

ON THE REVERSE CONVOLUTION INEQUALITIES FOR THE KONTOROVICH–LEBEDEV, FOURIER COSINE TRANSFORMS AND APPLICATIONS

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(Communicated by M. Praljak)

Abstract. In this paper, we investigate some reverse weighted L_p -norm ($p > 1$) inequalities for convolutions related to Kontorovich-Lebedev, Fourier cosine transforms. A class of integro-differential equations involving in Bessel operator are considered. The estimate of scattered acoustic field is established.

1. Introduction

For the Fourier transform, beside the fundamental Young's inequality, the weighted L_p -norm inequalities for Fourier convolution were considered by S. Saitoh *et al* (see [8, 11] and references there in). Inequalities of these types were considered by N. D. V. Nhan, D. T. Duc, V. K. Tuan (see [6] and references there in). Inequalities for Fourier cosine convolution was studied by N. T. Hong (see [3]). The reverse weighted L_p -norm convolution inequalities for Fourier transform and its applications also investigated in [9, 10].

PROPOSITION 1. ([9]) *Let F_1 and F_2 be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \leq F_1(x) \leq M_1^{\frac{1}{p}} < \infty, \quad 0 < m_2^{\frac{1}{p}} \leq F_2(x) \leq M_2^{\frac{1}{p}} < \infty, \quad p > 1, \quad x \in \mathbb{R}. \quad (1)$$

Then for any positive continuous functions ρ_1 and ρ_2 , we have the reverse L_p -weighted convolution inequality

$$\begin{aligned} & \|((F_1 \rho_1) *_F (F_2 \rho_2))(\rho_1 *_F \rho_2)^{\frac{1}{p}-1}\|_{L_p(\mathbb{R})} \\ & \geq A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right)^{-1} \|F_1\|_{L_p(\mathbb{R}; \rho_1)} \|F_2\|_{L_p(\mathbb{R}; \rho_2)}, \end{aligned} \quad (2)$$

where

$$\left(f *_F g \right) (x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad x \in \mathbb{R}. \quad (3)$$

Mathematics subject classification (2010): 35A22, 44A35, 26D15, 26D10, 45E10, 45J05, 33E30, 65R10.

Keywords and phrases: Integral transforms, weighted L_p inequality, reverse Hölder inequality, Kontorovich-Lebedev transform, integro-differential equations.

The main key to prove these results is the reverse Hölder’s inequality.

PROPOSITION 2. ([13]) *For two positive functions f and g satisfying $0 < m \leq \frac{f}{g} \leq M < \infty$ on the set X , and for $p, q > 1$, $p^{-1} + q^{-1} = 1$,*

$$\left(\int_X f d\mu \right)^{\frac{1}{p}} \left(\int_X g d\mu \right)^{\frac{1}{q}} \leq A_{p,q} \left(\frac{m}{M} \right) \int_X f^{\frac{1}{p}} g^{\frac{1}{q}} d\mu, \tag{4}$$

if the right hand side integral converges, where $A_{p,q}(t) = p^{-\frac{1}{p}} q^{-\frac{1}{q}} \frac{t^{-\frac{1}{pq}}(1-t)}{(1-t^{\frac{1}{p}})^{\frac{1}{p}}(1-t^{\frac{1}{q}})^{\frac{1}{q}}}$.

The convolution for Kontorovich-Lebedev transform was first introduced by V. A. Kakichev in [5]

$$(f \underset{\mathcal{KL}}{*} g)(x) = \int_0^\infty \int_0^\infty \frac{1}{2x} e^{-\frac{1}{2}(\frac{uv}{x} + \frac{ux}{v} + \frac{vx}{u})} f(u)g(v)dudv, \quad x > 0, \tag{5}$$

where \mathcal{KL} denotes the Kontorovich-Lebedev transform (see [14, 17])

$$\mathcal{KL}[f](y) = \int_0^\infty K_{iy}(x)f(x)dx, \quad y > 0. \tag{6}$$

Its kernel consists of the Macdonald function $K_\nu(x)$ of the pure imaginary index $\nu = iy$. This function satisfies the differential equation

$$z^2 \frac{d^2u}{dz^2} + z \frac{du}{dz} - (z^2 + \nu^2)u = 0. \tag{7}$$

The inequalities of Young’s type as well as the boundedness in weighted L_p for the Kontorovich-Lebedev transform and its convolution were investigated in [4, 15, 16]. Combining the Kontorovich-Lebedev transform together with the Fourier cosine transform, the following convolution was introduced (see [17])

$$(f \underset{1}{*} g)(x) = \int_0^\infty \int_0^\infty \frac{1}{2\pi x} [e^{-x \cosh(u+\nu)} + e^{-x \cosh(u-\nu)}] f(u)g(v)dudv, \quad x > 0. \tag{8}$$

In the present paper, we will establish the reverse weighted L_p -norm for convolutions (5), (8). In the mentioned applications, we are looking solution of integro-differential equations involing in Bessel operator in the convolution form and estimate it basing on the reverse norm inequalities of convolution. The estimate for the diffraction of an acoustic is given.

2. Reverse convolution inequalities for Kontorovich-Lebedev, Fourier cosine transforms

The aim of this section is drawing a parallel results of reverse weighted L_p -norm inequalities for convolutions related to Kontorovich-Lebedev, Fourier cosine transforms as the Fourier convolution (see [9, 10]).

THEOREM 3. *Let F_1 and F_2 be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \leq F_1(x) \leq M_1^{\frac{1}{p}} < \infty, \quad 0 < m_2^{\frac{1}{p}} \leq F_2(x) \leq M_2^{\frac{1}{p}} < \infty, \quad p > 1, \quad x > 0. \quad (9)$$

Then for any positive functions ρ_1 and ρ_2 we have the following reverse L_p -weighted convolution inequalities

$$\begin{aligned} & \|((F_1\rho_1) \underset{\mathcal{H}\mathcal{L}}{*} (F_2\rho_2))(\rho_1 \underset{\mathcal{H}\mathcal{L}}{*} \rho_2)^{\frac{1}{p}-1}\|_{L_p(\mathbb{R}_+)} \\ & \geq \left(\frac{1}{3}K_0\left(\frac{\sqrt{2}}{3}\right)\right)^{\frac{1}{p}} \left\{A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\right\}^{-1} \|F_1\|_{L_p(\mathbb{R}_+;\rho_1(u)\varphi(u))} \|F_2\|_{L_p(\mathbb{R}_+;\rho_2(v)\varphi(v))}, \end{aligned} \quad (10)$$

$$\begin{aligned} & \|((F_1\rho_1) \underset{1}{*} (F_2\rho_2))(\rho_1 \underset{1}{*} \rho_2)^{\frac{1}{p}-1}\|_{L_p(\mathbb{R}_+;\pi x)} \\ & \geq \left\{A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\right\}^{-1} \|F_1\|_{L_p(\mathbb{R}_+;\frac{\rho_1(u)}{\cosh u})} \|F_2\|_{L_p(\mathbb{R}_+;\frac{\rho_2(v)}{\cosh v})}. \end{aligned} \quad (11)$$

Proof. First, using the AM-GM inequality for three positive real $u, v, \frac{1}{x}$, we obtain $\frac{uv}{x} = uv\frac{1}{x} \leq u^3 + v^3 + \frac{1}{x^3}$. Similarly, we have $\frac{xu}{v} \leq x^3 + u^3 + \frac{1}{v^3}$; $\frac{xv}{u} \leq x^3 + v^3 + \frac{1}{u^3}$. Therefore,

$$\frac{1}{2} \left(\frac{uv}{x} + \frac{xu}{v} + \frac{xv}{u}\right) \leq \left(x^3 + \frac{1}{6x^3}\right) + \left(\frac{u^3}{3} + \frac{1}{6u^3}\right) + \left(\frac{v^3}{3} + \frac{1}{6v^3}\right).$$

So, we obtain the remarkable inequality

$$K(x, u, v) := \frac{1}{2x} e^{-\frac{1}{2}\left(\frac{uv}{x} + \frac{xu}{v} + \frac{xv}{u}\right)} \geq \frac{\varphi(x)\varphi(u)\varphi(v)}{2x}, \quad (12)$$

where $\varphi(t) = e^{-\left(\frac{t^3}{3} + \frac{1}{6t^3}\right)}$. Put $f(u, v) = F_1^p(u)F_2^p(v)K(x, u, v)\rho_1(u)\rho_2(v)$, and put $g(u, v) = K(x, u, v)\rho_1(u)\rho_2(v)$. The condition (9) implies

$$m_1m_2 \leq \frac{f(u, v)}{g(u, v)} \leq M_1M_2, \quad \forall u > 0, v > 0. \quad (13)$$

Applying the reverse Hölder’s inequality (4) for f and g on $X = \mathbb{R}_+^2$, we get

$$\begin{aligned}
 & A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) ((F_1 \rho_1) \underset{\mathcal{H}\mathcal{L}}{*} (F_2 \rho_2))(x) \\
 &= A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \int_0^\infty \int_0^\infty K(x, u, v) F_1(u) \rho_1(u) F_2(v) \rho_2(v) dudv \\
 &= A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \int_0^\infty \int_0^\infty f^{\frac{1}{p}}(u, v) g^{\frac{1}{q}}(u, v) dudv \\
 &\geq \left(\int_0^\infty \int_0^\infty f(u, v) dudv \right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^\infty g(u, v) dudv \right)^{\frac{1}{q}} \\
 &= \left(\int_0^\infty \int_0^\infty F_1^p(u) F_2^p(v) K(x, u, v) \rho_1(u) \rho_2(v) dudv \right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^\infty K(x, u, v) \rho_1(u) \rho_2(v) dudv \right)^{\frac{1}{q}}.
 \end{aligned} \tag{14}$$

Therefore,

$$\begin{aligned}
 & \left(((F_1 \rho_1) \underset{\mathcal{H}\mathcal{L}}{*} (F_2 \rho_2))(x) \right)^p \left((\rho_1 \underset{\mathcal{H}\mathcal{L}}{*} \rho_2)(x) \right)^{1-p} \\
 &\geq A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right)^{-p} \left(\int_0^\infty \int_0^\infty F_1^p(u) F_2^p(v) K(x, u, v) \rho_1(u) \rho_2(v) dudv \right) \\
 &\quad \times \left(\int_0^\infty \int_0^\infty K(x, u, v) \rho_1(u) \rho_2(v) dudv \right)^{\frac{p}{q}} \left(\int_0^\infty \int_0^\infty K(x, u, v) \rho_1(u) \rho_2(v) dudv \right)^{1-p} \\
 &\geq A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right)^{-p} \int_0^\infty \int_0^\infty \frac{\varphi(x) \varphi(u) \varphi(v)}{2x} F_1^p(u) F_2^p(v) \rho_1(u) \rho_2(v) dudv.
 \end{aligned} \tag{15}$$

From inequality (12) and using the Fubini theorem to interchange the order of intergration, we have

$$\begin{aligned}
 & \| ((F_1 \rho_1) \underset{\mathcal{H}\mathcal{L}}{*} (F_2 \rho_2)) (\rho_1 \underset{\mathcal{H}\mathcal{L}}{*} \rho_2)^{\frac{1}{p}-1} \|_{L_p(\mathbb{R}_+)}^p \\
 &= \int_0^\infty \left(((F_1 \rho_1) \underset{\mathcal{H}\mathcal{L}}{*} (F_2 \rho_2))(x) \right)^p \left((\rho_1 \underset{\mathcal{H}\mathcal{L}}{*} \rho_2)(x) \right)^{1-p} dx \\
 &\geq \left\{ A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \right\}^{-p} \int_0^\infty \left(\int_0^\infty \int_0^\infty F_1^p(u) F_2^p(v) K(x, u, v) \rho_1(u) \rho_2(v) dudv \right) dx
 \end{aligned}$$

$$\begin{aligned}
 &\geq \left\{ A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \right\}^{-p} \int_0^\infty \int_0^\infty \int_0^\infty \frac{\varphi(x)\varphi(u)\varphi(v)}{2x} F_1^p(u) F_2^p(v) \rho_1(u) \rho_2(v) dx du dv \\
 &\geq \left\{ A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \right\}^{-p} \int_0^\infty F_1^p(u) \rho_1(u) \varphi(u) du \int_0^\infty F_2^p(v) \rho_2(v) \varphi(v) dv \int_0^\infty \frac{\varphi(x)}{2x} dx \\
 &= \frac{1}{3} K_0 \left(\frac{\sqrt{2}}{3} \right) \left\{ A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \right\}^{-p} \int_0^\infty F_1^p(u) \rho_1(u) \varphi(u) du \int_0^\infty F_2^p(v) \rho_2(v) \varphi(v) \\
 &= \frac{1}{3} K_0 \left(\frac{\sqrt{2}}{3} \right) \left\{ A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \right\}^{-p} \|F_1\|_{L_p(\mathbb{R}_+; \rho_1(u)\varphi(u))}^p \|F_2\|_{L_p(\mathbb{R}_+; \rho_2(v)\varphi(v))}^p. \tag{16}
 \end{aligned}$$

To prove the inequality (11), we note that

$$\begin{aligned}
 T(x, u, v) &= \frac{1}{2\pi x} \left(e^{-x \cosh(u+v)} + e^{-x \cosh(u-v)} \right) \geq \frac{1}{2\pi x} \left(2\sqrt{e^{-x \cosh(u+v)} e^{-x \cosh(u-v)}} \right) \\
 &= \frac{1}{2\pi x} 2e^{-\frac{x}{2}(\cosh(u+v) + \cosh(u-v))} = \frac{1}{\pi x} e^{-x \cosh u \cosh v}. \tag{17}
 \end{aligned}$$

Let $f(u, v) = F_1^p(u) F_2^p(v) T(x, u, v) \rho_1(u) \rho_2(v)$ and $g(u, v) = T(x, u, v) \rho_1(u) \rho_2(v)$. We have

$$m_1 m_2 \leq \frac{f(u, v)}{g(u, v)} \leq M_1 M_2. \tag{18}$$

Applying the reverse Hölder’s inequality (4) for f and g on $X = \mathbb{R}_+^2$, we get

$$\begin{aligned}
 &A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) ((F_1 \rho_1) * (F_2 \rho_2))(x) \\
 &= A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \int_0^\infty \int_0^\infty T(x, u, v) F_1(u) \rho_1(u) F_2(v) \rho_2(v) dudv \\
 &= A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \int_0^\infty \int_0^\infty f^{\frac{1}{p}}(u, v) g^{\frac{1}{q}}(u, v) dudv \\
 &\geq \left(\int_0^\infty \int_0^\infty f(u, v) dudv \right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^\infty g(u, v) dudv \right)^{\frac{1}{q}} \\
 &= \left(\int_0^\infty \int_0^\infty F_1^p(u) F_2^p(v) T(x, u, v) \rho_1(u) \rho_2(v) dudv \right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^\infty T(x, u, v) \rho_1(u) \rho_2(v) dudv \right)^{\frac{1}{q}}. \tag{19}
 \end{aligned}$$

Therefore,

$$\begin{aligned} & \left(((F_1 \rho_1) * (F_2 \rho_2))(x) \right)^p \left((\rho_1 * \rho_2)(x) \right)^{1-p} \\ & \geq A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right)^{-p} \int_0^\infty \int_0^\infty \frac{1}{\pi x} e^{-x \cosh u \cosh v} F_1^p(u) F_2^p(v) \rho_1(u) \rho_2(v) du dv. \end{aligned} \tag{20}$$

Similarly, we obtain

$$\begin{aligned} & \| ((F_1 \rho_1) * (F_2 \rho_2)) (\rho_1 * \rho_2)^{\frac{1}{p}-1} \|_{L_p(\mathbb{R}_+; \pi x)}^p \\ & = \int_0^\infty \left(((F_1 \rho_1) * (F_2 \rho_2))(x) \right)^p \left((\rho_1 * \rho_2)(x) \right)^{1-p} \pi x dx \\ & \geq \left\{ A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \right\}^{-p} \int_0^\infty \left(\int_0^\infty \int_0^\infty F_1^p(u) F_2^p(v) T(x, u, v) \rho_1(u) \rho_2(v) du dv \right) \pi x dx \\ & \geq \left\{ A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \right\}^{-p} \int_0^\infty \int_0^\infty \int_0^\infty e^{-x \cosh u \cosh v} F_1^p(u) F_2^p(v) \rho_1(u) \rho_2(v) dx du dv \\ & \geq A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right)^{-p} \int_0^\infty F_1^p(u) \rho_1(u) du \int_0^\infty F_2^p(v) \rho_2(v) dv \int_0^\infty e^{-x \cosh u \cosh v} dx \\ & = \left\{ A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \right\}^{-p} \int_0^\infty F_1^p(u) \frac{\rho_1(u)}{\cosh u} du \int_0^\infty F_2^p(v) \frac{\rho_2(v)}{\cosh v} dv \\ & = \left\{ A_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \right\}^{-p} \|F_1\|_{L_p(\mathbb{R}_+; \frac{\rho_1(u)}{\cosh u})}^p \|F_2\|_{L_p(\mathbb{R}_+; \frac{\rho_2(v)}{\cosh v})}^p. \end{aligned} \tag{21}$$

The proof is complete. \square

In various problems, the solutions can be presented in the convolution form. In the cases of interest, the following reverse inequalities could be used to prove the lower boundedness of these solutions.

Denote $\mathbb{R}_\alpha = (0; \alpha)$ for $\alpha > 0$.

THEOREM 4. *Let f and g be positive functions satisfying*

$$0 < f(x) \leq M < \infty, \quad 0 < g(x) \leq N < \infty, \quad x > 0. \tag{22}$$

i) Assume that $f \in L_p(\mathbb{R}_+; \varphi)$, $g \in L_p(\mathbb{R}_+; \varphi)$, where $\varphi(t) = e^{-\left(\frac{t^3}{3} + \frac{1}{6t^3}\right)}$. We have the inequality

$$(f \underset{\mathcal{X}\mathcal{L}}{*} g)(x) \geq \frac{\varphi(x)}{(M.N)^{p-1} 2x} \|f\|_{L_p(\mathbb{R}_+; \varphi)}^p \|g\|_{L_p(\mathbb{R}_+; \varphi)}^p. \tag{23}$$

Moreover, the following inequality

$$\|(f \underset{\mathcal{H}\mathcal{L}}{*} g)\|_{L_r(\mathbb{R}_+)} \geq \left(\frac{2^{\frac{5+r}{6}} K_{\frac{1-r}{3}}(\frac{\sqrt{2}r}{3})}{3(MN)^{r(p-1)}} \right)^{\frac{1}{r}} \|f\|_{L_p(\mathbb{R}_+;\varphi)} \|g\|_{L_p(\mathbb{R}_+;\varphi)} \tag{24}$$

holds true for $r > p > 1$.

ii) Assume that $f \in L_p(\mathbb{R}_\alpha), g \in L_p(\mathbb{R}_\beta)$, where α, β are positive real number. We have the inequality

$$(f \underset{1}{*} g)(x) \geq \frac{e^{-x \cosh \alpha \cosh \beta}}{(MN)^{p-1} \pi x} \|f\|_{L_p(\mathbb{R}_\alpha)}^p \|g\|_{L_p(\mathbb{R}_\beta)}^p. \tag{25}$$

Moreover, the following inequality

$$\|(f \underset{1}{*} g)\|_{L_q(\mathbb{R}_r;x^\gamma)} \geq \left(\frac{r^{\gamma-q+1}}{\gamma-q+1} \right)^{\frac{1}{q}} \frac{e^{-r \cosh \alpha \cosh \beta}}{\pi(MN)^{p-1}} \|f\|_{L_p(\mathbb{R}_\alpha)}^p \|g\|_{L_p(\mathbb{R}_\beta)}^p \tag{26}$$

holds true for $\gamma \geq q > 1, r > 1$.

Proof. i) We have

$$\begin{aligned} (f^p \underset{\mathcal{H}\mathcal{L}}{*} g^p)(x) &= \int_0^\infty \int_0^\infty K(x, u, v) f^p(u) g^p(v) dudv \\ &\leq \int_0^\infty \int_0^\infty K(x, u, v) M^{p-1} N^{p-1} f(u) g(v) dudv = (MN)^{p-1} (f \underset{\mathcal{H}\mathcal{L}}{*} g)(x). \end{aligned} \tag{27}$$

On the other hand, we have

$$\begin{aligned} (f^p \underset{\mathcal{H}\mathcal{L}}{*} g^p)(x) &= \int_0^\infty \int_0^\infty K(x, u, v) f^p(u) g^p(v) dudv \\ &\geq \int_0^\infty \int_0^\infty \frac{\varphi(x)\varphi(u)\varphi(v)}{2x} f^p(u) g^p(v) dudv \\ &\geq \frac{\varphi(x)}{2x} \int_0^\infty \int_0^\infty f^p(u) \varphi(u) g^p(v) \varphi(v) dudv \\ &= \frac{\varphi(x)}{2x} \|f\|_{L_p(\mathbb{R}_+;\varphi)}^p \|g\|_{L_p(\mathbb{R}_+;\varphi)}^p. \end{aligned} \tag{28}$$

From (27) and (28), we obtain the estimate

$$(f \underset{\mathcal{H}\mathcal{L}}{*} g)(x) \geq \frac{\varphi(x)}{2x(M.N)^{p-1}} \|f\|_{L_p(\mathbb{R}_+;\varphi)}^p \|g\|_{L_p(\mathbb{R}_+;\varphi)}^p. \tag{29}$$

Therefore,

$$\begin{aligned}
 \|(f \underset{\mathcal{H}\mathcal{L}}{*} g)\|_{L_r(\mathbb{R}_+)}^r &= \int_0^\infty |f \underset{\mathcal{H}\mathcal{L}}{*} g|^r(x) dx \\
 &\geq \|f\|_{L_p(\mathbb{R}_+; \varphi)}^{pr} \|g\|_{L_p(\mathbb{R}_+; \varphi)}^{pr} \int_0^\infty \left(\frac{\varphi(x)}{2x(M.N)^{p-1}}\right)^r dx \\
 &= \frac{2^{\frac{5+r}{6}} K_{\frac{1-r}{3}}(\frac{\sqrt{2}r}{3})}{3(MN)^{r(p-1)}} \|f\|_{L_p(\mathbb{R}_+; \varphi)}^{pr} \|g\|_{L_p(\mathbb{R}_+; \varphi)}^{pr}.
 \end{aligned} \tag{30}$$

Hence, one can obtain (24).

ii) Similarly, we have

$$\begin{aligned}
 (f^p \underset{1}{*} g^p)(x) &= \int_0^\infty \int_0^\infty T(x, u, v) f^p(u) g^p(v) dudv \\
 &\leq \int_0^\infty \int_0^\infty T(x, u, v) (MN)^{p-1} f(u) g(v) dudv \\
 &= (MN)^{p-1} (f \underset{1}{*} g)(x).
 \end{aligned} \tag{31}$$

On the other hand,

$$\begin{aligned}
 (f^p \underset{1}{*} g^p)(x) &= \int_0^\infty \int_0^\infty T(x, u, v) f^p(u) g^p(v) dudv \\
 &\geq \int_0^\infty \int_0^\infty \frac{1}{\pi x} e^{-x \cosh u \cosh v} f^p(u) g^p(v) dudv \\
 &\geq \int_0^\beta \int_0^\alpha \frac{1}{\pi x} e^{-x \cosh \alpha \cosh \beta} f^p(u) g^p(v) dudv \\
 &= \frac{e^{-x \cdot \cosh \alpha \cosh \beta}}{\pi x} \|f\|_{L_p(\mathbb{R}_\alpha)}^p \|g\|_{L_p(\mathbb{R}_\beta)}^p.
 \end{aligned} \tag{32}$$

From (31) and (32), one can obtain the estimate respective to x on $(0; r)$.

$$(f \underset{1}{*} g)(x) \geq \frac{e^{-x \cosh \alpha \cosh \beta}}{(MN)^{p-1} \pi x} \|f\|_{L_p(\mathbb{R}_\alpha)}^p \|g\|_{L_p(\mathbb{R}_\beta)}^p. \tag{33}$$

Therefore,

$$\begin{aligned}
 \|(f *_1 g)\|_{L_q(\mathbb{R}_+; x^\gamma)} &= \left(\int_0^r (f *_1 g)^q(x) x^\gamma dx \right)^{\frac{1}{q}} \\
 &\geq \frac{e^{-r \cdot \cosh \alpha \cosh \beta}}{(MN)^{p-1} \pi} \|f\|_{L_p(\mathbb{R}_\alpha)}^p \|g\|_{L_p(\mathbb{R}_\beta)}^p \left(\int_0^r x^{\gamma-q} dx \right)^{\frac{1}{q}} \\
 &= \left(\frac{r^{\gamma-q+1}}{\gamma-q+1} \right)^{\frac{1}{q}} \frac{e^{-r \cdot \cosh \alpha \cosh \beta}}{\pi(MN)^{p-1}} \|f\|_{L_p(\mathbb{R}_\alpha)}^p \|g\|_{L_p(\mathbb{R}_\beta)}^p, \tag{34}
 \end{aligned}$$

where $\gamma \geq q > 1, r > 1$. \square

3. Applications

3.1. A class of integro-differential equations involving the Bessel operator

Inspite of having many useful applications, not many integro-differential equations can be solved in closed form. In this section, we consider a class of the integro-differential equation

$$f(x) + \frac{1}{2x} D \int_0^\infty \int_0^\infty e^{-\frac{1}{2} \left(\frac{uv}{x} + \frac{ux}{v} + \frac{vx}{u} \right)} f(u)h(v) dudv = g(x), \quad x > 0, \tag{35}$$

which are arisen naturally from the integral equation (see [4, 14, 15])

$$f(x) + \frac{1}{2x} \int_0^\infty \int_0^\infty e^{-\frac{1}{2} \left(\frac{uv}{x} + \frac{ux}{v} + \frac{vx}{u} \right)} f(u)h(v) dudv = g(x), \quad x > 0, \tag{36}$$

where D is a differential operator.

In [14], S. B. Yakubovich introduced the following function space

$$L^\alpha(\mathbb{R}_+) \equiv L(\mathbb{R}_+, K_\alpha(x)), \quad \alpha \geq 0, \tag{37}$$

with the norm

$$\|f\|_{L^\alpha(\mathbb{R}_+)} = \int_0^\infty K_\alpha(x) |f(x)| dx. \tag{38}$$

In this function space, the following analog of the Wiener theorem was established.

THEOREM 5. ([14]) *Let $f \in L^\alpha(\mathbb{R}_+)$. If $\mathcal{F}(s) = \lambda + \mathcal{H} \mathcal{L}[f](-is) \neq 0$ for all s in the closed strip $|\Re(s)| \leq \alpha$, including infinity then there is a unique q from L^α such that*

$$\frac{1}{\lambda + \mathcal{H} \mathcal{L}[f](-is)} = \lambda + \mathcal{H} \mathcal{L}[q](-is). \tag{39}$$

LEMMA 1. [16] Let $f \in L_2(\mathbb{R}_+; x)$ and $h \in L^0(\mathbb{R}_+)$. Then, $(f \underset{\mathcal{H}\mathcal{L}}{*} h) \in L_2(\mathbb{R}_+; x)$ and satisfies the factorization equation

$$\mathcal{H}\mathcal{L}[f \underset{\mathcal{H}\mathcal{L}}{*} h](y) = \mathcal{H}\mathcal{L}[f](y)\mathcal{H}\mathcal{L}[h](y). \tag{40}$$

Now, we consider the case operator D is the Bessel operator \mathcal{B} of the form

$$\mathcal{B}[\omega](x) := \left(x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - x^2 \right) \omega(x). \tag{41}$$

The equation (35) can be written in the form

$$f(x) + \frac{1}{2x} \mathcal{B} \left[\int_0^\infty \int_0^\infty e^{-\frac{1}{2} \left(\frac{uv}{v} + \frac{xv}{u} + \frac{uv}{x} \right)} f(u)h(v) dudv \right] = g(x). \tag{42}$$

We will find the solution of equation (42) in the $L_2(\mathbb{R}_+; x) \cap L^0(\mathbb{R}_+)$.

THEOREM 6. Let $g \in L_2(\mathbb{R}_+; x)$ and $h \in L^0(\mathbb{R}_+)$ are given functions, satisfying

$$1 - y^2 \mathcal{H}\mathcal{L}[h](y) \neq 0, \forall y > 0; \tag{43}$$

$$y^2 \mathcal{H}\mathcal{L}[h](y) \in L_2(\mathbb{R}_+; y \sinh \pi y); \mathcal{H}\mathcal{L}^{-1}(y^2 \mathcal{H}\mathcal{L}[h](y)) = h_1(y) \in L^0(\mathbb{R}_+). \tag{44}$$

The equation (42) has a unique solution in $L^0(\mathbb{R}_+) \cap L_2(\mathbb{R}_+; x)$ which can be presented in the convolution form

$$f(x) = g(x) + (g \underset{\mathcal{H}\mathcal{L}}{*} l)(x), \tag{45}$$

where l exists uniquely via the following

$$\frac{y^2 \mathcal{H}\mathcal{L}[f](y)}{1 - y^2 \mathcal{H}\mathcal{L}[h](y)} = \mathcal{H}\mathcal{L}[l](y). \tag{46}$$

Assuming that g, l belong to $L_p(\mathbb{R}_+; \varphi(x))$ and $0 < g(x) \leq M < \infty, 0 < l(x) \leq N < \infty, \forall x > 0$, we obtain

$$f(x) \geq g(x) + \frac{\varphi(x)}{(M.N)^{p-1} 2x} \|g\|_{L_p(\mathbb{R}_+; \varphi(x))} \|l\|_{L_p(\mathbb{R}_+; \varphi(x))}, \quad x > 0. \tag{47}$$

Furthermore,

i) the asymptotics of the solution

$$f(x) = g(x) \left(1 + \mathcal{O}\left(\frac{1}{x}\right) \right), \quad x \rightarrow \infty \tag{48}$$

is valid if we add the condition $g, l \in L_1(\mathbb{R}_+)$.

ii) a norm estimate

$$\|f\|_{L_{2p}(\mathbb{R}_+; x^p)} \geq \left[2(MN)^{1-p} \|g\|_{L_p(\mathbb{R}_+; \varphi(x))} \|g\|_{L_p(\mathbb{R}_+; (\varphi(x))^p)} \|l\|_{L_p(\mathbb{R}_+; \varphi(x))} \right]^{\frac{1}{2}} \tag{49}$$

holds on if we add condition $g \in L_p(\mathbb{R}_+; (\varphi(x))^p)$ and $f \in L_{2p}(\mathbb{R}_+; x^p)$.

Proof. Since $h \in L^0(\mathbb{R}_+)$, $g \in L_2(\mathbb{R}_+)$, then viture the (1), we have $\omega(x) := (f \underset{\mathcal{H}\mathcal{L}}{*} h)(x) \in L_2(\mathbb{R}_+; x)$. For $\omega(x) \in L_2(\mathbb{R}_+; x)$, by using the inverse formula for the Kontorovich-Lebedev transform between two spaces $L_2(\mathbb{R}_+; x)$ and $L_2(\mathbb{R}_+; \frac{\pi^2}{2}y \sinh \pi y)$, we have

$$\omega(x) = \mathcal{H}\mathcal{L}^{-1}[\Psi(y)] := \frac{2}{\pi^2 x} \int_0^\infty K_{iy}(x) y \sinh \pi y \Psi(y) dy, \tag{50}$$

therefore we obtain

$$2x\omega(x) = \frac{4}{\pi^2} \int_0^\infty K_{iy}(x) y \sinh \pi y \Psi(y) dy, \tag{51}$$

where $\Psi(y) = \mathcal{H}\mathcal{L}[\omega](y)$. We have

$$\mathcal{B}[2x\omega(x)] = \frac{4}{\pi^2} \int_0^\infty \mathcal{B}[K_{iy}(x)] y \sinh \pi y \Psi(y) dy, \tag{52}$$

if integral (51) and

$$\int_0^\infty \left(\frac{\partial}{\partial x} K_{iy}(x) \right) y \sinh \pi y \Psi(y) dy, \tag{53}$$

$$\int_0^\infty \left(\frac{\partial^2}{\partial x^2} K_{iy}(x) \right) y \sinh \pi y \Psi(y) dy, \tag{54}$$

converge uniformly on any compact subset of \mathbb{R}_+ . Since the Macdonald function is a solution of the Bessel equation, we obtain (see [4, 12, 16])

$$\mathcal{B}[K_{iy}(x)] = -y^2 K_{iy}(x). \tag{55}$$

Using (41), (50), (52), we have

$$\begin{aligned} \mathcal{H}\mathcal{L} \left[\frac{1}{2x} \mathcal{B}[2x\omega(x)] \right] (y) &= \int_0^\infty \frac{1}{2x} K_{iy}(x) \frac{4}{\pi^2} \int_0^\infty \mathcal{B}[K_{i\tau}(x)] \tau \sinh \pi \tau \Psi(\tau) d\tau dx \\ &= \mathcal{H}\mathcal{L} [\mathcal{H}\mathcal{L}^{-1}[-\tau^2 \Psi(\tau)]] (y, t) = -y^2 \Psi(y). \end{aligned} \tag{56}$$

Formula (56) holds true if

$$y^2 \Psi(y) \in L_2(\mathbb{R}_+; y \sinh \pi y). \tag{57}$$

By taking the Kontorovich-Lebedev transform to both side of (42), we obtain

$$\mathcal{H}\mathcal{L}[f](y) - y^2 \mathcal{H}\mathcal{L}[f](y) \mathcal{H}\mathcal{L}[h](y) = \mathcal{H}\mathcal{L}[g](y), \quad y > 0. \tag{58}$$

Combining with (43), we have

$$\mathcal{H}\mathcal{L}[f](y) = \frac{\mathcal{H}\mathcal{L}[g](y)}{1 - y^2\mathcal{H}\mathcal{L}[h](y)}. \tag{59}$$

The condition (44) implies that exists a unique h_1 belongs to $L^0(\mathbb{R}_+)$ such that $\mathcal{H}\mathcal{L}[h_1](y) = y^2\mathcal{H}\mathcal{L}[h](y)$. In virtue of Wiener-Levy theorem [14], there exists a unique function $l \in L^0(\mathbb{R}_+)$ satifying $\frac{\mathcal{H}\mathcal{L}[h_1](y)}{1 - \mathcal{H}\mathcal{L}[h_1](y)} = \mathcal{H}\mathcal{L}[l](y)$. The equation (59) can be rewritten $\mathcal{H}\mathcal{L}[f](y) = \mathcal{H}\mathcal{L}[g](y) + \mathcal{H}\mathcal{L}[g](y)\mathcal{H}\mathcal{L}[l](y)$. Therefore, the solution of equation (42) can be represented in the convolution form (45).

On the other hand, using formula (233), page 73 in [2]

$$z \frac{\partial K_\nu(z)}{\partial z} = \nu K_\nu(z) - zK_{\nu+1}(z), \tag{60}$$

we have

$$\frac{\partial K_{iy}(x)}{\partial x} = \frac{1}{x} (iyK_{iy}(x) - xK_{iy+1}(x)), \tag{61}$$

and

$$\frac{\partial^2 K_{iy}(x)}{\partial^2 x} = -\frac{iy+y^2}{x^2}K_{iy}(x) - \frac{2iy+1}{x}K_{iy+1}(x) + K_{iy+2}(x). \tag{62}$$

Applying the integral representation 9.6.25, page 376 in [1]

$$K_\nu(x) = \frac{2^\nu\Gamma(\nu+1/2)}{\sqrt{\pi}x^\nu} \int_0^\infty \cos(xt)(t^2+1)^{-\nu-1/2}dt, \quad \Re\nu > -1/2,$$

with $\nu = 1 + iy$ and $\nu = 2 + iy$, and recalling that

$$\Gamma(\nu+1) = \nu\Gamma(\nu), \quad |\Gamma(1/2+iy)| = \sqrt{\frac{\pi}{\cosh(\pi y)}},$$

we obtain

$$|K_{iy+1}(x)| \leq \frac{C(1+y)e^{-\pi y/2}}{x}, \quad |K_{iy+2}(x)| \leq \frac{C(1+y^2)e^{-\pi y/2}}{x^2}, \quad \text{for any } y > 0,$$

uniformly in x on any compact subset of \mathbb{R}_+ . For $K_{iy}(x)$ the following estimate holds [14]

$$|K_{iy}(x)| \leq e^{-\delta y}K_0(x\cos\delta), \quad 0 < \delta < \frac{\pi}{2}. \tag{63}$$

On the other hand,

$$|\mathcal{H}\mathcal{L}[f](y)| = \left| \int_0^\infty K_{iy}(\nu)f(\nu)d\nu \right| \leq \int_0^\infty K_0(\nu)|f(\nu)|d\nu = \|f\|_{L_1(\mathbb{R}_+;K_0(\nu))}. \tag{64}$$

We have

$$\begin{aligned} & \left| \int_0^\infty K_{iy+v}(x) y^m \sinh \pi y \Psi(y) dy \right| \\ & \leq C \int_0^\infty |K_{iy+v}(x)| y^m \sinh \pi y |\mathcal{H} \mathcal{L}[f](y)| |\mathcal{H} \mathcal{L}[h](y)| dy \\ & \leq C \int_0^\infty |K_{iy+v}(x)| y^m dy < C_{m,v}, \quad v = 0, 1, 2, \quad m = 0, 1, 2. \end{aligned} \tag{65}$$

Thus, integrals (51), (53), and (54) converge uniformly on any compact subset of \mathbb{R}_+ , and (52) holds. We have

$$\begin{aligned} \int_0^\infty |y^2 \Psi(y)|^2 y \sinh \pi y dy &= \int_0^\infty |\mathcal{H} \mathcal{L}[f](y)|^2 |y^2 \mathcal{H} \mathcal{L}[h](y)|^2 y \sinh \pi y dy \\ &\leq \|f\|_{L_1(\mathbb{R}_+; K_0(v))}^2 \int_0^\infty |\mathcal{H} \mathcal{L}[h_1](y)|^2 y \sinh \pi y dy \\ &= \|f\|_{L_1(\mathbb{R}_+; K_0(v))}^2 \|\mathcal{H} \mathcal{L}[h_1]\|_{L_2(\mathbb{R}_+; y \sinh \pi y)}^2 < \infty. \end{aligned}$$

Consequently, $y^2 \Psi(y) \in L_2(\mathbb{R}_+; y \sinh \pi y)$.

i) We obtain the inequality (47) by applying directly the inequality (29). Using the elementary inequality

$$a^2 + b^2 \geq 2ab, \tag{66}$$

we obtain

$$|K(x, u, v)| := \frac{1}{2x} e^{-\frac{1}{2}(\frac{uv}{x} + \frac{u^2+v^2}{uv}x)} \leq \frac{e^{-x}}{2x}. \tag{67}$$

Therefore,

$$f(x) = g(x) + (g \underset{\mathcal{H} \mathcal{L}}{*} l)(x) \leq g(x) + \int_0^\infty \int_0^\infty \frac{e^{-x}}{2x} g(u) l(v) dudv \tag{68}$$

$$= g(x) + \frac{e^{-x}}{2x} \|g\|_{L_1(\mathbb{R}_+)} \|l\|_{L_1(\mathbb{R}_+)}. \tag{69}$$

From (47) and (69), one can obtain (48).

ii) Recalling the inequality (66), (47) we have

$$\begin{aligned} & \left(\int_0^\infty |f(x)|^{2p} x^p dx \right)^{\frac{1}{2p}} = \left(\int_0^\infty |g(x) + (g \underset{\mathcal{H}\mathcal{L}}{*} I)(x)|^{2p} x^p dx \right)^{\frac{1}{2p}} \\ & \geq \left(\int_0^\infty 2^{2p} |g(x)|^p |(g \underset{\mathcal{H}\mathcal{L}}{*} I)(x)|^p dx \right)^{\frac{1}{2p}} \\ & \geq \left(\int_0^\infty 2^p |g(x)|^p \frac{(\varphi(x))^p}{(M.N)^{p(p-1)}} \|g\|_{L_p(\mathbb{R}_+; \varphi(x))}^{p^2} \|I\|_{L_p(\mathbb{R}_+; \varphi(x))}^{p^2} dy \right)^{\frac{1}{2p}} \\ & = \left[2(MN)^{1-p} \|g\|_{L_p(\mathbb{R}_+; \varphi(x))} \|g\|_{L_p(\mathbb{R}_+; (\varphi(x))^p)}^p \|I\|_{L_p(\mathbb{R}_+; \varphi(x))}^p \right]^{\frac{1}{2}}. \end{aligned}$$

The proof is complete. \square

Next, we consider the operator D is the differential operator of the infinite order, related to Bessel operator

$$\begin{aligned} D[\omega](x) &= \lim_{N \rightarrow \infty} \prod_{k=1}^N \left(-1 + \frac{\mathcal{B}}{(2k-1)^2} \right) [\omega](x) \\ &= - \lim_{N \rightarrow \infty} \prod_{k=1}^N \left(1 + \frac{x \left(x - \frac{d}{dx} - x \frac{d^2}{dx^2} \right)}{(2k-1)^2} \right) [\omega](x). \end{aligned} \tag{70}$$

The equation (35) can be rewritten in the form

$$f(x) - \frac{1}{2x} \lim_{N \rightarrow \infty} \prod_{k=1}^N \left(1 + \frac{x \left(x - \frac{d}{dx} - x \frac{d^2}{dx^2} \right)}{(2k-1)^2} \right) \int_0^\infty \int_0^\infty e^{-\frac{1}{2} \left(\frac{xu}{v} + \frac{xv}{u} + \frac{uv}{x} \right)} f(u)h(v)dudv = g(x). \tag{71}$$

Assuming that $h \in L^0(\mathbb{R}_+)$ and $g \in L_2(\mathbb{R}_+; x)$ are given functions. Recalling the results in [16], we have

$$\mathcal{H}\mathcal{L}[G](y) = \mathcal{H}\mathcal{L}[f](y) \mathcal{H}\mathcal{L}[h](y) \cosh \frac{\pi y}{2}, \tag{72}$$

where

$$G(x) = \frac{1}{2x} \lim_{N \rightarrow \infty} \prod_{k=1}^N \left(1 + \frac{x \left(x - \frac{d}{dx} - x \frac{d^2}{dx^2} \right)}{(2k-1)^2} \right) \int_0^\infty \int_0^\infty e^{-\frac{1}{2} \left(\frac{xu}{v} + \frac{xv}{u} + \frac{uv}{x} \right)} f(u)h(v)dudv. \tag{73}$$

By taking the Kontorovich-Lebedev transform to the both side of (71), we obtain

$$\mathcal{K}\mathcal{L}[f](y) - \mathcal{K}\mathcal{L}[f](y)\mathcal{K}\mathcal{L}[h](y) \cosh \frac{\pi y}{2} = \mathcal{K}\mathcal{L}[g](y), \quad x > 0. \quad (74)$$

Therefore,

$$\mathcal{K}\mathcal{L}[f](y) = \frac{\mathcal{K}\mathcal{L}[g](y)}{1 - \cosh \frac{\pi y}{2} \mathcal{K}\mathcal{L}[h](y)} \quad (75)$$

if

$$1 - \cosh \frac{\pi y}{2} \mathcal{K}\mathcal{L}[h](y) \neq 0, \forall y > 0. \quad (76)$$

Assuming that exists $q_1 \in L^0(\mathbb{R}_+)$ such that $\mathcal{K}\mathcal{L}[q_1](y) = \cosh \frac{\pi y}{2} \mathcal{K}\mathcal{L}[h](y)$. In virtue of Wiener-Levy theorem [14], there exists a unique function $q \in L^0(\mathbb{R}_+)$ satisfying $\frac{\mathcal{K}\mathcal{L}[q_1](y)}{1 - \mathcal{K}\mathcal{L}[q_1](y)} = \mathcal{K}\mathcal{L}[q](y)$, then

$$\frac{\cosh \frac{\pi y}{2} \mathcal{K}\mathcal{L}[h](y)}{1 - \cosh \frac{\pi y}{2} \mathcal{K}\mathcal{L}[h](y)} = \mathcal{K}\mathcal{L}[q](y). \quad (77)$$

The equation (75) can be rewritten

$$\mathcal{K}\mathcal{L}[f](y) = \mathcal{K}\mathcal{L}[g](y) + \mathcal{K}\mathcal{L}[g](y)\mathcal{K}\mathcal{L}[q](y).$$

Therefore, the solution of equation (71) can be represented in the convolution form

$$f(x) = g(x) + (g \underset{\mathcal{K}\mathcal{L}}{*} q)(x). \quad (78)$$

3.2. Estimate the diffraction of an acoustic via inequalities of generalized convolution

We will use the similar statement about a particular of the scattered acoustic field considered in [12] with the spectral potential function $g(u)$ is defined

$$(F_c g)(t) = \sinh(2\pi t)u(t). \quad (79)$$

Hence, we have

$$\begin{aligned} U(x) &= \frac{-\pi\sqrt{x}}{2} \frac{2}{\pi^2 x} \int_0^\infty t K_{it}(x) (F_c g)(t) (F_c h_1)(t) dt \\ &= \frac{-\pi}{2} \sqrt{x} (g \underset{1}{*} h_1)(x). \end{aligned} \quad (80)$$

Formula (80) gives us another representation of $U(x)$ in a form related to the generalized convolution (8). Assuming that g is positive function and $g \in L_p(\mathbb{R}_\alpha)$, $\alpha > 0$. By using the inequality (25), we have

$$|U(x)| = \frac{\pi}{2} \sqrt{x} (g *_1 h_1)(x) \geq \frac{e^{-x \cosh \alpha \cosh \beta}}{2(MN)^{p-1} \sqrt{x}} \|g\|_{L_p(\mathbb{R}_\alpha)}^p \|h_1\|_{L_p(\mathbb{R}_\beta)}^p, \beta > 0. \quad (81)$$

In particular, $p = 2$, we obtain $\int_0^\beta \frac{1}{\cosh^2 \frac{v}{2}} dv = 2 \operatorname{Tanh} \beta$, $\beta > 0$. Thus,

$$|U(x)| \geq 2 \operatorname{Tanh} \beta \frac{e^{-x \cosh \alpha \cosh \beta}}{(MN) \sqrt{x}} \|g\|_{L_2(\mathbb{R}_\alpha)}^2.$$

If $U \in L_2(\mathbb{R}_r; x^2)$, $r > 1$ and $g \in L_2(\mathbb{R}_\alpha)$, $\alpha > 0$, $\beta > 0$ then we have estimate

$$\begin{aligned} \|U\|_{L_2(\mathbb{R}_r; x^2)}^2 &= \int_0^r |U(x)|^2 x dx = \int_0^r \left| \frac{2U(x)}{\pi \sqrt{x}} \right|^2 \frac{\pi^2 x^2}{4} dx = \int_0^r [(g *_1 h_1)(x)]^2 \frac{\pi^2 x^2}{4} dx \\ &= \frac{\pi^2}{4} \|[(g *_1 h_1)]\|_{L_2(\mathbb{R}_r; x^2)}^2 \geq \frac{r}{\pi M} \operatorname{Tanh}^2 \beta e^{-2r \cosh \alpha \cosh \beta} \|g\|_{L_2(\mathbb{R}_\alpha)}^4, \end{aligned} \quad (82)$$

by using the inequality (26) with $\gamma = q = 2$, $p = 2$, $M = \sup_{x \in \mathbb{R}_+} g(x) < \infty$ and $0 < h_1(x) =$

$$\frac{1}{\cosh \frac{x}{2}} < N = 1, \quad x > 0.$$

Therefore,

$$\|U\|_{L_2(\mathbb{R}_r; x^2)} \geq \sqrt{\frac{r}{\pi M}} \operatorname{Tanh} \beta e^{-r \cosh \alpha \cosh \beta} \|g\|_{L_2(\mathbb{R}_\alpha)}^2.$$

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(Received November 1, 2017)

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