf{t})d\mathbf{t} = 0$. This proves (10), and thus ends the proof of Theorem 1. □

From Theorem 6, one can define $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$ as the space of functions $f\in L^1(\mathbb{R}^n)$ such that

$$\|f\|_* := \sum_{\mathbf{e}\in\mathbb{E}} \|\mathbf{H}_\mathbf{e} f\|_{L^1(\mathbb{R}^n)} < \infty. \tag{12}$$

Our last result is the following:

THEOREM 8. *$\mathscr{H}_\varphi$ is bounded on $(H^1(\mathbb{R}\times\cdots\times\mathbb{R}),\|\cdot\|_*)$ if and only if (1) holds. Moreover, in that case,*

$$\|\mathscr{H}_\varphi\|_{(H^1(\mathbb{R}\times\cdots\times\mathbb{R}),\|\cdot\|_*)\to(H^1(\mathbb{R}\times\cdots\times\mathbb{R}),\|\cdot\|_*)} = \int_0^\infty\cdots\int_0^\infty \varphi(t_1,\ldots,t_n)dt_1\cdots dt_n$$

and, for any $\mathbf{e}\in\mathbb{E}$, $\mathscr{H}_\varphi$ commutes with $\mathbf{H_e}$ on $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$.

In order to prove Theorem 8, we need the following two lemmas.

LEMMA 3. *Let φ be such that (1) holds. Then, for any $\mathbf{e}\in\mathbb{E}$, $\mathscr{H}_\varphi$ commutes with the Hilbert transform $\mathbf{H_e}$ on $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$.*

LEMMA 4. *Let φ be such that (1) holds. Then:*

(i) *$\mathscr{H}_\varphi$ is bounded on $(H^1(\mathbb{R}\times\cdots\times\mathbb{R}),\|\cdot\|_*)$, moreover,*

$$\|\mathscr{H}_\varphi\|_{(H^1(\mathbb{R}\times\cdots\times\mathbb{R}),\|\cdot\|_*)\to(H^1(\mathbb{R}\times\cdots\times\mathbb{R}),\|\cdot\|_*)} \leqslant \int_0^\infty\cdots\int_0^\infty \varphi(t_1,\ldots,t_n)dt_1\ldots dt_n.$$

(ii) *If supp $\varphi\subset[0,1]^n$, then*

$$\|\mathscr{H}_\varphi\|_{(H^1(\mathbb{R}\times\cdots\times\mathbb{R}),\|\cdot\|_*)\to(H^1(\mathbb{R}\times\cdots\times\mathbb{R}),\|\cdot\|_*)} = \int_0^1\cdots\int_0^1 \varphi(t_1,\ldots,t_n)dt_1\ldots dt_n.$$

Proof of Lemma 3. Since Theorem 1 and the fact that H_j's are bounded on $H^1(\mathbb{R}\times\cdots\times\mathbb{R})$, it suffices to prove

$$\mathscr{H}_\varphi H_j f = H_j\mathscr{H}_\varphi f \tag{13}$$

for all $j\in\{1,\ldots,n\}$ and all $f\in H^1(\mathbb{R}\times\cdots\times\mathbb{R})$. Indeed, thanks to the ideas from [1, 15, 16] and Lemma 1(i), for almost every $\mathbf{y}=(y_1,\ldots,y_n)\in\mathbb{R}^n$,

$$\begin{aligned}\widehat{\mathscr{H}_\varphi H_j f}(\mathbf{y}) &= \int_0^\infty\cdots\int_0^\infty \widehat{H_j f}(t_1y_1,\ldots,t_ny_n)\varphi(t_1,\ldots,t_n)dt_1\ldots dt_n\\ &= \int_0^\infty\cdots\int_0^\infty (-i\,\mathrm{sign}(t_jy_j))\hat{f}(t_1y_1,\ldots,t_ny_n)\varphi(t_1,\ldots,t_n)dt_1\ldots dt_n\\ &= (-i\,\mathrm{sign}\,y_j)\widehat{\mathscr{H}_\varphi f}(\mathbf{y}) = \widehat{H_j\mathscr{H}_\varphi f}(\mathbf{y}).\end{aligned}$$

This proves (13), and thus ends proof of Lemma 3, since the uniqueness of the Fourier transform. □

Proof of Lemma 4. (i) For all $f\in H^1(\mathbb{R}\times\cdots\times\mathbb{R})$ and all $\mathbf{e}\in\mathbb{E}$, by Lemma 3 and Theorem 2, we get

$$\begin{aligned}\|\mathbf{H_e}\mathscr{H}_\varphi f\|_{L^1(\mathbb{R}^n)} &= \|\mathscr{H}_\varphi\mathbf{H_e}f\|_{L^1(\mathbb{R}^n)}\\ &\leqslant \int_0^\infty\cdots\int_0^\infty \varphi(t_1,\ldots,t_n)dt_1\ldots dt_n\|\mathbf{H_e}f\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

This proves that

$$\|\mathscr{H}_\varphi\|_{(H^1(\mathbb{R}\times\cdots\times\mathbb{R}),\|\cdot\|_*)\to(H^1(\mathbb{R}\times\cdots\times\mathbb{R}),\|\cdot\|_*)} \leqslant \int_0^\infty \cdots \int_0^\infty \varphi(t_1,\dots,t_n)dt_1\dots dt_n.$$

(ii) The proof is similar to that of Lemma 2(ii) and will be omitted. The key point is the estimate (9) and the fact that $\|\cdot\|_* \sim \|\cdot\|_{H^1(\mathbb{R}\times\cdots\times\mathbb{R})}$. □

Proof of Theorem 8. The proof is similar to that of Theorem 1 by Lemma 4. We leave the details to the interested readers. □

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