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SOME PROPERTIES OF ZIPF–MANDELBROT LAW AND HURWITZ ζ–FUNCTION

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Abstract. In this paper we deal with analytical properties of the Zipf-Mandelbrot law. If total mass of this law is spread all over positive integers we come to Hurwitz ζ-function. As we show, it is very natural first to examine properties of Hurwitz ζ-function to derive properties of Zipf-Mandelbrot law. Using some well-known inequalities such as Chebyshev's and Lyapunov's inequality we are able to deduce a whole variety of theoretical characterizations that include, among others, log-convexity, log-subadditivity, exponential convexity.

1. Introduction

In 1918 [5] Jean-Baptiste Estoup first notice the hyperbolic nature of word use. The first explanation of this regularity is done by George Kingsley Zipf in 1935, today's known as Zipf's law [5]. Zipf in [12] observed in the study of human languages that when words are ranked according to their occurrence frequency in the descending order, the frequency is inversely proportional to its rank k according to

$$f(k) = \frac{C}{k^s}, \tag{1}$$

where s is near 1 and C is the normalizing constant.

The more general model introduced Benoit Mandelbrot [6, pp. 503–512], by using arguments on the fractal structure of lexical trees:

$$f(k) = \frac{C}{(k+q)^s}. \tag{2}$$

When $q=0$, we get indeed Zipf's law.

For $N \in \mathbb{N}$, $q \geqslant 0$, $s > 0$, $k \in \{1,2,\dots,N\}$, in a more clear form, Zipf-Mandelbrot law (probability mass function) is defined with

$$f(k,N,q,s) = \frac{1/(k+q)^s}{\zeta(N,s,q)}, \tag{3}$$

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where

$$\zeta(N,s,q)=\sum_{i=1}^{N}\frac{1}{(i+q)^{s}}, \tag{4}$$

$N\in\mathbb{N},\ q\geqslant 0,\ s>0,\ k\in\{1,2,\ldots,N\}$. If total number of words N tends to infinity we denote

$$f(k,q,s)=\frac{1/(k+q)^{s}}{\zeta(s,q)}, \tag{5}$$

where

$$\zeta(s,q)=\sum_{i=1}^{\infty}\frac{1}{(i+q)^{s}} \tag{6}$$

we recognize as Hurwitz ζ-function. This infinite case, when total mass is spread over all set of positive integers, particularly, is studied in [7]. Note here, that we use more suitable version of Hurwitz ζ function (see also [1]), since in the classical definition sum starts from zero and $q>0$. However, this fact does not alter our conclusions about Hurwitz ζ-function.

The are also quite different interpretation of Zipf-Mandelbrot law. As it is pointed out in [8] (see also [2], [11]), parameters in (3) can be interpreted in the following way: N is the number of species present and the parameters q and s have an ecological interpretation: q represents the diversity of the environment and s the predictability of the ecosystem, i.e. the average probability of the appearance of a species.

2. Monotonicity properties

As starting point, we use the next proposition on inequalities for sums of positive order ([9, pp. 36], [10, pp. 165]).

PROPOSITION 1. *If* $a_i\geqslant 0,\ i\in\mathbb{N}$ *then for* $0<t<s$

$$\left(\sum_{i=1}^{\infty}a_i^{s}\right)^{\frac{1}{s}}\leqslant\left(\sum_{i=1}^{\infty}a_i^{t}\right)^{\frac{1}{t}}. \tag{7}$$

THEOREM 1.

i) The function $s\mapsto[\zeta(N,s,q)]^{1/s}$ *is decreasing i.e. for* $s>t>0$

$$[\zeta(N,s,q)]^{1/s}\leqslant[\zeta(N,t,q)]^{1/t}.$$

ii) The function $s\mapsto[f(k,N,q,s)]^{1/s}$ *is increasing i.e. for* $s>t>0$

$$[f(k,N,q,s)]^{1/s}\geqslant[(f(k,N,q,t)]^{1/t}.$$

iii) The function $s\mapsto[\zeta(s,q)]^{1/s}$ *is decreasing i.e. for* $s>t>0$

$$[\zeta(s,q)]^{1/s}\leqslant[\zeta(t,q)]^{1/t}.$$

iv) The function $s \mapsto [f(k,q,s)]^{1/s}$ *is increasing i.e. for* $s > t > 0$

$$[f(k,q,s)]^{1/s} \geqslant [(f(k,q,t)]^{1/t}.$$

Proof. i) We use the Proposition 1, for

$$a_i = \begin{cases} \frac{1}{i+q}, & i = 1, \dots, N; \\ 0, & i > N. \end{cases}$$

ii) Follows from i)-part and

$$\frac{1}{f(k,N,q,s)} = \sum_{i=1}^{N} \left(\frac{k+q}{i+q} \right)^s = (k+q)^s \zeta(N,s,q). \tag{8}$$

iii) Use Proposition 1 for $a_i = \frac{1}{i+q}$, $i \in \mathbb{N}$.
iv) Follows from iii)-part and

$$\frac{1}{f(k,q,s)} = (k+q)^s \zeta(s,q). \quad \square \tag{9}$$

THEOREM 2. *The function*

$$s \mapsto (Nf(k,N,q,s))^{1/s} \tag{10}$$

is decreasing i.e. for $s > t > 0$

$$(Nf(k,N,q,s))^{1/s} \leqslant (Nf(k,N,q,t))^{1/t}. \tag{11}$$

Proof. From (8) it follows

$$\frac{1}{Nf(k,N,q,s)} = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{k+q}{i+q} \right)^s, \tag{12}$$

i.e.

$$(Nf(k,N,q,s))^{-1/s} = \left[\frac{1}{N} \sum_{i=1}^{N} \left(\frac{k+q}{i+q} \right)^s \right]^{1/s}. \tag{13}$$

Denote $x_i = \frac{k+q}{i+q}$, $i = 1, \dots, N$. Then the right-hand side of (13) is the power mean

$$M_N^{[s]}(x_{\overline{1,N}}) := \left[\frac{1}{N} \sum_{i=1}^{N} x_i^s \right]^{1/s}.$$

Using well-known fact, that $s \mapsto M_N^{[s]}(x_{\overline{1,N}})$ is increasing function (see for example [9, 10]) we conclude that the function

$$s \mapsto (Nf(k,N,q,s))^{1/s} \tag{14}$$

is decreasing. $\square$

3. Log-convexity and exponential convexity

Let us recall well-known Lyapunov inequality, for sequences ([9, pp. 34], [10, pp. 117]).

PROPOSITION 2. *If* $a_i \geqslant 0,\ i \in \mathbb{N},$ *then for* $0 < r < s < t$

$$\left(\sum_{i=1}^{\infty} a_i^s\right)^{t-r} \leqslant \left(\sum_{i=1}^{\infty} a_i^r\right)^{t-s} \left(\sum_{i=1}^{\infty} a_i^t\right)^{s-r}. \tag{15}$$

If we set $a_i = \frac{1}{i+q}$, $i \in \mathbb{N}$ in (15) we get

COROLLARY 1. *For* $1 < r < s < t$

$$\zeta^{t-r}(s,q) \leqslant \zeta^{t-s}(r,q)\zeta^{s-r}(t,q). \tag{16}$$

In the next theorem we prove, log-concavity of $s \mapsto f(k,N,q,s)$ and log-convexity of $s \mapsto \zeta(s,q)$.

THEOREM 3. *Let* $\lambda \in (0,1)$.

i) For $0 < r < t$,

$$\zeta(N, \lambda r + (1-\lambda)t, q) \leqslant \zeta^{\lambda}(N,r,q)\zeta^{1-\lambda}(N,t,q).$$

ii) For $0 < r < t$,

$$(f(k,N,q,\lambda r + (1-\lambda)t))^{-1} \leqslant (f(k,N,q,r))^{-\lambda} (f(k,N,q,t))^{-(1-\lambda)}.$$

iii) For $1 < r < t$,

$$\zeta(\lambda r + (1-\lambda)t, q) \leqslant \zeta^{\lambda}(r,q)\zeta^{1-\lambda}(t,q).$$

iv) For $1 < r < t$,

$$(f(k,q,\lambda r + (1-\lambda)t))^{-1} \leqslant (f(k,q,r))^{-\lambda} (f(k,q,t))^{-(1-\lambda)}.$$

Proof. i) For $0 < r < t$ and $\lambda \in (0,1)$ we set

$$a_i = \begin{cases} \frac{1}{i+q}, & i = 1,\dots,N; \\ 0, & i > N. \end{cases}$$

and $s = \lambda r + (1-\lambda)t$ in (15):

$$\left(\sum_{i=1}^{N}\left(\frac{1}{i+q}\right)^{\lambda r + (1-\lambda)t}\right)^{t-r} \leqslant \left(\sum_{i=1}^{N}\left(\frac{1}{i+q}\right)^{r}\right)^{\lambda(t-r)} \left(\sum_{i=1}^{N}\left(\frac{1}{i+q}\right)^{t}\right)^{(1-\lambda)(t-r)}.$$

ii) Follows from (8) and i)-part.
iii) We set $a_i = \frac{1}{i+q}$ and $s = \lambda r + (1-\lambda)t$ in (15).
iv) Follows from iii)-part and (9). □

We can conclude even more since this result can be extended to exponential convexity [4].

DEFINITION 1. A function $h : I \to \mathbb{R}$ is exponentially convex on an interval $I \subseteq \mathbb{R}$ if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$, $x_i \in I$, $i = 1, \ldots, n$.

THEOREM 4. *The function $s \mapsto \zeta(s,q)$ is exponentially convex function on $(1,\infty)$.*

Proof. For a given $n \in \mathbb{N}$ let $\xi_m \in \mathbb{R}$, $s_m \in (1,\infty)$ $(m = 1, \ldots, n)$ we have

$$\sum_{l,m=1}^{n} \xi_l \xi_m \zeta\left(\frac{s_l + s_m}{2}, q\right) = \sum_{l,m=1}^{n} \xi_l \xi_m \sum_{i=1}^{\infty} \frac{1}{(i+q)^{\frac{s_l+s_m}{2}}} \tag{17}$$

$$= \sum_{i=1}^{\infty} \sum_{l,m=1}^{n} \xi_l \xi_m \frac{1}{(i+q)^{\frac{s_l+s_m}{2}}} \tag{18}$$

$$= \sum_{i=1}^{\infty} \left(\sum_{m=1}^{n} \frac{1}{(i+q)^{\frac{s_m}{2}}} \right)^2 \geqslant 0. \tag{19}$$

Since the function $s \mapsto \zeta(s,q)$ is continuous function on $(1,\infty)$, we conclude its exponential convexity on $(1,\infty)$. □

Using (9) we have also the next corollary.

COROLLARY 2. *The function $s \mapsto (f(k,q,s))^{-1}$ is exponentially convex function on $(1,\infty)$.*

Proof. This is consequence of (9) and the fact that exponential convexity is closed under finite multiplication of exponentially convex functions. □

COROLLARY 3. *The matrices $\left[\left(\zeta\left(\frac{s_l+s_m}{2}, q\right)\right)\right]_{l,m=1}^{n}$ and $\left[\left(f\left(k,q,\frac{s_l+s_m}{2}\right)\right)^{-1}\right]_{l,m=1}^{n}$ are positive semi definite for all $n \in \mathbb{N}$, $s_1, \ldots, s_n$ in $(1,\infty)$.*

We can also deduce exponential convexity from diversity point of view, notion mentioned in the introduction.

THEOREM 5. *For any $s > 0$, $N \in \mathbb{N}$, the function*

$$q \mapsto \zeta(N,s,q)$$

is exponentially convex on $(0,\infty)$.

Proof. For $k = 1, \dots, N,$ using the Laplace transform,

$$\frac{1}{(k+q)^s} = \int_0^\infty e^{-(k+q)t} \frac{t^{s-1}}{\Gamma(s)} dt$$

and the fact

$$\sum_{i,j=1}^{n} \xi_i \xi_j \exp\left[-\left(k + \frac{q_i + q_j}{2}\right) t\right] = e^{-kt} \left(\sum_{i=1}^{n} \xi_i \exp\left(-\frac{q_i}{2} t\right)\right)^2 \geqslant 0,$$

we conclude exponential convexity of the function $q \mapsto \frac{1}{(k+q)^s}$ on $(0,\infty)$. Now $q \mapsto \zeta(N,s,q)$ is exponentially convex on $(0,\infty)$ as a finite sum of exponentially convex functions. $\square$

THEOREM 6. *For any* $s > 1,$ *the function*

$$q \mapsto \zeta(s,q)$$

is exponentially convex on $(0,\infty)$.

Proof. Using Mellin transformation

$$\zeta(s,q) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-(q+1)t}}{1 - e^{-t}} dt$$

and

$$\sum_{i,j=1}^{n} \xi_i \xi_j \exp\left(-\left(\frac{q_i + q_j}{2} + 1\right) t\right) = \left(\sum_{i=1}^{n} \xi_i \exp\left(-\frac{q_i + 1}{2} t\right)\right)^2 \geqslant 0,$$

we conclude exponential convexity of $q \mapsto \zeta(s,q)$ on $(0,\infty)$. $\square$

COROLLARY 4. *For* $s > 1,$ *the matrix* $\left[\zeta\left(s, \left(\frac{q_l + q_m}{2}\right)\right)\right]_{l,m=1}^{n}$ *is positive semi definite for all* $n \in \mathbb{N},$ $q_1, \dots, q_n \in (0,\infty)$.

COROLLARY 5. *For any* $s > 1,$ *the function*

$$q \mapsto \zeta(s,q)$$

is log-convex on $(0,\infty)$.

4. Log subadditivity

Let us recall Chebyshev's inequality (see [9, pp. 27], [10, pp. 197]).

THEOREM 7. *Let $(a_1,\dots,a_N)$ and $(b_1,\dots,b_N)$ be two N-tuples of real numbers such that*

$$(a_i-a_j)(b_i-b_j)\geqslant 0,\ \ \textit{for}\ \ i,j=1,\dots,N,$$

and $(w_1,\dots,w_N)$ be a positive n-tuple. Then

$$\left(\sum_{i=1}^{N} w_i\right)\left(\sum_{i=1}^{N} w_i a_i bi\right)\geqslant\left(\sum_{i=1}^{n} w_i a_i\right)\left(\sum_{i=1}^{N} w_i b_i\right). \tag{20}$$

THEOREM 8. *The function $s\mapsto Nf(k,N,q,s)$ is log subadditive, i.e. for $s,r>0$*

$$Nf(k,N,q,s+r)\leqslant[Nf(k,N,q,s)]\,[Nf(k,N,q,r)]. \tag{21}$$

Proof. We apply Chebyshev's inequality (20) for

$$a_i=\left(\frac{k+q}{i+q}\right)^s,\ b_i=\left(\frac{k+q}{i+q}\right)^r,\ w_i=\frac{1}{N};\quad i=1,\dots,N.$$

Hence we get

$$\frac{1}{N}\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^{s+r}\geqslant\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^{s}\right)\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^{r}\right)$$

$$\Rightarrow\frac{1}{Nf(k,N,q,s+r)}\geqslant\frac{1}{Nf(k,N,q,s)}\,\frac{1}{Nf(k,N,q,r)},$$

concluding (21). □

THEOREM 9. *The function $u\mapsto\left[f(k,N,q,u^{-1})\right]^{-u}$ is log-convex.*

Proof. Using Lyapunov inequality in Proposition 2, for $0<r<s<t$

$$\left(\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^{s}\right)^{t-r}\leqslant\left(\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^{r}\right)^{t-s}\left(\sum_{i=1}^{N}\left(\frac{k+q}{i+q}\right)^{t}\right)^{s-r}. \tag{22}$$

Using (13) we rewrite this as

$$[f(k,N,q,s)]^{-\frac{1}{s}}\leqslant\left\{[f(k,N,q,r)]^{-\frac{1}{r}}\right\}^{\frac{r}{s}\frac{t-s}{t-r}}\left\{[f(k,N,q,t)]^{-\frac{1}{t}}\right\}^{\frac{t}{s}\frac{s-r}{t-r}} \tag{23}$$

Now we substitute $t=1/x$, $r=1/y$, $\lambda=\frac{t}{s}\frac{s-r}{t-r}$ in (23), and since $1-\lambda=\frac{r}{s}\frac{t-s}{t-r}$, $s=[\lambda x+(1-\lambda)y]^{-1}$, we have

$$\left[f(k,N,q,[\lambda x+(1-\lambda)y]^{-1})\right]^{-[\lambda x+(1-\lambda)y]}$$

$$\leqslant\left\{\left[f(k,N,q,x^{-1})\right]^{-x}\right\}^{\lambda}\left\{\left[f(k,N,q,y^{-1})\right]^{-y}\right\}^{1-\lambda},$$

concluding log-convexity of the function $u\mapsto\left[f(k,N,q,u^{-1})\right]^{-u}$. □

5. Gini means and further monotonicity

For positive n-tuple $(a_1,\dots,a_n)$, $\alpha,\beta\in\mathbb{R}$, Gini means are defined with

$$G(\alpha,\beta)=\begin{cases}\left(\frac{\sum_{i=1}^{n}a_i^{\alpha}}{\sum_{i=1}^{n}a_i^{\beta}}\right)^{\frac{1}{\alpha-\beta}}, & \alpha\neq\beta;\\ \exp\left(\sum_{i=1}^{n}a_i^{\alpha}\ln a_i/\sum_{i=1}^{n}a_i^{\alpha}\right), & \alpha=\beta.\end{cases}\tag{24}$$

It is known then see [10, pp. 119],

$$G(\alpha_1,\beta_1)\leqslant G(\alpha_2,\beta_2),\tag{25}$$

for $\alpha_1\leqslant\alpha_2$, $\beta_1\leqslant\beta_2$, $\alpha_1\neq\beta$, $\alpha_2\neq\beta_2$.

If we choose $a_i=\frac{k+q}{i+q}$ in (24) we will get Zip-Mandelbrot means:

$$Z(\alpha,\beta)=\begin{cases}\left(\frac{f(k,N,q,\beta)}{f(k,N,q,\alpha)}\right)^{\frac{1}{\alpha-\beta}}, & \alpha\neq\beta;\\ \left[(k+q)^{\alpha}\zeta(N,s,\alpha)\right]^{\frac{(k+q)^{\alpha}H_{N,q,\alpha}}{\alpha f(k,N,q,\alpha)}}\exp\left(-\frac{(k+q)^{\alpha}}{\alpha f(k,N,q,\alpha)}E(k,N,q,\alpha)\right), & \alpha=\beta.\end{cases}\tag{26}$$

where

$$E(k,N,q,\alpha)=-\sum_{k=1}^{N}f(k,N,q,\alpha)\ln f(k,N,q,\alpha)$$

denotes Shannon entropy of the law (3) (for related results see also [3]).

Using (25) we can now formulate the next theorem.

THEOREM 10. *For* $0<\alpha_1\leqslant\alpha_2$, $0<\beta_1\leqslant\beta_2$, $\alpha_1\neq\beta$, $\alpha_2\neq\beta_2$;

$$Z(\alpha_1,\beta_1)\leqslant Z(\alpha_2,\beta_2).\tag{27}$$

The expectation of the Zipf-Mandelbrot law is

$$\sum_{k=1}^{N}kf(k,N,q,s)=\frac{1}{\zeta(N,s,q)}\sum_{k=1}^{N}\frac{k+q-q}{(k+q)^s}=\frac{\zeta(N,s-1,q)}{\zeta(N,s,q)}-q.$$

This is a decreasing function over s, as the next theorem shows.

THEOREM 11. *The function*

$$s\mapsto\frac{\zeta(N,s-1,q)}{\zeta(N,s,q)}$$

is decreasing on $\mathbb{R}_+$.

Proof. We set $a_i = \frac{1}{i+q}$, $i = 1,\ldots,N$ and $\alpha = s-1$, $\beta = s$ in (24). According (27), for $0 < s < t$, we have

$$\left(\frac{\zeta(N,s-1,q)}{\zeta(N,s,q)}\right)^{-1} \leqslant \left(\frac{\zeta(N,t-1,q)}{\zeta(N,t,q)}\right)^{-1}. \quad \square$$

Of course, result can be extended to Hurwitz ζ-function.

COROLLARY 6. *The function*

$$s \mapsto \frac{\zeta(s-1,q)}{\zeta(s,q)}$$

is decreasing on $\mathbb{R}_+$.

REMARK 1. General remark in this section is that parameters α, β in (26) could be any real numbers, so Theorems 10 and 11 are also valid on $\mathbb{R}^2$ and $\mathbb{R}$, respectively.

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REFERENCES

[1] G. BASTIEN, M. ROGALSKI, *Convexité, complète monotonie et inégalités sur les fonctions zêta et gamma, sur les fonctions de opérateurs de Baskakov et sur les fonctions arithmétiques*, Can. J. Math. **54**, 5 (2002), 916–944.

[2] S. FRONTIER, *Diversity and structure in aquatic ecosystems*, Mar. Biol. Ann. Rev. **23** (1985), 253–312.

[3] L. HORVÁTH, Đ. PEČARIĆ, J. PEČARIĆ, *Estimations of f and Rényi divergences by using a cyclic refinement of the Jensen's inequality*, Bull. Malays. Math. Sci. Soc. (accepted).

[4] J. JAKŠETIĆ, J. PEČARIĆ, *Exponential Convexity method*, J. Convex Anal. **20**, 1 (2013), 181–197.

[5] A. LELU, *Jean-Baptiste Estoup and the origins of Zipfs law: a stenographer with a scientific mind (1868–1950)*, Boletín de Estadística e Invenstigación Operativa 6–77, **30**, 1 (2014), 66–77.

[6] B. MANDELBROT, *An information theory of the statistical structure of language*, In Jackson, W., editor, Communication Theory, New York, Academic Press, 1953.

[7] D. Y. MANIN, *Mandelbrot's Model for Zipf's Law: Can Mandelbrot's Model Explain Zipf's Law for Language?*, **16**, 3 (2009), 274–285.

[8] D. MOUILLOT, A. LEPRETRE, *Introduction of relative abundance distribution (RAD) indices, estimated from the rank-frequency diagrams (RFD), to assess changes in community diversity*, Environmental Monitoring and Assessment **63**, 2 (2000), 279–295.

[9] J. PEČARIĆ, *Nejednakosti*, Hrvatsko matematičko društvo, Zagreb, 1996.

[10] J. PEČARIĆ, F. PROSHAN, Y. L. TONG, *Convex functions partial ordering and statistical applications mathematics in Science and Engineering*, vol. 187., Academic Press, 1992.

[11] J. B. WILSON, *Methods for fitting dominance/diversity curves*, **2** (1991), 35–46.
[12] G. K. ZIPF, *Human Behavior and the Principle of Least Effort*, Addison-Wesley, Reading, MA, 1949.

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