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MARCINKIEWICZ FUNCTIONS WITH HARDY SPACE KERNELS

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Abstract. In this paper we prove L^p estimates of Marcinkiewicz integral operators with kernels in the Hardy space and supported on general subvarieties. The considered subvarieties are of the type that caries partially the polynomial behavior as well as the behavior of convex functions. Results obtained in this paper improve as well as generalize known results.

1. Introduction and statement of results

Let $\mathbb{R}^n$, $n \geqslant 2$, be the n-dimensional Euclidean space and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. For non zero $y \in \mathbb{R}^n$, we let $y' = |y|^{-1}y$. Suppose that $\Omega \in L^1(\mathbb{S}^{n-1})$ is a homogeneous functions of degree zero on $\mathbb{R}^n$ and satisfies the cancellation condition

$$\int_{\mathbb{S}^{n-1}} \Omega(y')d\sigma(y') = 0. \tag{1}$$

For a suitable mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^d$ $(n,d \geqslant 2)$ and a measurable function $b : \mathbb{R}_+ \to \mathbb{R}$ consider the operator

$$\mu_{\Omega,\Phi,b}f(x) = \left(\int_{-\infty}^{\infty}\left|\int_{|y|\leqslant 2^t} f(x-\Phi(y))\frac{b(|y|)\Omega(y')}{|y|^{n-1}}dy\right|^2\frac{dt}{2^{2t}}\right)^{\frac{1}{2}}. \tag{2}$$

The main problem concerning the class of operators in (2) is to determine whether inequalities in the form $\left\|\mu_{\Omega,\Phi,b}f\right\|_p \leqslant C_p \left\|f\right\|_p$ hold for some $1 < p < \infty$. By specializing to the case $\Phi(y) = y$ and $b(t) = 1$, the resulting operator $\mu_\Omega = \mu_{\Omega,\Phi,b}$ reduces to the well known classical Marcinkiewicz integral operator introduced by E. M. Stein in [15]. In [15], E. M. Stein proved that μ_Ω is bounded on L^p for all $1 < p \leqslant 2$ provided that Ω is continuous and satisfies a Lip_α $(0 < \alpha \leqslant 1)$ condition on $\mathbb{S}^{n-1}$. Subsequently, A. Benedek, A. Calderón, and R. Panzone proved the L^p boundedness of μ_Ω for all $1 < p < \infty$ provided that Ω is continuously differentiable on $\mathbb{S}^{n-1}$ ([6]). In [10], Ding, Fan and Pan proved that μ_Ω is bounded on L^p for all $1 < p < \infty$ provided that Ω is in the Hardy space $H^1(\mathbb{S}^{n-1})$.

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When $\Phi(y) = \varphi(|y|)y'$ and $b(t) \equiv 1$, where φ satisfies heavy conditions such as the "finite doubling time condition" $\varphi'(Ct) \geqslant \varphi'(t)$ or "growth condition" $\varphi(2t) \leqslant c$ $\varphi(t)$, it can be shown using the oscillatory estimates in [14] that the special operator $\mu_{\Omega,\Phi,1}$ is bounded on $L^p(\mathbb{R}^n)$ provided that Ω is in the Hardy space $H^1(\mathbb{S}^{n-1})$. On the other hand, if $\Phi(y) = \varphi(|y|)y'$ where φ satisfies certain convexity assumptions, it shown in [1] that the special operator $\mu_{\Omega,\Phi,1}$ is bounded on L^p provided that $\Omega \in H^1(\mathbb{S}^{n-1})$.

However, for general mappings Φ and functions b, little is known about the boundedness of $\mu_{\Omega,\Phi,b}$ for kernels $\Omega \in H^1(\mathbb{S}^{n-1})$. It is our aim in this paper to consider this problem. More precisely, we seek L^p estimates of the operator $\mu_{\Omega,\Phi,b}$ when $b \in L^\infty(\mathbb{R}_+)$, the function Ω is in the space $H^1(\mathbb{S}^{n-1})$, and the mapping Φ is in the form $\Phi(y) = \varphi(|y|)y'$ with function φ caries partially the polynomial behavior as well as the behavior of convex functions. The examples per excellence are functions φ that behave like $\theta(t) = -t^3 + e^{-\frac{1}{t}}$. It is worth noticing that the function θ is neither convex nor polynomial.

For an integer $d \geqslant 0$, let $\mathscr{P}_d$ be the class of all real valued polynomials with degree at most d. Given $\lambda \in \mathbb{R}$. A function $\psi : [0,\infty) \to \mathbb{R}$ is said to belong to the class $\mathscr{PC}_\lambda(d)$ if there exists a polynomial $P \in \mathscr{P}_d$ and mapping $\varphi \in C^{d+1}([0,\infty))$ such that

i) $\psi(t) = P(t) + \lambda\varphi(t)$;
ii) $P(0) = 0$ and $\varphi^{(j)}(0) = 0$ for $0 \leqslant j \leqslant d$;
iii) $\varphi^{(j)}$ is positive nondecreasing on $(0,\infty)$ for $0 \leqslant j \leqslant d+1$.

A representation of the function ψ satisfying (i)–(iii) shall be referred to by the standard representation. It is clear that the class $\cup_{d\geqslant 0}(\mathscr{PC}_\lambda(d))$ contains properly the class of polynomials $\mathscr{P}_d$ as well as the class of convex increasing functions. In addition to the above stated example, one can easily verify that the function $\theta(t) = -t^2 + t^2\ln(1+t)$ is in $\mathscr{PC}_\lambda(2)$ which is neither convex nor polynomial.

Our main result in this paper is the following:

THEOREM 1.1. *Suppose that $\Phi(y) = \psi(|y|)y'$ with $\psi \in \mathscr{PC}_\lambda(d)$ for some $d \geqslant 0$ and $\lambda \in \mathbb{R}$. If $b \in L^\infty(\mathbb{R}_+)$ and $\Omega \in H^1(\mathbb{S}^{n-1})$ and satisfying (1), then the operator $\mu_{\Omega,\Phi,b}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ with L^p bounds independent of $\lambda \in \mathbb{R}$ and the coefficients of the particular polynomial involved in the standard representation of ψ.*

In light of the remark just before the statement of Theorem 1.1, it follows that Theorem 1.1 is a substantial improvement of the corresponding results in [1] and [10].

A similar result as in Theorem 1.1 can be obtained for the corresponding Marcinkiewicz integral operators that are related to area integral and Littlewood-Paley g_λ^* functions. In fact, we have the following:

THEOREM 1.2. *Suppose that $\Phi(y) = \psi(|y|)y'$ with $\psi \in \mathscr{PC}_\lambda(d)$ for some $d \geqslant 0$ and $\lambda \in \mathbb{R}$. If $b \in L^\infty(\mathbb{R}_+)$ and $\Omega \in H^1(\mathbb{S}^{n-1})$ and satisfying (1.1), then for $2 \leqslant p < \infty$ and $s > 1$, the operators $\widetilde{\mu}_{\Omega,\Phi,b}$ and $\mu^*_{\Omega,\Phi,b,s}$ satisfy*

$$\left\|\tilde{\mu}_{\Omega,\Phi,b}(f)\right\|_p \leqslant C_p \|f\|_p \tag{3}$$

$$\left\|\mu^{*}_{\Omega,\Phi,b,s}(f)\right\|_{p} \leqslant C_p \|f\|_p, \tag{4}$$

where

$$\tilde{\mu}_{\Omega,\Phi,b}f(x) = \left(\int_{\Upsilon(x)} \left|\int_{|y|\leqslant 2^t} f(z-\Phi(y))\frac{b(|y|)\Omega(y')}{|y|^{n-1}}dy\right|^2 \frac{dzdt}{2^{(2+n)t}}\right)^{\frac{1}{2}}, \tag{5}$$

$$\mu^{*}_{\Omega,\Phi,b,s}f(x) = \left(\int\int_{\mathbb{R}^{n+1}} \left|\int_{|y|\leqslant 2^t} f(z-\Phi(y))\frac{b(|y|)\Omega(y')}{|y|^{n-1}}dy\right|^2 \frac{2^{(ns-2-n)t}dzdt}{(2^t+|x-z|)^{ns}}\right)^{\frac{1}{2}}, \tag{6}$$

$\Upsilon(x) = \{(z,t) \in \mathbb{R}^{n+1} : |x-z| < 2^t\}$. *The constants C_p are independent of $\lambda \in \mathbb{R}$ and the coefficients of the particular polynomial involved in the standard representation of ψ.*

Throughout this paper, the letter C is a positive constant that may vary at each occurrence but it is independent of the essential variables. For any $x = (x_1, x_2, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n$, we shall let $\bar{x} = (x_1, x_2, \ldots, x_{n-1})$. Also, for any $\chi : \mathbb{R}^n \to \mathbb{R}$, we let $supp(\chi)$ denote the support of χ.

2. Oscillatory integral estimates

This section is devoted to obtain necessary estimates that we need to prove our results. We start by establishing the following main proposition:

PROPOSITION 2.1. *Let $\{\sigma_{t,\Omega,b} : t \in \mathbb{R}\}$ be the family of measures defined by*

$$(\sigma_{t,\Omega,b})\hat{}(\xi) = \frac{1}{2^t}\int_{2^{t-1}\leqslant|y|<2^t} \exp(-i(\xi\cdot\psi(|y|)y')\frac{b(|y|)\Omega(y')}{|y|^{n-1}}dy. \tag{7}$$

Suppose that there exists $\rho > 0$ such that

(a) $supp(\Omega) \subseteq \mathbb{S}^{n-1} \cap \{y \in \mathbb{R}^n : |y-e| < \rho\}$ *where* $e = (0,0,\ldots,1)$;

(b) $\|\Omega\|_\infty \leqslant \rho^{-(n-1)}$;

(c) $\int_{\mathbb{S}^{n-1}} \Omega(y')\ d\sigma(y') = 0$.

Then there exist a natural number d, a convex increasing function φ, and families of measures $\mathscr{F}_l = \{\upsilon^{(l)}_{t,\Omega} : t \in \mathbb{R}\}$, $0 \leqslant l \leqslant d+3$ such that

$$\upsilon_{t,\Omega}^{(d+3)} = \sigma_{t,\Omega,b} \tag{8}$$

$$\upsilon_{t,\Omega}^{(0)} = 0 \tag{9}$$

$$\left|(\upsilon_{t,\Omega}^{(d+3)})\hat{\ }(\xi)\right| \leqslant C\left|\lambda\varphi(2^t)\rho^{2\max\{0,\frac{1}{n}-\rho\}}\xi\right|^{-\frac{1}{4(d+1)}} \tag{10}$$

$$\left|(\upsilon_{t,\Omega}^{(d+3)})\hat{\ }(\xi)-(\upsilon_{t,\Omega}^{(d+2)})\hat{\ }(\xi)\right| \leqslant C\left|\lambda\varphi(2^{t+1})\rho^{2\max\{0,\frac{1}{n}-\rho\}}\xi\right| \tag{11}$$

$$\left|(\upsilon_{t,\Omega}^{(d+2)})\hat{\ }(\xi)\right| \leqslant C\left|\lambda\varphi(2^t)\rho^{\max\{0,\frac{1}{n}-\rho\}}\overline{\xi}\right|^{-\frac{1}{4(d+1)}} \tag{12}$$

$$\left|(\upsilon_{t,\Omega}^{(d+2)})\hat{\ }(\xi)-(\upsilon_{t,\Omega}^{(d+1)})\hat{\ }(\xi)\right| \leqslant C\left|\lambda\varphi(2^{t+1})\rho^{\max\{0,\frac{1}{n}-\rho\}}\overline{\xi}\right| \tag{13}$$

$$\left|(\upsilon_{t,\Omega}^{(d+1)})\hat{\ }(\xi)\right| \leqslant C\left|\lambda 2^{dt}\rho^{2\max\{0,\frac{1}{n}-\rho\}}\xi\right|^{-\frac{1}{4(d+1)}} \tag{14}$$

$$\left|(\upsilon_{t,\Omega}^{(d+1)})\hat{\ }(\xi)-(\upsilon_{t,\Omega}^{(d)})\hat{\ }(\xi)\right| \leqslant C\left|\lambda 2^{d(t+1)}\rho^{2\max\{0,\frac{1}{n}-\rho\}}\xi\right| \tag{15}$$

$$\left|(\upsilon_{t,\Omega}^{(l)})\hat{\ }(\xi)\right| \leqslant C\left|\lambda 2^{dt}\rho^{\max\{0,\frac{1}{n}-\rho\}}\overline{\xi}\right|^{-\frac{1}{4l}},\ \ 1\leqslant l\leqslant d \tag{16}$$

$$\left|(\upsilon_{t,\Omega}^{(l)})\hat{\ }(\xi)-(\upsilon_{t,\Omega}^{(l-1)})\hat{\ }(\xi)\right| \leqslant C\left|\lambda 2^{l(t+1)}\rho^{\max\{0,\frac{1}{n}-\rho\}}\overline{\xi}\right|,\ \ 1\leqslant l\leqslant d \tag{17}$$

Proof. We shall only prove the estimates (8)–(17) for the case $\rho < \frac{1}{n}$. The case $\rho \geqslant \frac{1}{n}$ follows by minor modifications but with simpler argument. Let $P \in \mathscr{P}_d$, $\varphi \in C^{d+1}([0,\infty))$, and $\lambda \in \mathbb{R}$ be as in the standard representation of ψ. Suppose that

$$P(t) = \sum_{k=1}^{d} c_k t^k.$$

For $1 \leqslant l \leqslant d$, let

$$P_l(t) = \sum_{k=1}^{l} c_k t^k$$

and

$$\psi_l(t) = \lambda\varphi(t) + P_l(t).$$

It is clear that

$$\psi_d(t) = \psi(t). \tag{18}$$

For $1 \leqslant l \leqslant d+3$, let $\Psi_{l,\lambda} : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\Psi_{d+3,\lambda}(y) = \psi(|y|)y', \tag{19}$$

$$\Psi_{d+2,\lambda}(y) = \lambda\varphi(|y|)(\tilde{y}',1) + P(|y|)y', \tag{20}$$

$$\Psi_{d+1,\lambda}(y) = \lambda\varphi(|y|)e + P(|y|)y' \tag{21}$$

and

$$\Psi_{l,\lambda}(y) = \lambda\varphi(|y|)e + P_l(|y|)(\tilde{y}',1)$$

for $1 \leqslant l \leqslant d$. Also, we let

$$\Psi_{0,\lambda}(y) = \lambda\varphi(|y|)e. \tag{22}$$

For $0 \leqslant l \leqslant d+3$, define the measures $\upsilon_{t,\Omega}^{(l)}$ by

$$(\upsilon_{t,\Omega}^{(l)})\hat{}(\xi) = \frac{1}{2^t} \int\limits_{2^{t-1} \leqslant |y| < 2^t} \exp(-i(\xi \cdot \Psi_{l,\lambda}(y))\frac{b(|y|)\Omega(y')}{|y|^{n-1}}dy. \tag{23}$$

Using the definition of $\Psi_{d+3,\lambda}$ and the cancellation property in (c), it follows that (8) and (9) hold trivially.

Now, we prove (10). Notice

$$\left|(\upsilon_{t,\Omega}^{(d+3)})\hat{}(\xi)\right|^2 \leqslant C\rho^{-2(n-1)} \int\limits_{Supp(\Omega)} \int\limits_{Supp(\Omega)} I(t,\xi',y',z')d\sigma(y')d\sigma(z') \tag{24}$$

where

$$I(t,\xi',y',z') = \left|\int_1^2 \exp(-i(|\xi|\,\psi(2^{t-1}r)(\xi'\cdot(y'-z'))))dr\right|. \tag{25}$$

By induction, it can be shown that

$$\varphi^{(d+1)}(t) \geqslant t^{-d-1}\varphi(t). \tag{26}$$

Thus,

$$\left|\frac{d^{d+1}}{dt^{d+1}}(\psi(t)))\right| = \left|\lambda\varphi^{(d+1)}(t)\right|. \tag{27}$$

Thus, by (26), (27), and Van der Corput's lemma [16], we get

$$I(t,\xi',y',z') \leqslant C\left|\varphi(2^t)\xi\cdot(y'-z')\right|^{-\frac{1}{d+1}};$$

when combined with the trivial estimate $I(t,\xi',y',z') \leqslant 1$, we get

$$I(t,\xi',y',z') \leqslant C\left|\varphi(2^t)\xi\cdot(y'-z')\right|^{-\frac{1}{2(d+1)}}. \tag{28}$$

By (24) and (28), we get

$$\left|(\upsilon_{t,\Omega}^{(d+3)})\hat{}(\xi)\right|^2 \leqslant \frac{C\rho^{-2n+2}}{|\varphi(2^t)\xi|^{\frac{1}{2(d+1)}}} \int\limits_{Supp(\Omega)} \sup_{\xi'\in\mathbb{S}^{n-1}} \int\limits_{Supp(\Omega)} \frac{d\sigma(y')}{|\xi'\cdot(y'-z')|^{\frac{1}{2(d+1)}}}d\sigma(z'). \tag{29}$$

By similar argument as in the proof of Lemma 5.12 in [2], we get

$$\sup_{\xi'\in\mathbb{S}^{n-1}} \int\limits_{Supp(\Omega)} \left|\xi'\cdot(y'-z')\right|^{-\frac{1}{2(d+1)}} d\sigma(y') \leqslant C\rho^{(n-1)}\left|\rho^2\right|^{-\frac{1}{2(d+1)}}; \tag{30}$$

when combined with (29) and the observation that $|Supp(\Omega)| \approx \rho^{n-1}$, we obtain

$$\left|(\upsilon_{t,\Omega}^{(d+3)})\hat{}(\xi)\right|^2 \leqslant C\left|\varphi(2^t)\rho^2\xi\right|^{-\frac{1}{2(d+1)}}$$

and hence (10).

Next, to see (11), we first notice that

$$\left|\Psi_{d+3,\lambda}(y) - \Psi_{d+2,\lambda}(y)\right| = |\lambda\varphi(|y|)|\left|y_n' - 1\right|.$$

Thus,

$$\begin{aligned}
&\left|(\upsilon_{t,\Omega}^{(d+3)})\hat{}(\xi) - (\upsilon_{t,\Omega}^{(d+2)})\hat{}(\xi)\right| \\
&\leqslant \left|\lambda\xi\varphi(2^{t+1})\right| \int_{Supp(\Omega)} \left|\Omega(y')\right| \left|y_n' - 1\right| d\sigma(y') \\
&\leqslant \left|\lambda\rho^2\xi\varphi(2^{t+1})\right| \|\Omega\|_1 \leqslant \left|\lambda\rho^2\xi\varphi(2^{t+1})\right|.
\end{aligned}$$

Now, we prove (12). By similar argument as that lead to (29), we get

$$\left|(\upsilon_{t,\Omega}^{(d+2)})\hat{}(\xi)\right|^2 \leqslant \frac{C\rho^{-2n+2}}{\left|\varphi(2^t)\overline{\xi}\right|^{\frac{1}{2(d+1)}}} \int_{Supp(\Omega)} \sup_{\xi'\in\mathbb{S}^{n-1}} \int_{Supp(\Omega)} \frac{d\sigma(y')}{\left|(\overline{\xi})'\cdot(\overline{y'}-\overline{z'})\right|^{\frac{1}{2(d+1)}}} d\sigma(z'). \tag{31}$$

By similar argument as in the proof of Lemma 5.12 in [2], we get

$$\sup_{(\overline{\xi})'\in\mathbb{S}^{n-2}} \int_{Supp(\Omega)} \left|(\overline{\xi})'\cdot(\overline{y'}-\overline{z'})\right|^{-\frac{1}{2(d+1)}} d\sigma(y') \leqslant C\rho^{(n-1)}|\rho|^{-\frac{1}{2(d+1)}}; \tag{32}$$

when combined with (31) implies (12). The proof of (13) follows by similar argument as that lead to (11). The verifications of (14)–(17) follow similar argument as that used in the verifications of the corresponding estimates (10)–(13). We omit details. □

We end this section by the following simple proposition which has its roots in [1]:

PROPOSITION 2.2. *Let $\varphi : [0,\infty) \to \mathbb{R}$ be convex increasing function with $\varphi(0) = 0$. Let $l,k \in \mathbb{Z}$. Suppose that r is a positive real number. For any real number t satisfying*

$$\log_2(2^{k-l-1}\varphi^{-1}(r^{-1})) < t < \log_2(2^{k-l+2}\varphi^{-1}(r^{-1})),$$

we have

$$\varphi(2^{t+l}) \leqslant 2^{k+2}r^{-1} \quad \textit{for} \quad k \leqslant -3 \tag{33}$$

and

$$\varphi(2^{t+l-1}) \geqslant 2^{-k-2}r^{-1} \quad \textit{for} \quad k \geqslant 3. \tag{34}$$

Proof. Since φ is convex increasing, it can be shown that

$$\varphi(2r) \geqslant 2\varphi(r). \tag{35}$$

In order to verify (33) and (34), one only needs to observe that

$$\varphi(2^{k-1}\varphi^{-1}(r^{-1})) \leqslant \varphi(2^{t+l}) \leqslant \varphi(2^{k+2}\varphi^{-1}(r^{-1})). \tag{36}$$

This completes the proof. $\square$

3. Maximal functions and the Hardy space $H^1(\mathbb{S}^{n-1})$

We start this section, by recalling the following lemma in [5]:

LEMMA 3.1. ([5]). *Let $\{\mu_k\}_{k\in\mathbf{Z}}$ and $\{\tau_k\}_{k\in\mathbb{Z}}$ be sequences of non negative Borel measures on $\mathbb{R}^n$. Let $L:\mathbb{R}^n\to\mathbb{R}^m$ be a linear transformation. Suppose that for all $k\in\mathbb{Z}$, $\xi\in\mathbb{R}^n$, for some $a\geqslant 2$, $\alpha, C>0$, and for some constant $B>1$, we have*
(i) $\|\mu_k\|\leqslant B$; $\|\tau_k\|\leqslant B$;
(ii) $|\hat{\mu}_k(\xi)|\leqslant CB(a^{kB}|L(\xi)|)^{-\frac{\alpha}{B}}$;
(iii) $|\hat{\mu}_k(\xi)-\hat{\tau}_k(\xi)|\leqslant CB(a^{kB}|L(\xi)|)^{\frac{\alpha}{B}}$;
(iv) Suppose that

$$\|\tau^*(f)\|_p\leqslant B\|f\|_p \text{ for all } 1<p\leqslant\infty \text{ and } f\in L^p(\mathbb{R}^n).$$

Then the inequality

$$\|\mu^*(f)\|_p\leqslant B\|f\|_p$$

holds for all $1<p\leqslant\infty$ and f in $L^p(\mathbb{R}^n)$. The constant C_p is independent of B and the linear transformation L.

Now, we prove the following result concerning maximal functions which is a generalization of Lemma 5.9 in [2]:

THEOREM 3.2. *Suppose that $\Psi:\mathbb{R}^n\to\mathbb{R}^d$ is a non constant mapping and that $\psi\in\mathscr{P}\mathscr{C}_\lambda(d)$ for some $d\geqslant 0$. Suppose also that $\lambda\in\mathbb{R}$. If $\Omega\in L^1(\mathbb{S}^{n-1})$ is homogeneous of degree zero on $\mathbb{R}^n$, then the maximal function $\mathscr{M}_{\Psi,\Omega}$ given by*

$$\mathscr{M}_{\Psi,\Omega}(f)(x)=\sup_{j\in\mathbb{Z}}\int\limits_{2^j\leqslant|y|\leqslant 2^{j+1}}\left|f(x-\psi(|y|)\Psi(y'))\right|\frac{|\Omega(y')|\,dy}{|y|^n} \tag{37}$$

satisfies

$$\left\|\mathscr{M}_{\Psi,\Omega}(f)\right\|_p\leqslant C_p\|\Omega\|_1\|f\|_p \tag{38}$$

for $1<p<\infty$. Here, the constant C_p is independent of λ, $\Psi(y')$, and the coefficients of the particular polynomial involved in the standard representation of ψ.

Proof. By change of variables, it can be shown that

$$\mathscr{M}_{\Psi,\Omega}(f)(x)\leqslant\int\limits_{\mathbb{S}^{n-1}}|\Omega(y')|\left(\sup_{j\in\mathbb{Z}}\int\limits_{2^j}^{2^{j+1}}\left|f(x-\psi(t)\Psi(y'))\right|\frac{dt}{t}\right)d\sigma(y').$$

Therefore, by generalized Minkowski's inequality we have

$$\left\|\mathscr{M}_{\Psi,\Omega}(f)\right\|_p \leqslant \int_{\mathbb{S}^{n-1}} \left|\Omega(y')\right| \left\| \left(\sup_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \left|f(x-\psi(t)\Psi(y'))\right| \frac{dt}{t} \right) \right\|_p d\sigma(y'). \tag{39}$$

Thus, to prove (38), we only need to prove that the family of maximal functions

$$\mathscr{M}_{\psi,z}(f)(x) = \sup_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \left|f(x-\psi(t)z)\right| \frac{dt}{t}, \ \ z\in\mathbb{R}^n$$

satisfy

$$\left\|\mathscr{M}_{\psi,z}(f)\right\|_p \leqslant C_p \left\|f\right\|_p \tag{40}$$

for $1<p<\infty$ with constant C_p independent of λ, $z\in\mathbb{R}^n$, and the coefficients of the particular polynomial involved in the standard representation of ψ.

In order to prove (40), we follow a classical argument which is based on an application of Lemma 3.1. In fact, since $\psi\in\mathscr{PC}_\lambda(d)$, there exist a polynomial $P\in\mathscr{P}_d$ and mapping $\varphi\in C^{d+1}([0,\infty))$ satisfying (i)–(iii) in the standard representation of ψ. For each $j\in\mathbb{Z}$, define the measures μ_j and τ_j by

$$\int f d\mu_j = \int_{2^j}^{2^{j+1}} f(\psi(t)z)\frac{dt}{t}$$

$$\int f d\tau_j = \int_{2^j}^{2^{j+1}} f(P(t)z)\frac{dt}{t}.$$

It is radially seen that

$$\left\|\mu_j\right\| \leqslant 1, \left\|\tau_j\right\| \leqslant 1. \tag{41}$$

Moreover, we have

$$\mathscr{M}_{\psi,z}(f)(x) = \sup_{j\in\mathbb{Z}} \left|(\mu_j * |f|)(x)\right|.$$

Let τ^* be the maximal function corresponding to the measures $\{\tau_j : j\in\mathbb{Z}\}$, i.e.,

$$\tau^*(f)(x) = \sup_{j\in\mathbb{Z}} \left|(\tau_j * |f|)(x)\right|.$$

The boundedness of the maximal function τ^* is known on $L^p(\mathbb{R}^n)$ for all $1<p<\infty$. In fact, by Proposition 1 on page 477 in [16], we have

$$\left\|\tau^*(f)\right\|_p \leqslant C_p \left\|f\right\|_p \tag{42}$$

for all $1<p<\infty$ with constant C_p independent of the coefficients of the polynomial P and the constant z.

Now, we verify that the condition (iii) in Lemma 3.1 holds. For each $z \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, let $L_{\lambda,z} : \mathbb{R}^n \to \mathbb{R}$ be the linear transformation given by $L_{\lambda,z}(\xi) = \lambda \xi \cdot z$. It is straightforward to see that

$$\left|\hat{\mu}_j(\xi) - \hat{\tau}_j(\xi)\right| \leqslant \int_{2^j}^{2^{j+1}} |\lambda \varphi(t) \xi \cdot z| \frac{dt}{t} \leqslant C \left|\varphi(2^{j+1}) L_{\lambda,z}(\xi)\right|. \tag{43}$$

On the other hand, by (26), (27), and Van der Corput's lemma [16], we get

$$\left|\hat{\mu}_j(\xi)\right| \leqslant C \left|L_{\lambda,z}(\xi) \varphi(2^j)\right|^{-\frac{1}{d+1}}. \tag{44}$$

Hence, by (42), (41), (22), (44), and Lemma 3.1, the proof is complete. □

We should point out here that a generalization of Theorem 3.2 will appear in a forthcoming paper. We end this section by recalling the definition of the hardy space $H^1(\mathbb{S}^{n-1})$. The hardy space $H^1(\mathbb{S}^{n-1})$ can be defined by using atoms:

DEFINITION 3.3. A function $\mathbf{a} : \mathbb{S}^{n-1} \to \mathbb{C}$ is called an H^1 atom if it satisfies the following:

(i) $supp(\mathbf{a}) \subseteq \mathbb{S}^{n-1} \cap \{y \in \mathbb{R}^n : |y - y_0| < \rho\}$ for some $y_0 \in \mathbb{S}^{n-1}$ and $\rho > 0$;
(ii) $\|\mathbf{a}\| \leqslant \rho^{-(n-1)}$;
(iii) $\int_{\mathbb{S}^{n-1}} \mathbf{a}(x) d\sigma(x) = 0$.

For sake of simplicity, we shall refer to ρ and y_0 in the above definition by $rad(\mathbf{a})$ and $cent(\mathbf{a})$ respectively.

DEFINITION 3.4. A function $\Omega : \mathbb{S}^{n-1} \to \mathbb{C}$ is in $H^1(\mathbb{S}^{n-1})$ if there are H^1 atoms $\mathbf{a}_1, \mathbf{a}_2, \ldots$ on $\mathbb{S}^{n-1}$, a sequence of complex numbers $\{c_j\}$ with $\sum_j |c_j| < \infty$, and $\Omega_0 \in L^\infty$ such that

$$\Omega = \Omega_0 + \sum_j c_j \mathbf{a}_j.$$

Here, $\sum_j |c_j| \leqslant \|\Omega\|_{H^1(\mathbb{S}^{n-1})}$.

4. Main lemma

In order to prove our main results, we prove the following introductory lemma:

LEMMA 4.1. *Suppose that $\Phi(y) = \psi(|y|)y'$ with $\psi \in \mathscr{PC}_\lambda(d)$ for some $d \geqslant 0$ and $\lambda \in \mathbb{R}$. Suppose also that $b \in L^\infty(\mathbb{R}_+)$ and that Ω is an H^1 atom on $\mathbb{S}^{n-1}$ with $Cent(\Omega) = e = (0, \ldots, 1)$ and $rad(\Omega) = \rho < \frac{1}{n}$. Let $\{\sigma_{t,\Omega,b} : t \in \mathbb{R}\}$ be the family of measures given by (7). For a fixed $j \in \mathbb{Z}$, let $\mu^{(j)}_{\Omega,\Phi,b}$ be the operator given by*

$$\mu^{(j)}_{\Omega,\Phi,b}(f)(x) = \left(\int_{-\infty}^{\infty} \left|\sigma_{t+j,\Omega,b} * f(x)\right|^2 dt\right)^{\frac{1}{2}}. \tag{45}$$

Then

$$\left\|\mu^{(j)}_{\Omega,\Phi,b}(f)\right\|_p \leqslant C_p \|f\|_p \tag{46}$$

for all $1<p<\infty$*, where* C_p *independent of* ρ*,* λ*, and* j*.*

Proof. Let d, φ, and $\upsilon^{(l)}_{t,\Omega}, 0\leqslant l\leqslant d+3$ be as in Proposition 2.1. For $1\leqslant s\leqslant d+3$, let n_s be such that

$$n_s=\begin{cases} n, & s=d+3,\ d+1\\ n-1, & s\neq d+3,\ d+1\end{cases}.$$

Let $L_s:\mathbb{R}^n\to\mathbb{R}^{n_s}$ be the linear transformation defined by

$$L_s(\xi)=\begin{cases}\lambda\rho^{2\max\{0,\frac{1}{n}-\rho\}}\xi, & s=d+3,\ d+1\\ \lambda\rho^{\max\{0,\frac{1}{n}-\rho\}}\overline{\xi}, & s\neq d+3,\ d+1\end{cases}.$$

Also, for $0\leqslant s\leqslant d+3$, we let $\{a_{t,s}: t\in\mathbb{R}\}$ be the lacunary sequence defined by

$$a_{t,s}=\begin{cases}\varphi(2^t), & s=d+3,\ d+2\\ 2^{dt}, & s=d+1\\ 2^{st}, & s\neq d+1,\ d+2,\ d+3\end{cases}.$$

Thus, by Proposition 2.1, we get

$$\upsilon^{(d+2)}_{t+j,\Omega}=\sigma_{t,\Omega,b} \tag{47}$$

$$\upsilon^{(0)}_{t,\Omega}=0 \tag{48}$$

$$\left|(\upsilon^{(s)}_{t+j,\Omega})\hat{\ }(\xi)\right|\leqslant C\left|a_{t+j,s}L_s(\xi)\right|^{-\frac{1}{4(s+1)}} \tag{49}$$

$$\left|(\upsilon^{(s)}_{t+j,\Omega})\hat{\ }(\xi)-(\upsilon^{(s-1)}_{t+j,\Omega})\hat{\ }(\xi)\right|\leqslant C\left|a_{t+j+1,s}L_s(\xi)\right|. \tag{50}$$

On the other hand, it is clear that

$$\left\|\upsilon^{(s)}_{t+j,\Omega}\right\|\leqslant C. \tag{51}$$

By Theorem 3.2, we obtain that the maximal functions

$$(\upsilon^{(s)}_{\Omega})^*(f)(x)=\sup_t\left|\left|\upsilon^{(s)}_{t,\Omega}\right|*f(x)\right| \tag{52}$$

satisfy

$$\left\|(\upsilon^{(s)}_{\Omega})^*(f)\right\|_p\leqslant C_p\|f\|_p \tag{53}$$

for all $1<p<\infty$ and for $1\leqslant s\leqslant d+3$.

By similar argument as in [12], there exists a family of measures $\{\tau_{t,\Omega}^{(s)}, 1 \leqslant s \leqslant d+3, t \in \mathbb{R}\}$ such that

$$\left\|\tau_{t+j,\Omega}^{(s)}\right\| \leqslant C. \tag{54}$$

$$\left|(\tau_{t+j,\Omega}^{(s)})\hat{}(\xi)\right| \leqslant C\left|a_{t+j,s}L_s(\xi)\right|^{-\frac{1}{4(s+1)}} \tag{55}$$

$$\left|(\tau_{t+j,\Omega}^{(s)})\hat{}(\xi)\right| \leqslant C\left|a_{t+j+1,s}L_s(\xi)\right| \tag{56}$$

$$\left\|(\tau_{\Omega}^{(s)})^*(f)\right\|_p \leqslant C_p\|f\|_p, 1<p<\infty \tag{57}$$

and

$$\sigma_{t,\Omega,b} = \sum_{s=1}^{d+3}\tau_{t,\Omega}^{(s)}. \tag{58}$$

By (58), we immediately obtain

$$\mu_{\Omega,\Phi,b}^{(j)}(f)(x) \leqslant \sum_{s=1}^{d+3}\mu_{\Omega,\Phi,b}^{(j,s)}(f)(x), \tag{59}$$

where $\mu_{\Omega,\Phi,b}^{(j,s)}$ is the operator that has the same definition as $\mu_{\Omega,\Phi,b}^{(j)}(f)$ with $\sigma_{t+j,\Omega,b}$ is replaced by $\tau_{t+j,\Omega}^{(s)}$. Since

$$\left\|\mu_{\Omega,\Phi,b}^{(j)}(f)\right\|_p \leqslant \sum_{s=1}^{d+3}\left\|\mu_{\Omega,\Phi,b}^{(j,s)}(f)\right\|_p$$

it suffices to prove that

$$\left\|\mu_{\Omega,\Phi,b}^{(j,s)}(f)\right\|_p \leqslant C_p\|f\|_p \tag{60}$$

for all $1<p<\infty$, where C_p independent of ρ, λ, and j.

Now, by an elementary procedure, choose a collection of $\mathscr{C}^\infty$ functions $\{\omega_k^{(s)}\}_{k\in\mathbb{Z}}$ on $(0,\infty)$ with the following properties:

$$supp(\omega_k^{(s)}) \subseteq \left[\frac{1}{a_{k+1,s}}, \frac{1}{a_{k-1,s}}\right]; \tag{61}$$

$$0 \leqslant \omega_k \leqslant 1; \tag{62}$$

$$\left|\frac{d^l\omega_k^{(s)}}{du^l}(u)\right| \leqslant \frac{C_l}{u^l}; \tag{63}$$

$$\sum_{k\in\mathbb{Z}}\omega_k^{(s)}(u) = 1. \tag{64}$$

Define the functions $\{\psi_k^{(s)} : k \in \mathbb{Z}\}$ on $\mathbb{R}^n$ by

$$(\psi_k^{(s)})\hat{}(\xi) = \omega_k^{(s)}(|L_s(\xi)|^2)$$

Then, by (64), we have

$$\mu_{\Omega,\Phi,b}^{(j,s)}(f)(x) \leqslant \sum_{k\in\mathbb{Z}} I_{j,k}^{(s)}(f)(x), \tag{65}$$

where

$$I_{j,k}^{(s)}(f)(x) = (\int_{-\infty}^{\infty} \left| f * \tau_{t+j,\Omega}^{(s)} * \psi_{\lfloor t+j \rfloor + k}(x) \right|^2 dt)^{\frac{1}{2}}. \tag{66}$$

Here, $\lfloor x \rfloor$ is the greatest integer function less than or equal to x. For $k \in \mathbb{Z}$, let S_k^j be the operator given by

$$S_k^j f(x) = (\int_{-\infty}^{\infty} \left| f * \psi_{\lfloor t+j \rfloor + k}(x) \right|^2 dt)^{\frac{1}{2}}. \tag{67}$$

By following similar argument as in [16], it can be shown that

$$\left\| S_k^j(f) \right\|_p \leqslant C_p \left\| f \right\|_p \tag{68}$$

for all $p \in (1,\infty)$ with constant C_p independent of the essential parameters.

Now, we estimate the L^p norm of the operator $I_{j,k}^{(s)}(f)$. For $p > 2$, set $q = (\frac{p}{2})'$ and choose a non-negative function $v \in L_+^q(\mathbb{R}^n)$ with $\|v\|_q = 1$ such that

$$\left\| I_{j,k}^{(s)}(f) \right\|_p^2 = \int_{\mathbf{R}^n} \int_{-\infty}^{\infty} \left| f * \tau_{t+j,\Omega}^{(s)} * \psi_{\lfloor t+j \rfloor + k}(x) \right|^2 v(x) dt dx.$$

By Hölder's inequality, (59), and (68), we have

$$\left\| I_{j,k}^{(s)}(f) \right\|_p \leqslant \left\| S_k^j(f) \right\|_p \left\| (\tau_\Omega^{(s)})^*(v) \right\|_q^{\frac{1}{2}} \leqslant C \left\| f \right\|_p; \tag{69}$$

when combined with duality argument, we get

$$\left\| I_{j,k}^{(s)}(f) \right\|_p \leqslant C \left\| f \right\|_p \tag{70}$$

for all $1 < p < \infty$.

Now, we estimate $\left\| I_{j,k}^{(s)}(f) \right\|_2$. First, we observe that the function 2^t satisfies the conclusion of Proposition 2.2 though it is not zero at the origin.

For $k \geqslant 3$, by Plancherel's theorem, (55), (33) with $r = |\xi|$, and (54), we have

$$\begin{aligned} \left\| I_{j,k}^{(s)}(f) \right\|_2^2 &\leqslant C \int_{\mathbb{R}^n} \left| \hat{f}(\xi) \right|^2 \int_{-\infty}^{\infty} \left| a_{t+j,s} L_s(\xi) \right| \left| \omega_{\lfloor t+j \rfloor + k}(|L_s(\xi)|) \right|^2 dt d\xi \\ &\leqslant C 2^{2(-k+3)} \int_{\mathbb{R}^n} \left| \hat{f}(\xi) \right|^2 d\xi. \end{aligned} \tag{71}$$

Hence,

$$\left\| I_{j,k}^{(s)}(f) \right\|_2 \leqslant C 2^{(-k+2)} \left\| f \right\|_2 \text{ for } k \geqslant 3. \tag{72}$$

Similarly, we obtain

$$\left\|I_{j,k}^{(s)}(f)\right\|_2 \leqslant C2^{\frac{k+2}{s+1}} \|f\|_2 \text{ for } k \leqslant -3. \tag{73}$$

By Plancherel's theorem along with (54), we obtain

$$\left\|I_{j,k}^{(s)}(f)\right\|_2 \leqslant \sqrt{3} \|f\|_2 \text{ for } k = -2,-1,0,1,2. \tag{74}$$

Now, by (70), (72)–(74), and an interpolation, we obtain

$$\left\|I_{j,k}^{(s)}(f)\right\|_p \leqslant CA^{-|k|\theta(p)} \|f\|_p, \tag{75}$$

for $1 < p < \infty$ and $\theta(p) > 0$ with constant C independent of ρ and j. Hence,

$$\left\|\mu_{\Omega,\Phi,b}^{(j,s)}(f)\right\|_p \leqslant \sum_{k\in\mathbf{Z}} \left\|I_{j,k}^{(s)}(f)\right\|_p \leqslant C\{\sum_{k\in\mathbf{Z}} A^{-|k|\theta(p)}\} \|f\|_p \leqslant C_p \|f\|_p,$$

where, C_p, C, and $\theta(p)$ are constants independent of ρ and j. This completes the proof. □

Now, by following exactly the same argument as in Lemma 4.2, we have

LEMMA 4.2. *Let φ, y_0, ρ, j, $\mu_{\Omega,\Phi,b}^{(j)}$ be as in Lemma 4.1. If Ω is an L^∞ function on $\mathbb{S}^{n-1}$ or an, $H^1(\mathbb{S}^{n-1})$ atom with rad$(\Omega) = \rho \geqslant \frac{1}{n}$, then*

$$\left\|\mu_{\Omega,\Phi,b}^{(j)}(f)\right\|_p \leqslant C_p \|f\|_p \tag{76}$$

for all $1 < p < \infty$, where C_p is a constant independent of λ. Moreover, it is also independent of ρ if Ω is an, $H^1(\mathbb{S}^{n-1})$ atom.

5. Proofs of main results

This section is devoted to prove our main results. Since the proof of Theorem 1.2 can be obtained using the estimates obtained in this paper and following similar argument as in [1]. We shall only present a proof of Theorem 1.1.

Proof of Theorem 1.1. Let ψ, Ω, λ, and $\mu_{\Omega,\Phi,b}$ be as in the statement of Theorem 1.1. Since $\Omega \in H^1(\mathbb{S}^{n-1})$, there are H^1 atoms $\mathbf{a}_1, \mathbf{a}_2, \ldots$ on $\mathbb{S}^{n-1}$, a sequence of complex numbers $\{c_j\}$ with $\sum_j |c_j| < \infty$, and $\Omega_0 \in L^\infty$ such that

$$\Omega = \Omega_0 + \sum_j c_j \mathbf{a}_j \tag{77}$$

with $\sum_j |c_j| \leqslant \|\Omega\|_{H^1(\mathbb{S}^{n-1})}$. Therefore,

$$\mu_{\Omega,\Phi,b}(f)(x) \leqslant \mu_{\Omega_0,\Phi,b}(f)(x) + \sum_j |c_j| \mu_{\mathbf{a}_j,\Phi,b}(f)(x). \tag{78}$$

By Lemma 4.2, we have

$$\left\|\mu_{\Omega_0,\Phi,b}(f)\right\|_p \leqslant C_p \|f\|_p \tag{79}$$

L^p for all $1 < p < \infty$.

On the other hand, for an $H^1(\mathbb{S}^{n-1})$ atom $\mathbf{a}_j$, by using a proper rotation on $\mathbb{S}^{n-1}$, we may assume that $cent(\mathbf{a}_j) = e = (0,0,\ldots,1)$.

It is radially seen that

$$\mu_{\mathbf{a}_j,\Phi,b}(f)(x) \leqslant \sum_{l=0}^{\infty} 2^{-l} \mu^{(l)}_{\mathbf{a}_j,\Phi,b}(f)(x). \tag{80}$$

By Lemma 4.1 and Lemma 4.2, we have

$$\left\|\mu^{(l)}_{\mathbf{a}_j,\Phi,b}(f)\right\|_p \leqslant C\|f\|_p \tag{81}$$

for all $1 < p < \infty$ with constant C independent of the atom $\mathbf{a}_j$ and the index l. Hence, by (79)–(81), we get

$$\left\|\mu_{\mathbf{a}_j,\Phi,b}(f)\right\|_p \leqslant C\|f\|_p \tag{82}$$

for all $1 < p < \infty$ with constant C independent of the atom $\mathbf{a}_j$. This completes the proof. $\square$

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