

## A NOTE ON THE HARDY–LITTLEWOOD INEQUALITIES FOR MULTILINEAR FORMS

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*Abstract.* The notion of entropy of the Hardy–Littlewood inequalities for multilinear forms was introduced and explored by Pellegrino and Teixeira in [14]. In this note, among other results, we introduce a related notion and obtain some new estimates.

### 1. Introduction

The Hardy–Littlewood inequalities [10] for  $m$ –linear forms (see, for instance, [3, 4, 7, 8, 9, 12, 15]) are natural extensions of the Bohnenblust–Hille inequality [6] when the sequence space  $c_0$  is replaced by the sequence space  $\ell_p$ . These inequalities assert that for any integer  $m \geq 2$  there exist constants  $C_{m,p}^{\mathbb{K}} \geq 1$  such that

$$\left( \sum_{j_1, \dots, j_m=1}^{\infty} |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{m,p}^{\mathbb{K}} \|T\|, \quad (1)$$

for all continuous  $m$ –linear forms  $T : \ell_p \times \dots \times \ell_p \rightarrow \mathbb{K}$  (here, and henceforth,  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ) and  $p \geq 2m$ .

A similar statement is: for any integer  $m \geq 2$  there exist constants  $C_{m,p}^{\mathbb{K}} \geq 1$  such that

$$\left( \sum_{j_1, \dots, j_m=1}^k |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{m,p}^{\mathbb{K}} \|T\|, \quad (2)$$

for all  $m$ –linear forms  $T : \ell_p^k \times \dots \times \ell_p^k \rightarrow \mathbb{K}$ , all positive integers  $k$ , and  $p \geq 2m$ .

The investigation of the optimal constants of the Hardy–Littlewood and related inequalities is closely related to the fashionable investigation of the optimal Bohnenblust–Hille inequality constants (see, for instance [11, 12, 13, 14] and the references therein).

The notion of entropy in the context of the Hardy–Littlewood inequalities was introduced by Pellegrino and Teixeira [14] and it essentially estimates the number of monomials needed to achieve the optimal constant  $C_{m,p}^{\mathbb{K}}$ . There are strong evidences that in general the entropy is finite (see [14]). In this note we introduce a similar notion,

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that we name dimensional-entropy; it estimates the minimum of the  $k$  in (2) needed to achieve the optimal constant  $C_{m,p}^{\mathbb{K}}$ . Our first main result improves an estimate of [14]. More precisely, in [14, Lemma 5.1] it is proved that

$$\left( \sum_{j_1, \dots, j_m=1}^k |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2p}{p-2m+2}} \right)^{\frac{p-2m+2}{2p}} \leq \|T\| \tag{3}$$

for all  $m$ -linear forms  $T : \ell_p^k \times \dots \times \ell_p^k \rightarrow \mathbb{K}$ , all positive integers  $k$ , and  $p > 2m$ . Using a recent result of Albuquerque and Rezende [1] we improve (3) by showing a sharper inequality in the anisotropic setting. In fact we show that

$$\left( \sum_{j_1=1}^k \left( \dots \left( \sum_{j_m=1}^k |T(e_{j_1}, \dots, e_{j_m})|^2 \right)^{\frac{1}{2}} \dots \right)^{\frac{2p}{p-2m+2}} \right)^{\frac{p-2m+2}{2p}} \leq \|T\|,$$

where the precise definition of the intermediary exponents will be clear along this note. We apply this result to explore the notion of entropy introduced in [14], with a different viewpoint. More precisely for each fixed  $k$  we define  $C_{m,p}^{\mathbb{K}}(k)$  as the sharp constant for the Hardy–Littlewood inequalities when we are restricted to  $T : \ell_p^k \times \dots \times \ell_p^k \rightarrow \mathbb{K}$ , and

$$ent_{HL}(\mathbb{K}) := \inf \left\{ k : C_{m,p}^{\mathbb{K}} = C_{m,p}^{\mathbb{K}}(k) \right\}.$$

We prove that

$$ent_{HL}(\mathbb{K}) \geq \left( C_{m,p}^{\mathbb{K}} \right)^{\frac{m^2-3m+p}{2p}}.$$

## 2. Results

Throughout this paper,  $X, Y$  shall stand for Banach spaces over the scalar field  $\mathbb{K}$  of real or complex numbers. The topological dual of  $X$  and its closed unit ball are denoted by  $X^*$  and  $B_{X^*}$ , respectively. For  $r, p \geq 1$ , a linear operator  $T : X \rightarrow Y$  is said  $(r; p)$ -summing if there exists a constant  $C > 0$  such that

$$\left( \sum_{j=1}^{\infty} \|T(x_j)\|^r \right)^{\frac{1}{r}} \leq C \|(x_j)_{j=1}^{\infty}\|_{w,p},$$

where

$$\|(x_j)_{j=1}^{\infty}\|_{w,p} := \sup_{\varphi \in B_{Y^*}} \left( \sum_{j=1}^{\infty} |\varphi(x_j)|^p \right)^{\frac{1}{p}} < \infty.$$

A natural anisotropic approach to summing operators is the following: for all  $\mathbf{r} = (r_1, \dots, r_m)$ ,  $\mathbf{p} = (p_1, \dots, p_m) \in [1, +\infty)^m$ , a multilinear operator  $T : X_1 \times \dots \times X_m \rightarrow Y$

is said to be multiple  $(\mathbf{r}, \mathbf{p})$ -summing if there exists a constant  $C > 0$  such that for all sequences  $x^k := (x_j^k)_{j \in \mathbb{N}}$ ,  $k = 1, \dots, m$ , we have

$$\left( \sum_{j_1=1}^{\infty} \left( \dots \left( \sum_{j_m=1}^{\infty} |T(x_j)|^{r_m} \right)^{\frac{r_{m-1}}{r_m}} \dots \right)^{\frac{r_1}{r_2}} \right)^{\frac{1}{r_1}} \leq C \prod_{k=1}^m \left\| (x_j^{(k)})_{j=1}^{\infty} \right\|_{w, p_k},$$

where  $T(x_j) := T(x_{j_1}^1, \dots, x_{j_m}^m)$ . The class of all multiple  $(\mathbf{r}, \mathbf{p})$ -summing operators is a Banach space with the norm defined by the infimum of all previous constants  $C > 0$ . The space of all such operators is denoted by  $\Pi_{(\mathbf{r}, \mathbf{p})}^m(X_1, \dots, X_m, Y)$ . When  $r_1 = \dots = r_m = r$ , we simply write  $(r; \mathbf{p})$ . For  $\mathbf{p} \in [1, +\infty)^m$  and each  $k \in \{1, \dots, m\}$ , we define

$$\left| \frac{1}{\mathbf{p}} \right|_{\geq k} := \frac{1}{p_k} + \dots + \frac{1}{p_m}.$$

Recently Albuquerque and Rezende [1, Theorem 3] have proved the following result, that generalizes recent results of Bayart [5] and Pellegrino–Santos–Serrano–Teixeira [13]:

**THEOREM 1.** (Albuquerque and Rezende) *Let  $m$  be a positive integer,  $r \geq 1$ , and  $\mathbf{s}, \mathbf{p}, \mathbf{q} \in [1, +\infty)^m$  be such that  $q_k \geq p_k$ , for  $k = 1, \dots, m$  and*

$$\frac{1}{r} - \left| \frac{1}{\mathbf{p}} \right| + \left| \frac{1}{\mathbf{q}} \right| > 0.$$

Then

$$\Pi_{(r; \mathbf{p})}^m(X_1, \dots, X_m, Y) \subset \Pi_{(\mathbf{s}; \mathbf{q})}^m(X_1, \dots, X_m, Y),$$

for any Banach spaces  $X_1, \dots, X_m$ , with

$$\frac{1}{s_k} - \left| \frac{1}{\mathbf{q}} \right|_{\geq k} = \frac{1}{r} - \left| \frac{1}{\mathbf{p}} \right|_{\geq k}$$

for each  $k \in \{1, \dots, m\}$ , and the inclusion operator has norm 1.

Using that all  $m$ -linear forms from  $X_1 \times \dots \times X_m$  to  $\mathbb{K}$  are multiple  $(2; 1, \dots, 1, p^*)$ -summing for  $p > 2m$  and, from Theorem 1 with

$$\begin{aligned} r &= 2 \\ \mathbf{p} &= (1, \dots, 1, p^*) \\ \mathbf{q} &= (p^*, \dots, p^*), \end{aligned}$$

we have

$$\Pi_{(2; 1, \dots, 1, p^*)}^m(E_1, \dots, E_m; \mathbb{K}) \subset \Pi_{(\mathbf{s}; \mathbf{q})}^m(E_1, \dots, E_m; \mathbb{K})$$

with

$$\begin{cases} \frac{1}{s_1} - \left(\frac{m}{p^*}\right) = \frac{1}{2} - \left(\frac{1}{1} + \dots + \frac{1}{1} + \frac{1}{p^*}\right) \\ \vdots \\ \frac{1}{s_m} - \frac{1}{p^*} = \frac{1}{2} - \left(\frac{1}{p^*}\right). \end{cases}$$

So, we have the following improvement of [14, Lemma 5.1]:

**THEOREM 2.** *For all  $m \geq 2$  we have*

$$\left( \sum_{j_1=1}^k \left( \dots \left( \sum_{j_m=1}^k |T(e_{j_1}, \dots, e_{j_m})|^{s_m} \right)^{\frac{1}{s_m}} \dots \right)^{\frac{1}{2} s_1} \right)^{\frac{1}{s_1}} \leq \|T\|$$

for all  $m$ -linear forms  $T : \ell_p^k \times \dots \times \ell_p^k \rightarrow \mathbb{K}$  and all positive integers  $k$  and  $p > 2m$ , with

$$s_j = \frac{2p}{p - 2m + 2j}$$

for all  $j = 1, \dots, m$ .

The above result is sharp at least when  $p = \infty$  because in this case  $s_j = 2$  for every  $j$  and the estimate

$$\left( \sum_{j_1, \dots, j_m}^k |T(e_{j_1}, \dots, e_{j_m})|^2 \right)^{\frac{1}{2}} \leq \|T\|,$$

for all  $m$ -linear forms  $T : \ell_\infty^k \times \dots \times \ell_\infty^k \rightarrow \mathbb{K}$ , and all positive integers  $k$ , is optimal.

Now we state and prove our second main result:

**THEOREM 3.** *For all  $m \geq 2$  we have*

$$ent_{HL}(\mathbb{K}) \geq \left( C_{m,p}^{\mathbb{K}} \right)^{\frac{m^2 - 3m + p}{2p}}.$$

*Proof.* Following the notation of Theorem 2, by the Hölder inequality for mixed norms (see [2]) and Theorem 2, we have

$$\begin{aligned}
 & \left( \sum_{j_1, \dots, j_m=1}^k |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \\
 & \leq \left( \sum_{j_1=1}^k \left( \dots \left( \sum_{j_m=1}^k |T(e_{j_1}, \dots, e_{j_m})|^{s_m} \right)^{\frac{1}{s_m}} \dots \right)^{s_1} \right)^{\frac{1}{s_1}} \\
 & \quad \times \left( \sum_{j_1=1}^k \left( \dots \left( \sum_{j_m=1}^k |1|^{t_m} \right)^{\frac{1}{t_m}} \dots \right)^{t_1} \right)^{\frac{1}{t_1}} \\
 & \leq k^{\frac{1}{t_1} + \dots + \frac{1}{t_m}} \|T\|
 \end{aligned} \tag{4}$$

with

$$\begin{cases} \frac{1}{\frac{2mp}{mp+p-2m}} = \frac{1}{s_m} + \frac{1}{t_m}, \\ \vdots \\ \frac{1}{\frac{2mp}{mp+p-2m}} = \frac{1}{s_1} + \frac{1}{t_1}. \end{cases}$$

Since

$$\begin{aligned}
 \frac{1}{s_1} + \dots + \frac{1}{s_m} &= \left(\frac{m}{2}\right) - \left(\frac{m(m-1)}{2} + m\left(1 - \frac{1}{p}\right)\right) + \left(1 - \frac{1}{p}\right) \left(\frac{m(m+1)}{2}\right) \\
 &= \frac{m(p-m+1)}{2p},
 \end{aligned}$$

we have

$$\frac{1}{t_1} + \dots + \frac{1}{t_m} = \frac{mp+p-2m}{2p} - \frac{m(p-m+1)}{2p} = \frac{1}{2p}(m^2-3m+p)$$

and, by (4), we conclude that

$$\left( \sum_{j_1, \dots, j_m=1}^k |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq \|T\| k^{\frac{1}{2p}(m^2-3m+p)}.$$

Thus

$$k^{\frac{1}{2p}(m^2-3m+p)} \geq C_{m,p}^{\mathbb{K}}$$

and

$$k \geq \left(C_{m,p}^{\mathbb{K}}\right)^{\frac{m^2-3m+p}{2p}}. \quad \square$$

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