

LANDEN INEQUALITIES FOR A CLASS OF HYPERGEOMETRIC FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, we study a class of Gaussian hypergeometric function ${}_2F_1(a, b; (a+b+1)/2; x)$ ($a, b > 0$), and find the maximal regions of ab plane in the first quadrant where the well-known Landen identities for the complete elliptic integrals of the first kind turn on respective inequalities valid for each $x \in (0, 1)$. Besides, the generalized Grötzsch ring function with two parameters $\mu_{a,b}(r)$ is introduced, and the analogs of duplication formula satisfied by Grötzsch ring function $\mu(r)$ for $\mu_{a,b}(r)$, in the form of inequalities, will be derived.

1. Introduction

For real numbers a, b , and c with $c \neq 0, -1, -2, \dots$, the Gaussian hypergeometric function is defined by [1, 8, 30, 40, 48, 49, 55, 72]

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}$$

for $x \in (-1, 1)$, where

$$(a, n) = a(a+1)(a+2)\cdots(a+n-1)$$

for $n = 1, 2, \dots$, and $(a, 0) = 1$ for $a \neq 0$. The function $F(a, b; c; x)$ is called zero-balanced when $c = a + b$. The limiting values for $F(a, b; c; x)$ at 1 are as follows (see [8, Theorem 1.19 and 1.48])

$$\begin{cases} F(a, b; c; 1) = \Gamma(c)\Gamma(c-a-b)/[\Gamma(c-a)\Gamma(c-b)], & a+b < c, \\ B(a, b)F(a, b; c; z) + \log(1-z) = R(a, b) + O((1-z)\log(1-z)), & a+b = c, \\ F(a, b; c; z) = (1-z)^{c-a-b}F(c-a, c-b; c; z), & a+b > c. \end{cases} \quad (1.1)$$

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Here $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ ($\text{Re}(x) > 0$) [60, 61, 68, 69, 70, 74, 75, 76] and $B(p, q) = [\Gamma(p)\Gamma(q)]/[\Gamma(p+q)]$ ($\text{Re}(p) > 0, \text{Re}(q) > 0$) are the classical gamma function and beta functions, respectively, and

$$R(a, b) = -\psi(a) - \psi(b) - 2\gamma, \quad R\left(\frac{1}{2}, \frac{1}{2}\right) = \log 16,$$

$\psi(z) = \Gamma'(z)/\Gamma(z)$ ($\text{Re}(z) > 0$) [66, 67, 77] and γ is the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right) = 0.577 \dots$$

As special cases of hypergeometric functions, the complete elliptic integral of the first kind $\mathcal{K}(r)$ and the second kind $\mathcal{E}(r)$ can be expressed as

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{-1/2} dt, \quad r \in [0, 1), \quad \mathcal{K}(1^-) = +\infty,$$

$$\mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{1/2} dt, \quad r \in [0, 1],$$

respectively. Elliptic integrals have been studied over three centuries. Many important applications were found in physics, engineering, theory of mean values, number theory and other related fields [1, 2, 13, 14, 16, 25, 26, 27, 28, 32, 37, 38, 41, 43, 47, 58]. Recently, they also occur frequently in geometric function theory, particularly, these integrals enable us to investigate a lot of special functions that are indispensable for the study of quasiconformal mappings [3, 4, 5, 6, 11, 33]. For more properties and inequalities for $\mathcal{K}(r)$ and $\mathcal{E}(r)$ see e.g. [15, 17, 18, 19, 20, 21, 22, 23, 24, 31, 45, 50, 51, 52, 53, 57, 62, 64, 65, 73]. Recall that some of the most important properties of the elliptic integral $\mathcal{K}(r)$ are Landen identities proved in 1771 [59, p. 507]

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), \quad \mathcal{K}\left(\frac{1-r}{1+r}\right) = \frac{1+r}{2}\mathcal{K}(r'). \tag{1.2}$$

Here and in what follows $r' \equiv \sqrt{1-r^2}$. Equation (1.2) is equivalent to

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{4r}{(1+r)^2}\right) = (1+r)F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right), \tag{1.3}$$

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{1-r}{1+r}\right)^2\right) = \frac{1+r}{2}F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-r^2\right).$$

In 1992, a natural problem related to Landen identities was proposed by Anderson et al. in [7, p. 79]:

PROBLEM 1.1. Find an analog of Landen’s transformation formulas in (1.2) for the zero-balanced hypergeometric function $F(a, b; a+b; x)$ ($a, b > 0$).

Later, Qiu and Vuorinen [35, Theorem 1.2] claimed that the function

$$s(r) = (1 + \sqrt{r})F(a, b; a + b; r) - F\left(a, b; a + b; \frac{4\sqrt{r}}{(1 + \sqrt{r})^2}\right)$$

is increasing in $r \in (0, 1)$ for $(a, b) \in \{a, b > 0 | a + b \leq 1\}$, and derived the Landen inequality

$$F\left(a, b; a + b; \frac{4r}{(1 + r)^2}\right) \leq (1 + r)F(a, b; a + b; r^2) \tag{1.4}$$

for each $r \in (0, 1)$ and $(a, b) \in \{a, b > 0 | a + b \leq 1\}$. Unfortunately, there is an error in the proof of the monotonicity of $s(r)$. Simić and Vuorinen [39, Lemma 3.2] gave a correct proof, and showed that $s(r)$ is also increasing on $(0, 1)$ for $(a, b) \in \{a, b > 0 | a + b > 1, ab \leq 1/4\}$, and decreasing for $(a, b) \in \{a, b > 0 | 1/a + 1/b \leq 4\}$. Besides, for each $r \in (0, 1)$, they found the maximal regions of ab plane in the first quadrant where equalities (1.2) become (1.4) or its reversed inequality.

Let \mathbf{B}^2 be the unit disk in the plane and $\mu(r)$ the modulus of the plane Grötzsch ring $\mathbf{B}^2 \setminus [0, r]$ for $r \in (0, 1)$. Then $\mu(r)$, called the Grötzsch ring function, has the following explicit expression in terms of complete elliptic integrals (see [29, p. 60])

$$\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \quad r \in (0, 1). \tag{1.5}$$

By (1.2) and (1.5), it is easy to see that

$$\mu(r) = 2\mu\left(\frac{2\sqrt{r}}{1+r}\right), \quad \mu(r) = \frac{1}{2}\mu\left(\frac{1-r'}{1+r'}\right). \tag{1.6}$$

The Grötzsch ring function plays a very important role in quasi-conformal theory and quasi-regular theory (see [8, 42]). Moreover, using $\mu(r)$, the classical modular equation of signature 2 and degree p ($p > 1$) in number theory also can be rewritten as (see [43])

$$\mu(s) = p\mu(r), \quad 0 < r < 1.$$

Because of its importance, numerous properties and inequalities of $\mu(r)$ have been obtained by many authors in the recent years (see [8, 10, 34]).

On the other hand, in the 1990's, Anderson, Vuorinen, Vamanamurthy and Qiu et al. began to study a natural generalization of $\mu(r)$ defined by

$$\mu_a(r) = \frac{\pi}{2 \sin(\pi a)} \frac{F(a, 1 - a; 1; 1 - r^2)}{F(a, 1 - a; 1; r^2)}, \quad r \in (0, 1)$$

in the theory of Ramanujan generalized modular equation, and a lot of beautiful properties and inequalities of $\mu(r)$ have been extended to the function $\mu_a(r)$ [12, 34, 36, 44, 46, 71]. For instance, Qiu and Vuorinen [34, 36] derived the analogs of the duplication formula and logarithmic inequalities satisfied by $\mu(r)$ for $\mu_a(r)$. The infinite-product representations for $\mu_{1/3}(r)$ and $\mu_{1/4}(r)$ which involves only r are established in [54, 56].

The objective of this paper is twofold. The first one is to study a class of hypergeometric function $F(a, b; (a + b + 1)/2; x)$ with $a, b > 0$, and prove a generalization of Landen identities for it in the form of inequalities analogously. The other one is to introduce the generalized Grötzsch ring function with two parameters defined by

$$\mu_{a,b}(r) = \frac{B(a,b)}{2} \frac{F(a, b; (a + b + 1)/2; 1 - r^2)}{F(a, b; (a + b + 1)/2; r^2)}, \quad r \in (0, 1), \tag{1.7}$$

and find the analogs of identities (1.6) for $\mu_{a,b}(r)$.

According to the definition of $\mu_{a,b}(r)$ in (1.7), it follows that $\mu_a(r) = \mu_{a,1-a}(r)$, $\mu(r) = \mu_{1/2}(r) = \mu_{1/2,1/2}(r)$, and the limiting values of $\mu_{a,b}(r)$ at 0 and 1 are

$$\begin{aligned} \mu_{a,b}(0^+) &= \lim_{r \rightarrow 0^+} \frac{B(a,b)}{2} F\left(a, b; \frac{a+b+1}{2}; 1-r^2\right) \\ &= \begin{cases} \frac{B(a,b)}{2} H(a,b), & a+b < 1; \\ +\infty, & a+b \geq 1, \end{cases} \end{aligned} \tag{1.8}$$

$$\mu_{a,b}(1^-) = \lim_{r \rightarrow 1^-} \frac{B(a,b)}{2F(a, b; \frac{a+b+1}{2}; r^2)} = \begin{cases} \frac{B(a,b)}{2H(a,b)}, & a+b < 1; \\ 0, & a+b \geq 1, \end{cases} \tag{1.9}$$

respectively. Here and in what follows

$$H(a, b) = \frac{B(\frac{a+b+1}{2}, \frac{1-a-b}{2})}{B(\frac{1+b-a}{2}, \frac{1+a-b}{2})}.$$

Moreover, due to [9, Lemmas 4.5, 4.6 and 4.8], $\mu_{a,b}(r)$ satisfies the following derivative formula

$$\frac{d\mu_{a,b}(r)}{dr} = -\frac{\Gamma(\frac{a+b+1}{2})^2}{\Gamma(a+b)} \frac{1}{r^{a+b} r^{a+b+1} F(a, b; (a + b + 1)/2; r^2)^2}. \tag{1.10}$$

2. Main results

For convenience, we first list some regions in $\{(a, b) \in \mathbb{R}^2 | a > 0, b > 0\}$ (see Figure 1):

$$\begin{aligned} D_1 &= \{(a, b) | a, b > 0, a + b \leq 1\}, \\ D_2 &= \left\{ (a, b) | a, b > 0, a + b > 1, ab + \frac{a+b}{2} - \frac{3}{4} < 0 \right\}, \\ D_3 &= \left\{ (a, b) | a, b > 0, ab + \frac{a+b}{2} - \frac{3}{4} \geq 0, ab - \frac{a+b+1}{8} < 0 \right\}, \\ D_4 &= \left\{ (a, b) | a, b > 0, ab - \frac{a+b+1}{8} \geq 0 \right\}, \end{aligned}$$

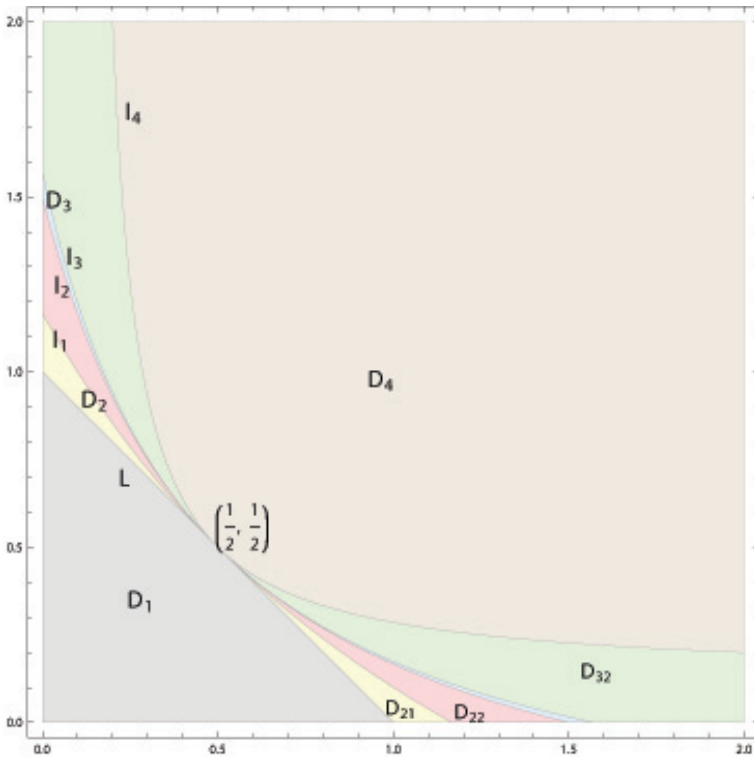


Figure 1: The regions $D_1, D_2, D_3, D_4, D_{21}, D_{22}, D_{31},$ and D_{32} , where $L: a + b = 1, l_1: ab + 3(a+b)/2 - 7/4 = 0, l_2: ab + (a+b)/2 - 3/4 = 0, l_3: ab + 7(a+b)/16 - 11/16 = 0, l_4: ab - (a+b+1)/8 = 0$.

$$\begin{aligned}
 D_{21} &= \left\{ (a, b) \mid a, b > 0, a + b > 1, ab + \frac{3}{2}(a + b) - \frac{7}{4} < 0 \right\}, \\
 D_{22} &= \left\{ (a, b) \mid a, b > 0, ab + \frac{3}{2}(a + b) - \frac{7}{4} \geq 0, ab + \frac{a + b}{2} - \frac{3}{4} < 0 \right\}, \\
 D_{31} &= \left\{ (a, b) \mid a, b > 0, ab + \frac{a + b}{2} - \frac{3}{4} \geq 0, ab + \frac{7}{16}(a + b) - \frac{11}{16} < 0 \right\}, \\
 D_{32} &= \left\{ (a, b) \mid a, b > 0, ab + \frac{7}{16}(a + b) - \frac{11}{16} \geq 0, ab - \frac{a + b + 1}{8} < 0 \right\}.
 \end{aligned}$$

Clearly, $D_1 \cup D_2 \cup D_3 \cup D_4 = \{(a, b) \in \mathbb{R}^2 \mid a > 0, b > 0\}$, $D_{21} \subset D_2, D_{22} \subset D_2, D_{31} \subset D_3$ and $D_{32} \subset D_3$.

THEOREM 2.1. Let $a, b \in (0, \infty)$. Then the inequality

$$F\left(a, b; \frac{a + b + 1}{2}; \frac{4r}{(1 + r)^2}\right) \leq (1 + r)F\left(a, b; \frac{a + b + 1}{2}; r^2\right) \tag{2.1}$$

holds for all $r \in (0, 1)$ if and only if $(a, b) \in D_1$, and the reversed inequality

$$F\left(a, b; \frac{a+b+1}{2}; \frac{4r}{(1+r)^2}\right) \geq (1+r)F\left(a, b; \frac{a+b+1}{2}; r^2\right) \tag{2.2}$$

takes place for each $r \in (0, 1)$ if and only if $(a, b) \in D_4$, with equality in each instance if and only if $(a, b) = (1/2, 1/2)$.

In the remaining region $(a, b) \in D_2 \cup D_3$, neither of the above inequalities holds for each $r \in (0, 1)$.

THEOREM 2.2. For $a, b > 0$ with $(a, b) \neq (1/2, 1/2)$, define the function f on $(0, 1)$ by

$$f(r) = (1 + \sqrt{r})F\left(a, b; \frac{a+b+1}{2}; r\right) - F\left(a, b; \frac{a+b+1}{2}; \frac{4\sqrt{r}}{(1+\sqrt{r})^2}\right).$$

Then the following statements are true:

- (1) $f(r)$ is strictly increasing from $(0, 1)$ onto $(0, [R(a, b) - \log 16]/B(a, b))$ when $(a, b) \in L = \{(a, b) | a, b > 0, a + b = 1\}$;
- (2) $f(r)$ is strictly increasing from $(0, 1)$ onto $(0, H(a, b))$ when $(a, b) \in D_1 \setminus L$;
- (3) $f(r)$ is strictly decreasing from $(0, 1)$ onto $(-\infty, 0)$ when $(a, b) \in D_4$.

Moreover, for $(a, b) \in D_1 \setminus L$, the inequality

$$\begin{aligned} F\left(a, b; \frac{a+b+1}{2}; \frac{4r}{(1+r)^2}\right) &\leq (1+r)F\left(a, b; \frac{a+b+1}{2}; r^2\right) \\ &\leq F\left(a, b; \frac{a+b+1}{2}; \frac{4r}{(1+r)^2}\right) + H(a, b) \end{aligned} \tag{2.3}$$

holds for all $r \in (0, 1)$, and for $(a, b) \in L$, the following inequality is valid for each $r \in (0, 1)$,

$$\begin{aligned} F\left(a, b; \frac{a+b+1}{2}; \frac{4r}{(1+r)^2}\right) &\leq (1+r)F\left(a, b; \frac{a+b+1}{2}; r^2\right) \\ &\leq F\left(a, b; \frac{a+b+1}{2}; \frac{4r}{(1+r)^2}\right) + \frac{R(a, b) - \log 16}{B(a, b)}. \end{aligned} \tag{2.4}$$

THEOREM 2.3. For $(a, b) \in \{(a, b) | a, b > 0, ab \geq a + b - 10/9, a + b \geq 2\}$, define the function g on $(0, 1)$ by

$$g(r) = 2\mu_{a,b}\left(\frac{2\sqrt{r}}{1+r}\right) - \mu_{a,b}(r).$$

Then g is strictly increasing from $(0, 1)$ onto $(-\infty, 0)$. In particular, the inequality

$$2\mu_{a,b}\left(\frac{2\sqrt{r}}{1+r}\right) < \mu_{a,b}(r) \tag{2.5}$$

holds for each $r \in (0, 1)$ with $(a, b) \in \{(a, b) | a, b > 0, ab \geq a + b - 10/9, a + b \geq 2\}$.

REMARK 2.4. If $a + b = 1$, then our Theorems 2.1 and 2.2 lead to Theorem 2.1 and Lemma 3.2 in [39], respectively.

3. Proofs of Theorems 2.1 and 2.2

Thought out this section, we let

$$F(x) = F\left(a, b; \frac{a+b+1}{2}; x\right), \quad G(x) = F\left(a+1, b+1; \frac{a+b+3}{2}; x\right) \tag{3.1}$$

with $a, b > 0$ and $(a, b) \neq (1/2, 1/2)$, and

$$F^*(x) = F\left(\frac{1}{2}, \frac{1}{2}; 1; x\right), \quad G^*(x) = F\left(\frac{3}{2}, \frac{3}{2}; 2; x\right). \tag{3.2}$$

LEMMA 3.1. ([63, Theorem 2.1]) *Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ with $b_n > 0$ for all $n \in \{0, 1, 2, \dots\}$. Let $h(x) = f(x)/g(x)$ and $H_{f,g} = (f'/g')g - f$, then the following statements are true:*

(1) *If the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing), then $h(x)$ is strictly increasing (decreasing) on $(0, r)$;*

(2) *If the non-constant sequence $\{a_n/b_n\}$ is increasing (decreasing) for $0 < n \leq n_0$ and decreasing (increasing) for $n > n_0$, then the function h is strictly increasing (decreasing) on $(0, r)$ if and only if $H_{f,g}(r^-) \geq (\leq) 0$. While if $H_{f,g}(r^-) < (>) 0$, then there exists $x_0 \in (0, r)$ such that $h(x)$ is strictly increasing (decreasing) on $(0, x_0)$ and strictly decreasing (increasing) on (x_0, r) .*

LEMMA 3.2. (1) *The function $\phi(x) = F(x)/F^*(x)$ is strictly decreasing on $(0, 1)$ if $(a, b) \in D_1$ and strictly increasing on $(0, 1)$ if $(a, b) \in D_4$. Moreover, if $(a, b) \in D_2 \cup D_3$, then there exists $x_0 \in (0, 1)$ such that $\phi(x)$ is strictly decreasing on $(0, x_0)$ and strictly increasing on $(x_0, 1)$;*

(2) *The function $\varphi(x) = G(x)/G^*(x)$ is strictly decreasing on $(0, 1)$ if $(a, b) \in D_1$ and strictly increasing on $(0, 1)$ if $(a, b) \in D_4 \cup D_{32}$. In the remaining region, namely $x \in D_2 \cup D_{31}$, $\varphi(x)$ is piecewise monotone on $(0, 1)$.*

Proof. For part (1), let

$$A_n = \frac{(a, n)(b, n)}{\left(\frac{a+b+1}{2}, n\right) n!}, \quad A_n^* = \frac{(1/2, n)(1/2, n)}{(1, n)(n)!},$$

then

$$\phi(x) = \frac{F(x)}{F^*(x)} = \frac{\sum_{n=0}^{\infty} A_n x^n}{\sum_{n=0}^{\infty} A_n^* x^n}. \tag{3.3}$$

Simple computations show that the monotonicity of $\{A_n/A_n^*\}$ depends on the sign of

$$H_n = \left(\frac{a+b-1}{2}\right)n^2 + \left(ab + \frac{a+b}{2} - \frac{3}{4}\right)n + ab - \frac{a+b+1}{8}. \tag{3.4}$$

We divide the proof into three cases.

Case 1 $(a, b) \in D_1$. Then $H_n < 0$ for $n = 0, 1, 2, \dots$ follows from (3.4). Therefore, (3.3) and Lemma 3.1(1) lead to that $\phi(x)$ is strictly decreasing on $(0, 1)$.

Case 2 $(a, b) \in D_4$. Then (3.4) implies $H_n > 0$ for $n = 0, 1, 2, \dots$, and $\phi(x)$ is strictly increasing on $(0, 1)$ by (3.3) and Lemma 3.1(1).

Case 3 $(a, b) \in D_2 \cup D_3$. Then from (3.4) we conclude that the sequence $\{A_n/A_n^*\}$ is decreasing for $n \in \{1, 2, \dots, n_0\}$ and increasing for $n \geq n_0$, for some integer n_0 . Moreover, using (1.1) and the derivative formula of hypergeometric function

$$\frac{dF(a, b; c; x)}{dx} = \frac{ab}{c}F(a+1, b+1; c+1; x),$$

we get

$$\begin{aligned} H_{F, F^*}(x) &= \frac{8ab}{a+b+1}(1-x) \frac{F(a+1, b+1; (a+b+3)/2; x)}{F(1/2, 1/2; 2; x)} F\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \\ &\quad - F\left(a, b; \frac{a+b+1}{2}; x\right) \\ &= \frac{8ab}{a+b+1}(1-x) \frac{(1-x)^{-(a+b+1)/2} F\left(\frac{b-a+1}{2}, \frac{a-b+1}{2}, \frac{a+b+3}{2}; x\right)}{F(1/2, 1/2; 2; x)} F\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \\ &\quad - (1-x)^{\frac{1-a-b}{2}} F\left(\frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+1}{2}; x\right) \\ &= (1-x)^{\frac{1-a-b}{2}} \left[\frac{8ab}{a+b+1} \frac{F\left(\frac{b-a+1}{2}, \frac{a-b+1}{2}, \frac{a+b+3}{2}; x\right)}{F(1/2, 1/2; 2; x)} F\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \right. \\ &\quad \left. - F\left(\frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+1}{2}; x\right) \right], \\ \lim_{x \rightarrow 1^-} H_{F, F^*}(x) &= +\infty. \tag{3.5} \end{aligned}$$

Combining (3.5), (3.3) and Lemma 3.1(2) together with (1.1), we have that there exists $x_0 \in (0, 1)$ such that $\phi(x)$ is strictly decreasing on $(0, x_0)$ and strictly increasing on $(x_0, 1)$.

For part (2), let

$$B_n = \frac{(a+1, n)(b+1, n)}{\left(\frac{a+b+3}{2}, n\right) n!}, \quad B_n^* = \frac{(3/2, n)(3/2, n)}{(2, n)(n)!},$$

then

$$\phi(x) = \frac{G(x)}{G^*(x)} = \frac{\sum_{n=0}^{\infty} B_n x^n}{\sum_{n=0}^{\infty} B_n^* x^n}. \tag{3.6}$$

Note that the monotonicity of $\{B_n/B_n^*\}$ depends on the sign of

$$H_n^* = \left(\frac{a+b-1}{2}\right)n^2 + \left[ab + \frac{3(a+b)}{2} - \frac{7}{4}\right]n + 2ab + \frac{7(a+b)}{8} - \frac{11}{8}. \tag{3.7}$$

Moreover, for $(a, b) \in D_2 \cup D_{31}^*$, we have

$$\begin{aligned} H_{G,G^*}(x) &= \frac{2(a+1)(b+1)}{a+b+3} \frac{F(a+2, b+2; \frac{a+b+5}{2}; x)}{9F(\frac{5}{2}, \frac{5}{2}; 3; x)/8} F\left(\frac{3}{2}, \frac{3}{2}; 2; x\right) \\ &\quad - F\left(a+1, b+1; \frac{a+b+3}{2}; x\right) \\ &= \frac{16(a+1)(b+1)}{9(a+b+3)} (1-x)^{-\frac{a+b+1}{2}} \frac{F(\frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+5}{2}; x)}{F(\frac{1}{2}, \frac{1}{2}; 3; x)} F\left(\frac{1}{2}, \frac{1}{2}; 2; x\right) \\ &\quad - (1-x)^{-\frac{a+b+1}{2}} F\left(\frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+3}{2}; x\right) \\ &= (1-x)^{-\frac{a+b+1}{2}} \left[\frac{16(a+1)(b+1)}{9(a+b+3)} \frac{F(\frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+5}{2}; x)}{F(\frac{1}{2}, \frac{1}{2}; 3; x)} F\left(\frac{1}{2}, \frac{1}{2}; 2; x\right) \right. \\ &\quad \left. - F\left(\frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+3}{2}; x\right) \right], \end{aligned}$$

$$\begin{aligned} &\lim_{x \rightarrow 1^-} (1-x)^{\frac{a+b+1}{2}} H_{G,G^*}(x) \\ &= \frac{16(a+1)(b+1)}{9(a+b+3)} \frac{\Gamma(\frac{a+b+5}{2}) \Gamma(\frac{a+b+3}{2}) \Gamma(2)\Gamma(1)}{\Gamma(a+2)\Gamma(b+2)} \frac{\Gamma(\frac{5}{2})^2}{\Gamma(\frac{3}{2})^2} \frac{\Gamma(3)\Gamma(2)}{\Gamma(a+1)\Gamma(b+1)} \\ &= \left(\frac{a+b-1}{2}\right) \frac{\Gamma(\frac{a+b+3}{2}) \Gamma(\frac{a+b+1}{2})}{\Gamma(a+1)\Gamma(b+1)} > 0, \end{aligned}$$

and thereby,

$$\lim_{x \rightarrow 1^-} H_{G,G^*}(x) = +\infty. \tag{3.8}$$

It follows from (3.6)-(3.8) together with the similar arguments in part (1) that the assertions of part (2) hold true. Here we omit the details. \square

Proof of Theorem 2.1. Since $y(r) = 4r/(1+r)^2 > r^2$ for $0 < r < 1$, Lemma 3.2(1) shows that $\phi(r^2) > \phi(y(r))$ for $(a, b) \in D_1$ and $\phi(r^2) < \phi(y(r))$ for $(a, b) \in D_4$. Therefore, it follows from (1.2) that

$$\frac{F(r^2)}{F^*(r^2)} > \frac{F(y(r))}{F^*(y(r))}, \quad F(y(r)) < \frac{F^*(y(r))}{F^*(r^2)} F(r^2) = (1+r)F(r^2)$$

for $(a, b) \in D_1$, which implies (2.1).

Inequality (2.2) can be derived analogously. The remaining conclusions easily follow from Lemma 3.2(1). \square

Proof of Theorem 2.2. Clearly $f(0^+) = 0$. From (1.1) we get

$$f(1^-) = \begin{cases} H(a, b), & a + b < 1, \\ \frac{R(a, b) - \log 16}{B(a, b)}, & a + b = 1, \\ -\infty, & a + b > 1. \end{cases} \quad (3.9)$$

Let $x = x(r) = 4\sqrt{r}/(1 + \sqrt{r})^2$. Then

$$1 - x = \left(\frac{1 - \sqrt{r}}{1 + \sqrt{r}} \right)^2, \quad \frac{dx}{dr} = \frac{2(1 - \sqrt{r})}{\sqrt{r}(1 + \sqrt{r})^3}. \quad (3.10)$$

Differentiating f yields

$$\begin{aligned} 2\sqrt{r}(1 - \sqrt{r})f'(r) &= (1 - \sqrt{r})F(r) + 2\sqrt{r}(1 - r) \frac{2ab}{a + b + 1} G(r) \\ &\quad - \frac{2ab}{a + b + 1} \frac{4(1 - x)}{1 + \sqrt{r}} G(x). \end{aligned} \quad (3.11)$$

On the other hand, differentiating Landen identity (1.2) with replacing r by \sqrt{r} , we get

$$\frac{1}{4} \frac{2(1 - \sqrt{r})}{\sqrt{r}(1 + \sqrt{r})^3} G^*(x) = \frac{1}{2\sqrt{r}} F^*(r) + \frac{1}{4} (1 + \sqrt{r}) G^*(r),$$

namely,

$$\frac{1 - x}{1 + \sqrt{r}} \frac{G^*(x)}{G^*(r)} = \frac{(1 - \sqrt{r})F^*(r)}{G^*(r)} + \frac{\sqrt{r}}{2} (1 - r). \quad (3.12)$$

Since φ defined as in Lemma 3.2(2) is strictly decreasing in $(0, 1)$ on D_1 , we conclude that $\varphi(r) > \varphi(x)$ for all $r \in (0, 1)$, that is

$$G(x) < \frac{G^*(x)}{G^*(r)} G(r). \quad (3.13)$$

Equations (3.11) and (3.12) together with inequality (3.13) lead to

$$\begin{aligned} &2\sqrt{r}(1 - \sqrt{r})f'(r) \\ &= (1 - \sqrt{r})F(r) + 2\sqrt{r}(1 - r) \frac{2ab}{a + b + 1} G(r) - \frac{2ab}{a + b + 1} \frac{4(1 - x)}{1 + \sqrt{r}} G(x) \\ &> (1 - \sqrt{r})F(r) + 2\sqrt{r}(1 - r) \frac{2ab}{a + b + 1} G(r) - \frac{2ab}{a + b + 1} \frac{4(1 - x)}{1 + \sqrt{r}} \frac{G^*(x)}{G^*(r)} G(r) \\ &= (1 - \sqrt{r})F(r) + 2\sqrt{r}(1 - r) \frac{2ab}{a + b + 1} G(r) - \frac{8ab}{a + b + 1} \left[\frac{(1 - \sqrt{r})F^*(r)}{G^*(r)} + \frac{\sqrt{r}}{2} (1 - r) \right] G(r) \\ &= (1 - \sqrt{r}) \left[F(r) - \frac{8ab}{a + b + 1} \frac{F^*(r)}{G^*(r)} G(r) \right] \\ &= (1 - \sqrt{r}) \frac{4F(r)^2}{G^*(r)} \left(\frac{F^*(r)}{F(r)} \right)'. \end{aligned} \quad (3.14)$$

Lemma 3.2(1) shows that $r \rightarrow F^*(r)/F(r)$ is strictly increasing in $(0, 1)$ on $(a, b) \in D_1$. Thus by (3.14) one has $f'(r) > 0$ for all $r \in (0, 1)$ when $(a, b) \in D_1$, and f is strictly increasing in $(0, 1)$.

On the other hand, since φ defined as in Lemma 3.2(2) is strictly increasing in $(0, 1)$ on D_4 , we have $\varphi(x) > \varphi(r)$, namely

$$G(x) > \frac{G^*(x)}{G^*(r)} G(r).$$

With the similar argument, one has

$$\sqrt{r} f'(r) < \frac{2F(r)^2}{G^*(r)} \left(\frac{F^*(r)}{F(r)} \right)' < 0,$$

because $F(r)/F^*(r)$ is strictly increasing in $(0, 1)$ on D_4 by Lemma 3.2(1). Hence $f(r)$ is strictly decreasing in $(0, 1)$ on D_4 . It is now obvious that the remaining conclusions hold. \square

4. Proof of Theorem 2.3

LEMMA 4.1. Let $x' = \sqrt{1-x^2}$ for $0 < x < 1$. Then the function

$$f(x) = \frac{(xx')^{(a+b-1)/2} F\left(a, b; \frac{a+b+1}{2}; x^2\right)}{F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right)} \tag{4.1}$$

is strictly increasing on $(0, 1)$ when $(a, b) \in \{(a, b) | a, b > 0, ab \geq a + b - 10/9, a + b \geq 2\}$.

Proof. Differentiating f gives

$$\begin{aligned} f'(x) = & \frac{1}{F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right)^2} \left\{ \left[\frac{a+b-1}{2} (xx')^{\frac{a+b-3}{2}} \left(\frac{1-2x^2}{x'} \right) F\left(a, b; \frac{a+b+1}{2}; x^2\right) \right. \right. \\ & + (xx')^{\frac{a+b-1}{2}} \frac{2ab}{a+b+1} F\left(a+1, b+1; \frac{a+b+3}{2}; x^2\right) \cdot 2x \Big] F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right) \\ & \left. - (xx')^{\frac{a+b-1}{2}} F\left(a, b; \frac{a+b+1}{2}; x^2\right) \cdot \frac{x}{2} F\left(\frac{3}{2}, \frac{3}{2}; 2; x^2\right) \right\}, \end{aligned}$$

and thereby

$$\begin{aligned} f_1(x) & \equiv x'(xx')^{-\frac{a+b-3}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right)^2 f'(x) \\ & = \frac{a+b-1}{2} (1-2x^2) F\left(a, b; \frac{a+b+1}{2}; x^2\right) F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right) \end{aligned}$$

$$\begin{aligned}
& + \frac{4ab}{a+b+1} x^2 x'^2 F\left(a+1, b+1; \frac{a+b+3}{2}; x^2\right) F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right) \\
& - \frac{x^2 x'^2}{2} F\left(a, b; \frac{a+b+1}{2}; x^2\right) F\left(\frac{3}{2}, \frac{3}{2}; 2; x^2\right). \tag{4.2}
\end{aligned}$$

Since $x'^2 F(3/2, 3/2; 2; x^2) = F(1/2, 1/2; 2; x^2) \leq F(1/2, 1/2; 1; x^2)$ for $x \in (0, 1)$, it follows from (4.2) that

$$\begin{aligned}
f_1(x) & \geq \frac{a+b-1}{2} (1-2x^2) F\left(a, b; \frac{a+b+1}{2}; x^2\right) F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right) \\
& + \frac{4ab}{a+b+1} x^2 x'^2 F\left(a+1, b+1; \frac{a+b+3}{2}; x^2\right) F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right) \\
& - \frac{x^2}{2} F\left(a, b; \frac{a+b+1}{2}; x^2\right) F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right) \\
& = F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right) \left[\frac{1}{2} (a+b-1 - (2a+2b-1)x^2) F\left(a, b; \frac{a+b+1}{2}; x^2\right) \right. \\
& \quad \left. + \frac{4ab}{a+b+1} x^2 (1-x^2) F\left(a+1, b+1; \frac{a+b+3}{2}; x^2\right) \right]. \tag{4.3}
\end{aligned}$$

Let $f_2(x) = f_1(x)/F(1/2, 1/2; 1; x^2)$, then from (4.3) we get

$$\begin{aligned}
f_2(x) & \geq \frac{a+b-1}{2} \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{\binom{a+b+1}{2}, n} \frac{x^{2n}}{n!} - (a+b-\frac{1}{2}) \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{\binom{a+b+1}{2}, n} \frac{x^{2n+2}}{n!} \\
& + \frac{4ab}{a+b+1} \sum_{n=0}^{\infty} \frac{(a+1,n)(b+1,n)}{\binom{a+b+3}{2}, n} \frac{x^{2n+2}}{n!} \\
& - \frac{4ab}{a+b+1} \sum_{n=0}^{\infty} \frac{(a+1,n)(b+1,n)}{\binom{a+b+3}{2}, n} \frac{x^{2n+4}}{n!} \\
& = \frac{a+b-1}{2} + \left[\frac{ab(a+b-1)}{a+b+1} - (a+b-\frac{1}{2}) + \frac{4ab}{a+b+1} \right] x^2 \\
& + \frac{a+b-1}{2} \sum_{n=0}^{\infty} \frac{(a,n+2)(b,n+2)}{\binom{a+b+1}{2}, n+2} \frac{x^{2n+4}}{(n+2)!} \\
& - (a+b-\frac{1}{2}) \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{\binom{a+b+1}{2}, n+1} \frac{x^{2n+4}}{(n+1)!} \\
& + 2 \sum_{n=0}^{\infty} \frac{(a,n+2)(b,n+2)}{\binom{a+b+1}{2}, n+2} \frac{x^{2n+4}}{(n+1)!} - 2 \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{\binom{a+b+1}{2}, n+1} \frac{x^{2n+4}}{n!} \\
& = \frac{a+b-1}{2} \left(1 - \frac{x^2}{a+b+1} \right) + \frac{ab(a+b-1) - (a-b)^2}{a+b+1} x^2 \\
& + \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{\binom{a+b+1}{2}, n+2} \frac{C_n}{(n+2)!} x^{2n+4}, \tag{4.4}
\end{aligned}$$

where

$$\begin{aligned}
 C_n &= \frac{a+b-1}{2}(n+a+1)(n+b+1) - (n+2) \left(a+b - \frac{1}{2}\right) \left(n+1 + \frac{a+b+1}{2}\right) \\
 &\quad + 2(n+a+1)(n+b+1)(n+2) - 2(n+1)(n+2) \left(n+1 + \frac{a+b+1}{2}\right) \\
 &= \left(\frac{a+b-2}{2}\right)n^2 + \frac{1}{4}(a+b+8ab-9)n \\
 &\quad + \frac{1}{2}[ab(a+b+3) - (a-b)^2 - (a+b+2)]. \tag{4.5}
 \end{aligned}$$

If $(a, b) \in \{(a, b) | a, b > 0, ab \geq a+b-10/9, a+b \geq 2\}$, then by elementary computations one has

$$\begin{aligned}
 &ab(a+b-1) - (a-b)^2 \geq \left(a+b - \frac{10}{9}\right)(a+b-1) - (a-b)^2 \\
 &= 4ab - \frac{19(a+b)}{9} + \frac{10}{9} \geq \frac{17(a+b)}{9} - \frac{30}{9} \geq \frac{4}{9}, \\
 &a+b+8ab-9 \geq a+b+8\left(a+b - \frac{10}{9}\right) - 9 = 9(a+b) - \frac{161}{9} \geq \frac{1}{9}, \\
 &\quad ab(a+b+3) - (a-b)^2 - (a+b+2) \\
 &\geq \left(a+b - \frac{10}{9}\right)(a+b+3) - (a-b)^2 - (a+b+2) \\
 &= 4ab + \frac{8(a+b)}{9} - \frac{16}{3} \geq 4\left(a+b - \frac{10}{9}\right) + \frac{8(a+b)}{9} - \frac{16}{3} \\
 &= \frac{44(a+b)}{9} - \frac{88}{9} \geq 0.
 \end{aligned}$$

Thus it follows from (4.4) and (4.5) that $f_2(x) > 0$ for all $x \in (0, 1)$. Therefore by (4.2) and (4.3), f is strictly increasing on $(0, 1)$ when $(a, b) \in \{(a, b) | a, b > 0, ab \geq a+b-10/9, a+b \geq 2\}$. \square

REMARK 4.2. The main purpose of Lemma 4.1 is to find some $a, b > 0$ such that the function f defined as in (4.1) is strictly monotone on $(0, 1)$. In fact, for $a+b < 1$, we have that $f(0^+) = f(1^-) = +\infty$, so that f is not monotone on $(0, 1)$; While for $a+b = 1$, Lemma 3.2(1) shows that f is strictly decreasing on $(0, 1)$. In the remaining region $(a, b) \in \{a+b > 1\}$, note that $f(0^+) = 0$ and $f(1^-) = +\infty$, we find its subregion such that f is strictly increasing on $(0, 1)$ in Lemma 4.1.

COROLLARY 4.3. For $(a, b) \in \{(a, b) | a, b > 0, ab \geq a+b-10/9, a+b \geq 2\}$, let $x = x(r) = 2\sqrt{r}/(1+r)$, then the Landen-type inequality

$$(xx')^{(a+b-1)/2} F\left(a, b; \frac{a+b+1}{2}; x^2\right) > (1+r)(rr')^{(a+b-1)/2} F\left(a, b; \frac{a+b+1}{2}; r^2\right) \tag{4.6}$$

holds for all $r \in (0, 1)$.

Proof. Since $x = x(r) = 2\sqrt{r}/(1+r) > r$ for all $r \in (0, 1)$, Lemma 4.1 implies that $f(x) > f(r)$ for $0 < r < 1$, and consequently inequality (4.6) follows. \square

Proof of Theorem 2.3. Note that $g(1^-) = 0$ by (1.9), and

$$\begin{aligned} & \lim_{r \rightarrow 0^+} g(r) \\ &= \lim_{r \rightarrow 0^+} \frac{B(a, b)}{2} \left[2F \left(a, b; \frac{a+b+1}{2}; \left(\frac{1-r}{1+r} \right)^2 \right) - F \left(a, b; \frac{a+b+1}{2}; 1-r^2 \right) \right] \\ &= \lim_{r \rightarrow 0^+} \left\{ B(a, b) \left[1 - \left(\frac{1-r}{1+r} \right)^{2(1-a-b)/2} F \left(\frac{-a+b+1}{2}, \frac{a-b+1}{2}; \frac{a+b+1}{2}; \left(\frac{1-r}{1+r} \right)^2 \right) \right] \right. \\ & \quad \left. - \frac{B(a, b)}{2} r^{1-a-b} F \left(\frac{-a+b+1}{2}, \frac{a-b+1}{2}; \frac{a+b+1}{2}; 1-r^2 \right) \right\} \\ &= \frac{B(a, b)}{2} \frac{\Gamma(\frac{a+b+1}{2})\Gamma(\frac{a+b-1}{2})}{\Gamma(a)\Gamma(b)} \lim_{r \rightarrow 0^+} \left[2 \cdot \left(\frac{2\sqrt{r}}{1+r} \right)^{1-a-b} - r^{1-a-b} \right] = -\infty. \end{aligned}$$

Let $x = 2\sqrt{r}/(1+r)$, then

$$x' = \sqrt{1-x^2} = \frac{1-r}{1+r}, \quad \frac{dx}{dr} = \frac{x'(1+x')^2}{2x}.$$

Differentiating g yields

$$\begin{aligned} g'(r) &= -2 \frac{\Gamma(\frac{a+b+1}{2})^2}{\Gamma(a+b)} \frac{1}{x^{a+b} x'^{a+b+1} F(a, b; \frac{a+b+1}{2}; x^2)^2} \cdot \frac{x'(1+x')^2}{2x} \\ & \quad + \frac{\Gamma(\frac{a+b+1}{2})^2}{\Gamma(a+b)} \cdot \frac{1}{r^{a+b} r'^{a+b+1} F(a, b; \frac{a+b+1}{2}; r^2)^2} \\ &= \frac{\Gamma(\frac{a+b+1}{2})^2}{\Gamma(a+b)} \frac{(1+x')^2}{(1+r)^2 x^{a+b+1} x'^{a+b} F(a, b; \frac{a+b+1}{2}; x^2)^2} \\ & \quad \times \left[\frac{(xx')^{a+b-1} F(a, b; \frac{a+b+1}{2}; x^2)^2}{(rr')^{a+b-1} F(a, b; \frac{a+b+1}{2}; r^2)^2} - (1+r)^2 \right]. \end{aligned} \tag{4.7}$$

Therefore, the monotonicity of g directly follows from (4.6) and (4.7). Inequality (2.5) is clear. \square

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