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INFINITE SERIES FORMULA FOR HÜBNER UPPER BOUND FUNCTION WITH APPLICATIONS TO HERSCH–PFLUGER DISTORTION FUNCTION

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Abstract. In this paper, we find an infinite series formula for the Hübner upper bound function, present the bounds for the Hersch-Pfluger distortion function, and improve the well known results on the quasiconformal Schwarz lemma and the solutions of the Ramanujan modular equations. Besides, the submultiplicative and power submultiplicative properties of the Hersch-Pfluger distortion function are also discussed.

1. Introduction

In 1952, Hersch and Pfluger [24, 27] generalized the classical Schwarz Lemma for analytic functions to the class $QC_K(\mathbb{B})$ $(K \geqslant 1)$ of K-quasiconformal mappings of the unit disk $\mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}$ onto itself with the origin fixed. They proved that there exists strictly increasing function $\varphi_K : [0,1] \to [0,1]$ such that

$$|f(z)| \leqslant \varphi_K(|z|) \tag{1.1}$$

for each $f \in QC_K(\mathbb{B})$ and $z \in \mathbb{B}$. Inequality (1.1) is known as the Schwarz Lemma for Quasiconformal mappings and the function φ_K is said to be Hersch-Pfluger distortion function defined by

$$\begin{cases} \varphi_K(r) = \mu^{-1}\left(\dfrac{\mu(r)}{K}\right), \\ \mu(r) = \dfrac{\pi}{2}\dfrac{\mathscr{K}'(r)}{\mathscr{K}(r)}, \\ \mathscr{K} = \mathscr{K}(r) = \displaystyle\int_0^{\pi/2} (1-r^2\sin^2\theta)^{-1/2}d\theta, \quad \mathscr{K}'(r) = \mathscr{K}(r') \end{cases} \tag{1.2}$$

for $r \in (0,1)$ and $K \in (0,\infty)$, $\varphi_K(0) = \varphi_K(1) - 1 = 0$, where $r' = \sqrt{1-r^2}$, $0 < r < 1$, $\mu(r)$ is the conformal modulus of the Grötzsch ring $\mathbb{B} \setminus [0,r]$ and $\mathscr{K} = \mathscr{K}(r)$ is the

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classical complete elliptic integral of the first kind [1, 3, 7, 11, 12, 13, 14, 21, 22, 23, 37, 38, 39, 42, 43, 44, 46, 48, 52]. For later reference, we recall that the complete elliptic integrals of the second kind [15, 16, 17, 18, 19, 20, 34, 40, 41, 45, 49, 50, 51] is defined by

$$\mathscr{E}=\mathscr{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2\theta}d\theta,\quad \mathscr{E}'(r)=\mathscr{E}(r'),\quad r\in(0,1).$$

The bounds for the Hersch-Pfluger distortion function $\varphi_K(r)$ have been studied by a lot of mathematicians for many years [6, 8, 25, 26, 28, 29, 31, 36]. For example, in 1960, Wang [36] obtained that

$$\varphi_K(r)\leqslant 4^{1-1/K}r^{1/K}$$

for all $r\in(0,1)$. Later, Hübner [25] proved

$$\varphi_K(r)\leqslant \exp\{\mathscr{M}(r)(1-1/K)\}r^{1/K}$$

for all $r\in(0,1)$, where $\mathscr{M}(r)=2r'^2\mathscr{K}(r)\mathscr{K}(r')/\pi+\log r$ is the Hübner upper bound function [30, 31]. In 1999, Qiu, Vamanamurthy and Vuorinen [31] showed that the Hübner upper bound function is the best possible function in terms of exponent form, namely, they proved that, for $a(r)$ is a real function on $(0,1)$, inequality

$$\varphi_K(r)<\exp[(1-1/K)a(r)]r^{1/K} \tag{1.3}$$

takes place for all $r\in(0,1)$ if and only if $a(r)\geqslant\mathscr{M}(r)$. Moreover, if we let $b(r)$ and $c(r)$ be real functions defined on $(0,1)$, then [31] also showed that inequalities

$$\varphi_{1/K}(r)<\exp[(1-K)b(r)]r^K \tag{1.4}$$

and

$$\varphi_{1/K}(r)>\exp[(1-K)c(r)]r^K \tag{1.5}$$

hold for all $r\in(0,1)$ and $K\in(1,\infty)$ if and only if $b(r)\leqslant\mathscr{M}(r)$ and $c(r)\geqslant\mathscr{U}(r)=\mu(r)+\log r$, respectively, where $\mu(\cdot)$ is the Grötzsch ring function defined as in (1.2).

On the other hand, Vuorinen [35] found that the solutions of Ramanujan's modular equation in number theory are related to the Hersch-Pfluger distortion function. Indeed, the classical modular equation of degree p can be expressed by $\mu(s)=p\mu(r)$ $(0<r<1)$ and its solution is given by

$$s=\varphi_K(r),\ K=1/p.$$

For these, and more properties and inequalities for $\varphi_K(r)$ can be found in the literature [2, 4, 5, 9, 10, 32, 47, 53].

The main purpose of this paper is to find an infinite series formula and present new bounds for the Hübner upper bound function, discuss the submultiplicative and power submultiplicative properties for the Hersch-Pfluger distortion function, and improve the well known results on the quasiconformal Schwarz lemma and the solutions of the Ramanujan modular equations.

Throughout this paper, we let $r'=\sqrt{1-r^2}$ for $0<r<1$, and $\mathscr{U}(r)=\mu(r)+\log r$, $\mathscr{M}(r)=2r'^2\mathscr{K}(r)\mathscr{K}(r')/\pi+\log r$. We now state our main results as follows.

THEOREM 1.1. *Let* $r_0 = r' \equiv \sqrt{1-r^2}$, $r_1 = 2\sqrt{r'}/(1+r') = \varphi_2(r_0)$, $r_2 = 2\sqrt{r_1}/(1+r_1) = \varphi_2(r_1) = \varphi_4(r_0), \cdots$, $r_n = 2\sqrt{r_{n-1}}/(1+r_{n-1}) = \varphi_2(r_{n-1}) = \varphi_{2^n}(r_0)$. *Then*

$$\mathscr{M}(r) = \frac{1}{2}\left[(1-r_0)\log(1-r_0)+(1+r_0)\log(1+r_0)\right] + \frac{1}{2}\sum_{k=1}^{\infty}\left\{\left[\prod_{m=0}^{k-1}\left(\frac{r_m}{2}\right)\right]\left[(1-r_k)\log(1-r_k)+(1+r_k)\log(1+r_k)\right]\right\}.$$

THEOREM 1.2. *Let the functions* H, J *and* I *be defined on* $(0,1)$ *by*

$$H(r) = \frac{1}{2}[(1-r)\log(1-r)+(1+r)\log(1+r)],$$

$$J(r) = H(r') + \log 2\left(\frac{1+\sqrt{r'}}{1+r'}\right)^2 r'^{3/2}$$

and

$$I(r) = H(r') + \frac{r'}{2}H\left(\frac{2\sqrt{r'}}{1+r'}\right) + \frac{r'^{3/2}}{2(1+r'-\sqrt{r'})}H\left(\frac{2\sqrt{2}(1+r')^{1/2}r'^{1/4}}{(1+\sqrt{r'})^2}\right),$$

respectively. Then the inequalities

$$I(r) < \mathscr{M}(r) < J(r) \tag{1.6}$$

hold for all $r \in (0,1)$.

Applying Theorem 1.2 and inequalities (1.3) and (1.4), we get the following corollary immediately.

COROLLARY 1.3. (1) *For all* $r \in (0,1)$ *and* $K \in (1,\infty)$,

$$\varphi_{1/K}(r) < \exp[(1-K)I(r)]r^K$$

and

$$\varphi_K(r) < \exp[(1-1/K)J(r)]r^{1/K},$$

where $I(r)$ *and* $J(r)$ *are defined as in Theorem 1.2.*

(2) *If* $f \in QC_K(\mathbb{B})$, *then for each* $z \in \mathbb{B}$,

$$|f(z)| \leqslant \exp\{\mathscr{M}(|z|)(1-1/K)\}|z|^{1/K} \leqslant \exp\{J(|z|)(1-1/K)\}|z|^{1/K},$$

that is

$$|f(z)| \leqslant \left\{4^{\frac{[1+(1-|z|^2)^{1/2}](1-|z|^2)^{3/8}}{1+(1-|z|^2)^{1/2}}}\left[1-(1-|z|^2)^{1/2}\right]^{\frac{1-(1-|z|^2)^{1/2}}{2}} \times\left[1+(1-|z|^2)^{1/2}\right]^{\frac{1+(1-|z|^2)^{1/2}}{2}}\right\}^{1-1/K}|z|^{1/K}.$$

THEOREM 1.4. *Let* $a(r,t)$, $b(r,t)$ *and* $c(r,t)$ *be the functions defined on* $(0,1)\times(0,1)$. *Then*

(1) *The inequality*

$$\varphi_K(r)\varphi_K(t)/\varphi_K(rt) < e^{a(r,t)(1-1/K)} \tag{1.7}$$

holds for all $r,t\in(0,1)$ *and* $K\in[1,\infty)$ *if and only if* $a(r,t)\geqslant\mathscr{M}(r)+\mathscr{M}(t)-\mathscr{M}(rt)$.

(2) *The inequality*

$$\varphi_{1/K}(r)\varphi_{1/K}(t)/\varphi_{1/K}(rt) < e^{b(r,t)(1-K)} \tag{1.8}$$

holds for all $r,t\in(0,1)$ *and* $K\in(1,\infty)$ *if and only if* $b(r,t)\leqslant\mathscr{M}(r)+\mathscr{M}(t)-\mathscr{M}(rt)$.

(3) *The inequality*

$$\varphi_{1/K}(r)\varphi_{1/K}(t)/\varphi_{1/K}(rt) > e^{c(r,t)(1-K)} \tag{1.9}$$

holds for all $r,t\in(0,1)$ *and* $K\in(1,\infty)$ *if and only if* $c(r,t)\geqslant\mathscr{U}(r)+\mathscr{U}(t)-\mathscr{U}(rt)$.

THEOREM 1.5. *Let* $A(r)$, $B(r)$ *and* $C(r)$ *be the functions defined on* $(0,1)$. *Then the following statements are true:*

(1) *If* $0<p\leqslant 1$, *then the inequalities*

$$\varphi_K(r)^p/\varphi_K(r^p) > e^{A(r)(1-1/K)}, \tag{1.10}$$

$$\varphi_{1/K}(r)^p/\varphi_{1/K}(r^p) > e^{B(r)(1-K)}, \tag{1.11}$$

$$\varphi_{1/K}(r)^p/\varphi_{1/K}(r^p) < e^{C(r)(1-K)} \tag{1.12}$$

hold for all $r\in(0,1)$ *and* $K\in(1,\infty)$ *if and only if* $A(r)\leqslant p\mathscr{M}(r)-\mathscr{M}(r^p)$, $B(r)\geqslant p\mathscr{M}(r)-\mathscr{M}(r^p)$, $C(r)\leqslant p\mathscr{U}(r)-\mathscr{U}(r^p)$, *respectively.*

(2) *If* $p>1$, *then the inequalities*

$$\varphi_K(r)^p/\varphi_K(r^p) < e^{A(r)(1-1/K)}, \tag{1.13}$$

$$\varphi_{1/K}(r)^p/\varphi_{1/K}(r^p) > e^{B(r)(1-K)}, \tag{1.14}$$

$$\varphi_{1/K}(r)^p/\varphi_{1/K}(r^p) < e^{C(r)(1-K)} \tag{1.15}$$

hold for all $r\in(0,1)$ *and* $K\in(1,\infty)$ *if and only if* $A(r)\geqslant p\mathscr{M}(r)-\mathscr{M}(r^p)$, $B(r)\geqslant p\mathscr{U}(r)-\mathscr{U}(r^p)$, $C(r)\leqslant p\mathscr{M}(r)-\mathscr{M}(r^p)$, *respectively.*

2. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. It follows from $r_0 = r' \equiv \sqrt{1-r^2}$, $r_1 = 2\sqrt{r'}/(1+r') = \varphi_2(r_0)$, $r_2 = 2\sqrt{r_1}/(1+r_1) = \varphi_2(r_1) = \varphi_4(r_0), \cdots$, $r_n = 2\sqrt{r_{n-1}}/(1+r_{n-1}) = \varphi_2(r_{n-1}) = \varphi_{2^n}(r_0)$ that

$$\begin{aligned}\mathscr{M}(r_1') =& \frac{2}{\pi}\left(\frac{2\sqrt{r'}}{1+r'}\right)^2 \mathscr{K}\left(\frac{2\sqrt{r'}}{1+r'}\right)\mathscr{K}\left(\frac{1-r'}{1+r'}\right) + \log\left(\frac{1-r'}{1+r'}\right)\\ =& \frac{2}{\pi}\frac{4r'}{(1+r')^2}(1+r')\mathscr{K}'(r)\left(\frac{1+r'}{2}\right)\mathscr{K}(r) + \log\left(\frac{1-r'}{1+r'}\right)\\ =& \frac{2}{r'}\left[\frac{2}{\pi}r'^2\mathscr{K}(r)\mathscr{K}'(r) + \log r\right] + \log\left(\frac{1-r'}{1+r'}\right) - \frac{1}{r'}\log(1-r'^2)\\ =& \frac{2}{r'}\mathscr{M}(r) - \frac{(1-r')\log(1-r') + (1+r')\log(1+r')}{r'},\end{aligned}$$

that is,

$$\frac{r_0}{2}\mathscr{M}(r_1') = \mathscr{M}(r) - \frac{1}{2}\left[(1-r_0)\log(1-r_0) + (1+r_0)\log(1+r_0)\right]. \tag{2.1}$$

Putting r_1 into (2.1) instead of r_0, one has

$$\frac{r_1}{2}\mathscr{M}(r_2') = \mathscr{M}(r_1') - \frac{1}{2}\left[(1-r_1)\log(1-r_1) + (1+r_1)\log(1+r_1)\right]. \tag{2.2}$$

From (2.1) and (2.2) we clearly see that

$$\begin{aligned}&\mathscr{M}(r) - \frac{1}{2}\left[(1-r_0)\log(1-r_0) + (1+r_0)\log(1+r_0)\right]\\ &- \frac{1}{2}\left(\frac{r_0}{2}\right)\left[(1-r_1)\log(1-r_1) + (1+r_1)\log(1+r_1)\right]\\ =& \frac{r_0 r_1}{2^2}\mathscr{M}(r_2').\end{aligned}$$

Similarly, replacing r_2 with r_0 in (2.1), we also have

$$\frac{r_2}{2}\mathscr{M}(r_3') = \mathscr{M}(r_2') - \frac{1}{2}\left[(1-r_2)\log(1-r_2) + (1+r_2)\log(1+r_2)\right],$$

which leads to

$$\begin{aligned}&\mathscr{M}(r) - \frac{1}{2}\left[(1-r_0)\log(1-r_0) + (1+r_0)\log(1+r_0)\right]\\ &- \frac{1}{2}\left(\frac{r_0}{2}\right)\left[(1-r_1)\log(1-r_1) + (1+r_1)\log(1+r_1)\right]\\ &- \frac{1}{2}\left(\frac{r_0}{2}\cdot\frac{r_1}{2}\right)\left[(1-r_2)\log(1-r_2) + (1+r_2)\log(1+r_2)\right]\\ =& \frac{r_0 r_1 r_2}{2^3}\mathscr{M}(r_3').\end{aligned}$$

Generally, by induction we can obtain

$$\begin{aligned}
&\mathscr{M}(r)-\frac{1}{2}[(1-r_0)\log(1-r_0)+(1+r_0)\log(1+r_0)]\\
&\quad-\frac{1}{2}\sum_{k=1}^{n}\left\{\left[\prod_{m=0}^{k-1}\left(\frac{r_m}{2}\right)\right][(1-r_k)\log(1-r_k)+(1+r_k)\log(1+r_k)]\right\}\\
&=\prod_{m=0}^{n}\left(\frac{r_m}{2}\right)\mathscr{M}(r'_{n+1}).
\end{aligned}\tag{2.3}$$

From (2.3) and the fact that $\mathscr{M}(r)$ is strictly decreasing from $(0,1)$ onto $(0,\log 4)$ we know that the double inequality

$$\begin{aligned}
0<&\mathscr{M}(r)-\frac{1}{2}[(1-r_0)\log(1-r_0)+(1+r_0)\log(1+r_0)]\\
&-\frac{1}{2}\sum_{k=1}^{n}\left\{\left[\prod_{m=0}^{k-1}\left(\frac{r_m}{2}\right)\right][(1-r_k)\log(1-r_k)+(1+r_k)\log(1+r_k)]\right\}\\
<&\prod_{m=0}^{n}\left(\frac{r_m}{2}\right)\log 4
\end{aligned}$$

holds for all $r\in(0,1)$. Letting $n\to\infty$, then $\prod_{m=0}^{n}(r_m/2)=0$, and thereby

$$\begin{aligned}
\mathscr{M}(r)=&\frac{1}{2}[(1-r_0)\log(1-r_0)+(1+r_0)\log(1+r_0)]\\
&+\frac{1}{2}\sum_{k=1}^{\infty}\left\{\left[\prod_{m=0}^{k-1}\left(\frac{r_m}{2}\right)\right][(1-r_k)\log(1-r_k)+(1+r_k)\log(1+r_k)]\right\}.
\end{aligned}$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. For $r\in(0,1)$, let $r_0=r'=\sqrt{1-r^2}$, $r_1=2\sqrt{r_0}/(1+r_0)$, $r_2=2\sqrt{r_1}/(1+r_1)$. Then from Theorem 1.1 and the inequality

$$H(r)=\sum_{n=0}^{\infty}r^{2n+2}/[(2n+1)(2n+2)]<(\log 2)r^2$$

for $r\in(0,1)$ we conclude that

$$\begin{aligned}
\mathscr{M}(r)&\leqslant H(r')+\frac{r_0}{2}\left[(\log 2)r_1^2\right]+\frac{r_0}{2}\frac{r_1}{2}\left[(\log 2)r_2^2\right]+\cdots+\prod_{m=0}^{n}\left(\frac{r_m}{2}\right)\left[(\log 2)r_m^2\right]+\cdots\\
&\leqslant H(r')+\log 2\left[\frac{r_0r_1^2}{2}+\frac{r_0r_1}{2^2}+\cdots+\frac{r_0r_1}{2^n}+\cdots\right]\\
&=H(r')+\log 2\left[\frac{r_0r_1^2}{2}-\frac{r_0r_1}{2}+\frac{r_0r_1}{2}+\frac{r_0r_1}{2^2}+\cdots+\frac{r_0r_1}{2^n}+\cdots\right]\\
&=H(r')+\left(\frac{1}{2}\log 2\right)r_0r_1(1+r_1)=H(r')+\log 2\left(\frac{1+\sqrt{r'}}{1+r'}\right)^2 r'^{3/2}=J(r)
\end{aligned}$$

and

$$\begin{aligned}\mathscr{M}(r) \geqslant & H(r') + \frac{r_0}{2}H(r_1) + \frac{r_0}{2}\frac{r_1}{2}H(r_2) + \frac{r_0}{2}\left(\frac{r_1}{2}\right)^2 H(r_2) + \frac{r_0}{2}\left(\frac{r_1}{2}\right)^3 H(r_2) \\ & + \cdots + \frac{r_0}{2}\left(\frac{r_1}{2}\right)^n H(r_2) + \cdots \\ = & H(r') + \frac{r_0}{2}H(r_1) + \frac{r_0}{2}H(r_2)\left[\frac{r_1}{2} + \left(\frac{r_1}{2}\right)^2 + \cdots + \left(\frac{r_1}{2}\right)^n + \cdots\right] \\ = & H(r') + \frac{r'}{2}H(r_1) + \frac{r_0 r_1}{2(2-r_1)}H(r_2) = I(r). \quad \square\end{aligned}$$

3. Proofs of Theorems 1.4 and 1.5

In [6], the following submultiplicative and power submultiplicative properties of $\varphi_K(r)$ were proved for $K \in (1,\infty)$, $p \in (1,\infty)$ and $r,t \in (0,1)$ by Anderson, Vamanamurthy and Vuorinen (also see [7, Exercises 10.19 (1), (5)]):

$$\varphi_K(r)\varphi_K(t) \leqslant 4^{1-1/K}\varphi_K(rt)$$

and

$$\varphi_K(r)^p < 4^{(p-1)(1-1/K)}\varphi_K(r^p).$$

In this section, we shall refine the above inequalities and establish some new sharp inequalities concerning the submultiplicative and power submultiplicative properties for $\varphi_K(r)$. In order to prove our main results we need several formulas and lemmas, which we present in this section.

For $0 < r < 1$, denote $s = \varphi_K(r)$ $(K \in (0,\infty))$, the following formulas were presented in [7, Equations(3.6)–(3.8),(5.9) and (10.6)] or [4, Theorem 4.1]:

$$\frac{d\mathscr{K}}{dr} = \frac{\mathscr{E} - r'^2\mathscr{K}}{rr'^2}, \quad \frac{d\mathscr{E}}{dr} = \frac{\mathscr{E} - \mathscr{K}}{r}, \quad \frac{d(\mathscr{E} - r'^2\mathscr{K})}{dr} = r\mathscr{K}, \quad \frac{d\mu(r)}{dr} = \frac{-\pi^2}{4rr'^2\mathscr{K}(r)^2},$$

$$\frac{\partial s}{\partial r} = \frac{ss'^2\mathscr{K}(s)\mathscr{K}'(s)}{rr'^2\mathscr{K}(r)\mathscr{K}'(r)} = \frac{ss'^2\mathscr{K}(s)^2}{Krr'^2\mathscr{K}(r)^2},$$

$$\frac{\partial s}{\partial K} = \frac{2}{\pi K}ss'^2\mathscr{K}(s)\mathscr{K}'(s) = \frac{2ss'^2\mathscr{K}(s)^2}{\pi K^2}\frac{\mathscr{K}'(r)}{\mathscr{K}(r)}.$$

The following Lemma 3.1 follows from Theorem 3.21 (7), Exercises 3.43 (11), (29), (32), Theorem 3.30 (1) and Theorem 5.13 (2) in [7].

LEMMA 3.1. (1) $r'^{1/2}\mathscr{K}$ *is strictly decreasing from* $(0,1)$ *onto* $(0,\pi/2)$*;*
(2) $(\mathscr{K} - \mathscr{E})/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4,\infty)$*;*
(3) $(\mathscr{K} - \mathscr{E})/(r^2\mathscr{K})$ *is strictly increasing from* $(0,1)$ *onto* $(1/2,1)$*;*
(4) $[\mathscr{K} - \mathscr{E} - (\mathscr{E} - r'^2\mathscr{K})]/r^4$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/16,\infty)$*;*
(5) $\mathscr{U}(r)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,\log 4)$*;*
(6) $\mathscr{M}(r)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,\log 4)$.

LEMMA 3.2. *Let* $K\in(0,\infty)$, $0<x<r<1$, $s=\varphi_K(r)$ *and* $y=\varphi_K(x)$. *Then the following statements are true:*

(1) $f(K)=s'^2\mathscr{K}(s)^3/(y'^2\mathscr{K}(y)^3)$ *is strictly decreasing from* $(0,\infty)$ *onto* $(0,1)$;

(2) $g(K)=[\mathscr{K}(s)-\mathscr{E}(s)]/[\mathscr{K}(y)-\mathscr{E}(y)]$ *is strictly decreasing from* $(0,\infty)$ *onto* $(\mu(x)/\mu(r),\infty)$.

Proof. For part (1), by logarithmic differentiation, we have

$$\begin{aligned}\frac{1}{f(K)}\frac{df(K)}{dK}=&-\frac{4}{\pi K}s^2\mathscr{K}(s)\mathscr{K}'(s)+\frac{6}{\pi K}\mathscr{K}'(s)[\mathscr{E}(s)-s'^2\mathscr{K}(s)]\\&+\frac{4}{\pi K}y^2\mathscr{K}(y)\mathscr{K}'(y)-\frac{6}{\pi K}\mathscr{K}'(y)[\mathscr{E}(y)-y'^2\mathscr{K}(y)]\\=&\frac{2}{\pi K}[f_1(y)-f_1(s)],\end{aligned}\tag{3.1}$$

where $f_1(r)=\mathscr{K}'(r)[\mathscr{K}(r)-\mathscr{E}(r)]+\mathscr{K}'(r)\{\mathscr{K}(r)-\mathscr{E}(r)-[\mathscr{E}(r)-r'^2\mathscr{K}(r)]\}$. By Lemma 3.1 (1), (2) and (4), we clearly see that $f_1(r)$ is strictly increasing on $(0,1)$. Then equation (3.1) leads to the conclusion that $df(K)/dK<0$ for $K\in(0,\infty)$ since $y<s$ and $f(K)$ is strictly decreasing on $(0,\infty)$. Moreover, $f(0^+)=1$ and by Lemma 3.1 (5),

$$\begin{aligned}\lim_{K\to+\infty}f(K)=&\lim_{K\to+\infty}\frac{\mu(y)^3}{\mu(s)^3}\frac{s'^2}{y'^2}=\frac{\mu(x)^3}{\mu(r)^3}\lim_{K\to+\infty}\left(\frac{e^{\log s'+\mu(s')}}{e^{\log y'+\mu(y')}}e^{\mu(y')-\mu(s')}\right)^2\\=&\frac{\mu(x)^3}{\mu(r)^3}\lim_{K\to+\infty}\left(e^{\mu(x')-\mu(r')}\right)^{2K}=0.\end{aligned}$$

For part (2), logarithmic differentiating gives

$$\begin{aligned}\frac{1}{g(K)}\frac{dg(K)}{dK}=&\frac{2}{\pi K}\frac{s^2\mathscr{K}(s)\mathscr{K}'(s)\mathscr{E}(s)}{\mathscr{K}(s)-\mathscr{E}(s)}-\frac{2}{\pi K}\frac{y^2\mathscr{K}(y)\mathscr{K}'(y)\mathscr{E}(y)}{\mathscr{K}(y)-\mathscr{E}(y)}\\=&\frac{2}{\pi K}[g_1(s)-g_1(y)],\end{aligned}\tag{3.2}$$

where $g_1(r)=\mathscr{K}'(r)\mathscr{E}(r)[r^2\mathscr{K}(r)/(\mathscr{K}(r)-\mathscr{E}(r))]$ is strictly decreasing on $(0,1)$ by Lemma 3.1 (3). Then equation (3.2) leads to the conclusion that $dg(K)/dK<0$ for $K\in(0,\infty)$ since $y<s$ and $g(K)$ is strictly decreasing on $(0,\infty)$. Moreover,

$$\begin{aligned}\lim_{K\to0}g(K)=&\lim_{K\to0}\frac{s^2}{y^2}=\lim_{K\to0}\left(\frac{e^{\log s+\mu(s)}}{e^{\log y+\mu(y)}}e^{\mu(y)-\mu(s)}\right)^2\\=&\lim_{K\to0}\left(e^{\mu(x)-\mu(r)}\right)^{2/K}=+\infty.\end{aligned}$$

$$\lim_{K\to+\infty}g(K)=\lim_{K\to+\infty}\frac{K\mathscr{K}'(s)\mathscr{K}(r)/\mathscr{K}'(r)-\mathscr{E}(s)}{K\mathscr{K}'(y)\mathscr{K}(x)/\mathscr{K}'(x)-\mathscr{E}(y)}=\frac{\mu(x)}{\mu(r)}.\quad\square$$

LEMMA 3.3. *The inequalities*

$$\mathscr{M}(r)+\mathscr{M}(t)>\mathscr{M}(rt) \tag{3.3}$$

$$\mathscr{U}(r)+\mathscr{U}(t)>\mathscr{U}(rt) \tag{3.4}$$

hold for all $r,t\in(0,1)$.

Proof. Inequality (3.4) follows directly from [7, Theorem 5.12 (2)]. For inequality (3.3), if we let $f(r)=\mathscr{M}(r)+\mathscr{M}(t)-\mathscr{M}(rt)$ for fixed $t\in(0,1)$, then differentiating f leads to

$$\frac{df}{dr}=\frac{4}{\pi r}\left[\mathscr{K}'(x)(\mathscr{K}(x)-\mathscr{E}(x))-\mathscr{K}'(r)(\mathscr{K}(r)-\mathscr{E}(r))\right]<0 \tag{3.5}$$

for all $r\in(0,1)$, as the function $r\to\mathscr{K}'(r)[\mathscr{K}(r)-\mathscr{E}(r)]$ is strictly increasing on $(0,1)$ by Lemma 3.1 (1) and (2). Then (3.5) leads to the conclusion that $f(r)<f(1^-)=0$ for all $r\in(0,1)$, and inequality (3.3) follows. □

LEMMA 3.4. *Let* $p\in(0,\infty)$, *then the following statements are true:*

(1) *The function*

$$f(r)=\mathscr{M}(r^p)-p\mathscr{M}(r)$$

is strictly decreasing from $(0,1)$ *onto* $(0,(1-p)\log 4)$ *if* $0<p<1$, *is strictly increasing from* $(0,1)$ *onto* $((1-p)\log 4,0)$ *if* $p>1$ *and is the constant function* $f(r)=0$ *if* $p=1$;

(2) *The function*

$$g(r)=\mathscr{U}(r^p)-p\mathscr{U}(r)$$

is strictly decreasing from $(0,1)$ *onto* $(0,(1-p)\log 4)$ *if* $0<p<1$, *is strictly increasing from* $(0,1)$ *onto* $((1-p)\log 4,0)$ *if* $p>1$ *and is the constant function* $g(r)=0$ *if* $p=1$.

Proof. For part (1), clearly, $f(r)=0$ if $p=1$. Let $x=r^p$, then differentiation gives

$$\frac{df(r)}{dr}=\frac{4p}{\pi r}\left[\mathscr{K}'(r)\left(\mathscr{K}(r)-\mathscr{E}(r)\right)-\mathscr{K}'(x)\left(\mathscr{K}(x)-\mathscr{E}(x)\right)\right]. \tag{3.6}$$

Since the function $r\to\mathscr{K}'(r)[\mathscr{K}(r)-\mathscr{E}(r)]$ is strictly increasing on $(0,1)$, equation (3.6) implies $df(r)/dr<0$ if $0<p<1$ and $df(r)/dr>0$ if $p>1$. Moreover, by Lemma 3.1 (6) we have $f(0^+)=\lim_{r\to 0^+}\mathscr{M}(r^p)-p\mathscr{M}(r)=(1-p)\log 4$ and $f(1^-)=0$.

For part (2), $g(r)=0$ if $p=1$. if we also let $x=r^p$, then by differentiation we have

$$dg(r)/dr=\frac{p\pi^2}{4rr'^2x'^2\mathscr{K}(r)^2\mathscr{K}(x)^2}[x'^2\mathscr{K}(x)^2-r'^2\mathscr{K}(r)^2]. \tag{3.7}$$

Lemma 3.1 (1) shows that the function $r\to r'\mathscr{K}(r)$ is strictly decreasing on $(0,1)$. Hence by (3.7) we conclude that $dg(r)/dr<0$ if $0<p<1$ and $dg(r)/dr>0$ if $p>1$. Moreover, by Lemma 3.1 (5) we get $g(0^+)=\lim_{r\to 0^+}\mathscr{U}(r^p)-p\mathscr{U}(r)=(1-p)\log 4$ and $g(1^-)=0$. □

THEOREM 3.5. *Let $a(r,t)$ and $b(r,t)$ be the functions defined on $(0,1)\times(0,1)$. For each $(r,t)\in(0,1)\times(0,1)$, we define the functions f and g on $[0,\infty)$ by*

$$f(K)=\varphi_K(r)\varphi_K(t)\left(e^{a(r,t)}\right)^{1/K}/\varphi_K(rt)$$

and

$$g(K)=\varphi_{1/K}(r)\varphi_{1/K}(t)\left(e^{b(r,t)}\right)^{K}/\varphi_{1/K}(rt).$$

Then

(1) *f is strictly decreasing on $(0,1]$ if and only if*

$$a(r,t)\geqslant \mathscr{U}(r)+\mathscr{U}(t)-\mathscr{U}(rt)$$

for all $r,t\in(0,1)$, and strictly increasing on $(0,1]$ if and only if

$$a(r,t)\leqslant \mathscr{M}(r)+\mathscr{M}(t)-\mathscr{M}(rt)$$

for all $r,t\in(0,1)$. Moreover, f is strictly decreasing on $[1,\infty)$ if and only if

$$a(r,t)\geqslant \mathscr{M}(r)+\mathscr{M}(t)-\mathscr{M}(rt) \tag{3.8}$$

for all $r,t\in(0,1)$.

(2) *g is strictly increasing on $(0,1]$ if and only if*

$$b(r,t)\geqslant \mathscr{M}(r)+\mathscr{M}(t)-\mathscr{M}(rt)$$

for all $r,t\in(0,1)$. For $K\in[1,\infty)$, g is strictly decreasing on $[1,\infty)$ if and only if

$$b(r,t)\leqslant \mathscr{M}(r)+\mathscr{M}(t)-\mathscr{M}(rt) \tag{3.9}$$

for all $r,t\in(0,1)$, and g is strictly increasing on $[1,\infty)$ if and only if

$$b(r,t)\geqslant \mathscr{U}(r)+\mathscr{U}(t)-\mathscr{U}(rt) \tag{3.10}$$

for all $r,t\in(0,1)$.

Proof. Let $x=rt$, $s=\varphi_K(r)$, $u=\varphi_K(t)$, and $y=\varphi_K(x)$. Then simple computations lead to

$$\begin{aligned}\frac{1}{f}\frac{\partial f}{\partial K}=&\frac{1}{s}\frac{2}{\pi K^2}ss'^2\mathscr{K}(s)^2\frac{\mathscr{K}'(r)}{\mathscr{K}(r)}+\frac{1}{u}\frac{2}{\pi K^2}uu'^2\mathscr{K}(u)^2\frac{\mathscr{K}'(t)}{\mathscr{K}(t)}\\&-\frac{1}{y}\frac{2}{\pi K^2}yy'^2\mathscr{K}(y)^2\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}-\frac{1}{K^2}a(r,t)\\=&\frac{1}{K^2}\left[h(K,r,t)-a(r,t)\right],\end{aligned} \tag{3.11}$$

where

$$h(K,r,t)=\frac{2}{\pi}s'^2\mathscr{K}(s)^2\frac{\mathscr{K}'(r)}{\mathscr{K}(r)}+\frac{2}{\pi}u'^2\mathscr{K}(u)^2\frac{\mathscr{K}'(t)}{\mathscr{K}(t)}-\frac{2}{\pi}y'^2\mathscr{K}(y)^2\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}.$$

Differentiating h gives

$$\begin{aligned}
\frac{\pi}{2}\frac{\partial h}{\partial K} =& 2\frac{\mathscr{K}'(r)}{\mathscr{K}(r)}s'\mathscr{K}(s)\left[-\frac{s}{s'}\mathscr{K}(s)+s'\frac{\mathscr{E}(s)-s'^2\mathscr{K}(s)}{ss'^2}\right]\frac{2}{\pi K^2}ss'^2\mathscr{K}(s)^2\frac{\mathscr{K}'(r)}{\mathscr{K}(r)}\\
&+2\frac{\mathscr{K}'(t)}{\mathscr{K}(t)}u'\mathscr{K}(u)\left[-\frac{u}{u'}\mathscr{K}(u)+u'\frac{\mathscr{E}(u)-u'^2\mathscr{K}(u)}{uu'^2}\right]\frac{2}{\pi K^2}uu'^2\mathscr{K}(u)^2\frac{\mathscr{K}'(t)}{\mathscr{K}(t)}\\
&-2\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}y'\mathscr{K}(y)\left[-\frac{y}{y'}\mathscr{K}(y)+y'\frac{\mathscr{E}(y)-y'^2\mathscr{K}(y)}{yy'^2}\right]\frac{2}{\pi K^2}yy'^2\mathscr{K}(y)^2\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}\\
=&\frac{4}{\pi K^2}y'^2\mathscr{K}(y)^3\left[\mathscr{K}(y)-\mathscr{E}(y)\right]h_1(K,r,t), \qquad (3.12)
\end{aligned}$$

where

$$\begin{aligned}
h_1(K,r,t) =& \left(\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}\right)^2-\left(\frac{\mathscr{K}'(r)}{\mathscr{K}(r)}\right)^2\frac{s'^2\mathscr{K}(s)^3\left[\mathscr{K}(s)-\mathscr{E}(s)\right]}{y'^2\mathscr{K}(y)^3\left[\mathscr{K}(y)-\mathscr{E}(y)\right]}\\
&-\left(\frac{\mathscr{K}'(t)}{\mathscr{K}(t)}\right)^2\frac{u'^2\mathscr{K}(u)^3\left[\mathscr{K}(u)-\mathscr{E}(u)\right]}{y'^2\mathscr{K}(y)^3\left[\mathscr{K}(y)-\mathscr{E}(y)\right]}. \qquad (3.13)
\end{aligned}$$

It follows from Lemma 3.2(1) and (2) that the function $K\to h_1(K,r,t)$ is strictly increasing on $[0,\infty)$, and

$$\lim_{K\to 0^+} h_1(K,r,t) = -\infty, \qquad (3.14)$$

$$\lim_{K\to 1} h_1(K,r,t) = \frac{h_2(r,t)}{x'^2\mathscr{K}(x)^3\left[\mathscr{K}(x)-\mathscr{E}(x)\right]}, \qquad (3.15)$$

$$\lim_{K\to\infty} h_1(K,r,t) = \left(\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}\right)^2, \qquad (3.16)$$

where

$$\begin{aligned}
h_2(r,t) =& x'^2\mathscr{K}(x)\mathscr{K}'(x)^2\left[\mathscr{K}(x)-\mathscr{E}(x)\right]-r'^2\mathscr{K}(r)\mathscr{K}'(r)^2\left[\mathscr{K}(r)-\mathscr{E}(r)\right]\\
&-t'^2\mathscr{K}(t)\mathscr{K}'(t)^2\left[\mathscr{K}(t)-\mathscr{E}(t)\right]\\
<& x'^2\mathscr{K}(x)\mathscr{K}'(x)^2\left[\mathscr{K}(x)-\mathscr{E}(x)\right]-r'^2\mathscr{K}(r)\mathscr{K}'(r)\mathscr{K}'(x)\left[\mathscr{K}(x)-\mathscr{E}(x)\right]\\
&-t'^2\mathscr{K}(t)\mathscr{K}'(t)\mathscr{K}'(x)\left[\mathscr{K}(x)-\mathscr{E}(x)\right]\\
=&\frac{\pi}{2}\left[\mathscr{M}(x)-\mathscr{M}(r)-\mathscr{M}(t)\right]\mathscr{K}'(x)\left[\mathscr{K}(x)-\mathscr{E}(x)\right]<0 \qquad (3.17)
\end{aligned}$$

for all $r,t\in(0,1)$ by (3.3).

Equations (3.14)–(3.16), inequality (3.17) and the monotonicity of the function $K\to h_1(K,r,t)$ lead to the conclusion that there exists $K_0\in(1,\infty)$ such that $h_1(K,r,t)<0$ for $K\in(0,K_0)$ and $h_1(K,r,t)>0$ for $K\in(K_0,\infty)$. Then from (3.12) and (3.13) we know that the function $K \to h(K,r,t)$ is strictly decreasing on $(0,K_0)$, and strictly increasing on (K_0,∞). Moreover, by (3.4) one has

$$\lim_{K\to 0^+} h(K,r,t) = \mathscr{U}(r)+\mathscr{U}(t)-\mathscr{U}(rt) > 0, \tag{3.18}$$

$$\lim_{K\to 1} h(K,r,t) = \mathscr{M}(r)+\mathscr{M}(t)-\mathscr{M}(rt) > 0, \tag{3.19}$$

$$\lim_{K\to\infty} h(K,r,t) = 0. \tag{3.20}$$

It follows from (3.11), (3.18), (3.19) and (3.20) together with the piecewise monotonicity of the function $K \to h(K,r,t)$ that

$$\partial f/\partial K \geqslant 0 \text{ for all } K \in (0,1] \Longleftrightarrow a(r,t) \leqslant \inf_{K\in(0,1]} h(K,r,t) = \mathscr{M}(r)+\mathscr{M}(t)-\mathscr{M}(rt),$$

$$\partial f/\partial K \leqslant 0 \text{ for all } K \in (0,1] \Longleftrightarrow a(r,t) \geqslant \sup_{K\in(0,1]} h(K,r,t) = \mathscr{U}(r)+\mathscr{U}(t)-\mathscr{U}(rt),$$

$$\partial f/\partial K \leqslant 0 \text{ for all } K \in [1,\infty) \Longleftrightarrow a(r,t) \geqslant \sup_{K\in[1,\infty)} h(K,r,t) = \mathscr{M}(r)+\mathscr{M}(t)-\mathscr{M}(rt).$$

Thus part (1) holds true.

Part (2) follows from part (1) and the fact that $g(K) = f(1/K)$. □

Proof of Theorem 1.4. By Theorem 3.5, the "if" parts are clear. We only need to prove the "only if" for part (1), since the others are similar. Denote $s = \varphi_K(r)$, $u = \varphi_K(t)$, and $y = \varphi_K(x)$. In (1.7), taking logarithm, rasing to power $K/(K-1)$ and letting $K \to 1$, we get

$$\lim_{K\to 1} \frac{\log s + \log u - \log y}{1-1/K} \leqslant a(r,t).$$

By l'Hôpital rule, we have

$$\lim_{K\to 1} \frac{\log s + \log u - \log y}{1-1/K} = \mathscr{M}(r)+\mathscr{M}(t)-\mathscr{M}(rt),$$

hence, part (1) follows. □

REMARK 3.6. Let $t \to 0$ in Theorems 3.5 and 1.4, then Theorems 3.5 and 1.4 reduce to Theorem 6 and Corollary 1 (namely, inequalities (1.3)–(1.5) in [31], respectively.

THEOREM 3.7. *Let $A(r)$ and $B(r)$ be the functions defined on $(0,1)$. For each $r \in (0,1)$ and $p \in (0,\infty)$, we define the functions F and G on $[0,\infty)$ by*

$$F(K) = \varphi_K(r)^p \left(e^{A(r)}\right)^{1/K} / \varphi_K(r^p)$$

and

$$G(K) = \varphi_{1/K}(r)^p \left(e^{B(r)}\right)^K / \varphi_{1/K}(r^p).$$

Then the following statements are true:

(1) *For* $0 < p \leqslant 1$, F *is strictly increasing on* $(0,1]$ *if and only if* $A(r) \leqslant p\mathscr{U}(r) - \mathscr{U}(r^p)$, *and strictly decreasing on* $(0,1]$ *if and only if* $A(r) \geqslant p\mathscr{M}(r) - \mathscr{M}(r^p)$. *Moreover,* F *is strictly increasing on* $[1,\infty)$ *if and only if*

$$A(r) \leqslant p\mathscr{M}(r) - \mathscr{M}(r^p) \tag{3.21}$$

for all $r \in (0,1)$*;*

For $p > 1$, F *is strictly increasing on* $(0,1]$ *if and only if* $A(r) \leqslant p\mathscr{M}(r) - \mathscr{M}(r^p)$, *and strictly decreasing on* $(0,1]$ *if and only if* $A(r) \geqslant p\mathscr{U}(r) - \mathscr{U}(r^p)$. *Moreover,* F *is strictly decreasing on* $[1,\infty)$ *if and only if*

$$A(r) \geqslant p\mathscr{M}(r) - \mathscr{M}(r^p) \tag{3.22}$$

for all $r \in (0,1)$.

(2) *For* $0 < p \leqslant 1$, G *is strictly decreasing on* $(0,1]$ *if and only if* $B(r) \leqslant p\mathscr{M}(r) - \mathscr{M}(r^p)$ *for all* $r \in (0,1)$. *Moreover,* G *is strictly increasing on* $[1,\infty)$ *if and only if*

$$B(r) \geqslant p\mathscr{M}(r) - \mathscr{M}(r^p) \tag{3.23}$$

for all $r \in (0,1)$, *and* G *is strictly decreasing on* $[1,\infty)$ *if and only if*

$$B(r) \leqslant p\mathscr{U}(r) - \mathscr{U}(r^p) \tag{3.24}$$

for all $r \in (0,1)$.

For $p > 1$, G *is strictly increasing on* $(0,1]$ *if and only if* $B(r) \geqslant p\mathscr{M}(r) - \mathscr{M}(r^p)$. *Moreover,* G *is strictly increasing on* $[1,\infty)$ *if and only if*

$$B(r) \geqslant p\mathscr{U}(r) - \mathscr{U}(r^p) \tag{3.25}$$

for all $r \in (0,1)$, *and* G *is strictly decreasing on* $[1,\infty)$ *if and only if*

$$B(r) \leqslant p\mathscr{M}(r) - \mathscr{M}(r^p) \tag{3.26}$$

for all $r \in (0,1)$.

Proof. Since $G(K) = F(1/K)$, it suffices to prove the assertion of part (1). Let $x = r^p$, $s = \varphi_K(r)$ and $y = \varphi_K(x)$. Then logarithmic differentiation of F gives

$$\begin{aligned}\frac{1}{F(K)}\frac{\partial F}{\partial K} =& p\frac{1}{s}\frac{2ss'^2\mathscr{K}(s)^2}{\pi K^2}\frac{\mathscr{K}'(r)}{\mathscr{K}(r)} - \frac{1}{y}\frac{2yy'^2\mathscr{K}(y)^2}{\pi K^2}\frac{\mathscr{K}'(x)}{\mathscr{K}(x)} - \frac{1}{K^2}A(r)\\ =& \frac{1}{K^2}[F_1(K) - A(r)],\end{aligned} \tag{3.27}$$

where

$$F_1(K)=\frac{2p}{\pi}\frac{\mathscr{K}'(r)}{\mathscr{K}(r)}s'^2\mathscr{K}(s)^2-\frac{2}{\pi}\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}y'^2\mathscr{K}(y)^2. \tag{3.28}$$

Simple computations lead to

$$\lim_{K\to 0^+}F_1(K)=p\mathscr{U}(r)-\mathscr{U}(r^p), \tag{3.29}$$

$$\lim_{K\to 1}F_1(K)=p\mathscr{M}(r)-\mathscr{M}(r^p), \tag{3.30}$$

$$\lim_{K\to \infty}F_1(K)=0, \tag{3.31}$$

$$\begin{aligned}\frac{\pi}{2}\frac{\partial F_1}{\partial K}=&2p\frac{\mathscr{K}'(r)}{\mathscr{K}(r)}s'\mathscr{K}(s)\left[-\frac{s}{s'}\mathscr{K}(s)+s'\frac{\mathscr{E}(s)-s'^2\mathscr{K}(s)}{ss'^2}\right]\frac{2}{\pi K^2}ss'^2\mathscr{K}(s)^2\frac{\mathscr{K}'(r)}{\mathscr{K}(r)}\\&-2\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}y'\mathscr{K}(y)\left[-\frac{y}{y'}\mathscr{K}(y)+y'\frac{\mathscr{E}(y)-y'^2\mathscr{K}(y)}{yy'^2}\right]\frac{2}{\pi K^2}yy'^2\mathscr{K}(y)^2\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}\\=&\frac{4}{\pi K^2}y'^2\mathscr{K}(y)^3\left[\mathscr{K}(y)-\mathscr{E}(y)\right]F_2(K),\end{aligned} \tag{3.32}$$

where

$$F_2(K)=\left(\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}\right)^2-p\left(\frac{\mathscr{K}'(r)}{\mathscr{K}(r)}\right)^2\frac{s'^2\mathscr{K}(s)^3\left[\mathscr{K}(s)-\mathscr{E}(s)\right]}{y'^2\mathscr{K}(y)^3\left[\mathscr{K}(y)-\mathscr{E}(y)\right]}. \tag{3.33}$$

Next, we divide the proof into two cases.

Case 1 $p>1$. Then $x<r$ and thereby $y<s$. Equation (3.33) and Lemma 3.2(1) and (2) show that $F_2(K)$ is strictly increasing on $(0,\infty)$. Moreover

$$\lim_{K\to 0^+}F_2(K)=-\infty, \tag{3.34}$$

$$\lim_{K\to 1}F_2(K)=\left(\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}\right)^2-p\frac{r'^2\mathscr{K}(r)\mathscr{K}'(r)^2\left[\mathscr{K}(r)-\mathscr{E}(r)\right]}{x'^2\mathscr{K}(x)^3\left[\mathscr{K}(x)-\mathscr{E}(x)\right]}<0, \tag{3.35}$$

$$\lim_{K\to \infty}F_2(K)=\left(\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}\right)^2, \tag{3.36}$$

since

$$\begin{aligned}&\left[x'^2\mathscr{K}(x)^3\left[\mathscr{K}(x)-\mathscr{E}(x)\right]\right]\left[\left(\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}\right)^2-p\frac{r'^2\mathscr{K}(r)\mathscr{K}'(r)^2\left[\mathscr{K}(r)-\mathscr{E}(r)\right]}{x'^2\mathscr{K}(x)^3\left[\mathscr{K}(x)-\mathscr{E}(x)\right]}\right]\\=&x'^2\mathscr{K}(x)\mathscr{K}'(x)^2\left[\mathscr{K}(x)-\mathscr{E}(x)\right]-pr'^2\mathscr{K}(r)\mathscr{K}'(r)^2\left[\mathscr{K}(r)-\mathscr{E}(r)\right]\\<&x'^2\mathscr{K}(x)\mathscr{K}'(x)^2\left[\mathscr{K}(x)-\mathscr{E}(x)\right]-pr'^2\mathscr{K}(r)\mathscr{K}'(r)\mathscr{K}'(x)\left[\mathscr{K}(x)-\mathscr{E}(x)\right]\\=&\frac{\pi}{2}\left[\mathscr{M}(r^p)-p\mathscr{M}(r)\right]\mathscr{K}'(x)\left[\mathscr{K}(x)-\mathscr{E}(x)\right]<0.\end{aligned}$$

It follows from (3.34)–(3.36) and the monotonicity of $F_2(K)$ that there exists $K_1 \in (1,\infty)$ such that $F_2(K) < 0$ for $K \in (0,K_1)$, and $F_2(K) > 0$ for $K \in (K_1,\infty)$. By (3.32) we clearly see that $F_1(K)$ is strictly decreasing on $(0,K_1)$, and strictly increasing on (K_1,∞). Thus from (3.27)–(3.31) and Lemma 3.4 one has

$$\partial F/\partial K \geqslant 0 \text{ for all } K \in (0,1] \Longleftrightarrow A(r) \leqslant \inf_{K\in(0,1]} F_1(K) = p\mathscr{M}(r) - \mathscr{M}(r^p),$$

$$\partial F/\partial K \leqslant 0 \text{ for all } K \in (0,1] \Longleftrightarrow A(r) \geqslant \sup_{K\in(0,1]} F_1(K) = p\mathscr{U}(r) - \mathscr{U}(r^p),$$

$$\partial F/\partial K \leqslant 0 \text{ for all } K \in [1,\infty) \Longleftrightarrow A(r) \geqslant \sup_{K\in[1,\infty)} F_1(K) = p\mathscr{M}(r) - \mathscr{M}(r^p).$$

The assertion of part (1) for $p > 1$ is clear.

Case 2 $0 < p \leqslant 1$. Then $x > r$ and thereby $y > s$. Equation (3.33) and Lemma 3.2(1) and (2) imply that $F_2(K)$ is strictly decreasing on $(0,\infty)$. Moreover

$$\lim_{K\to 0^+} F_2(K) = \left(\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}\right)^2, \tag{3.37}$$

$$\lim_{K\to 1} F_2(K) = \left(\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}\right)^2 - p\frac{r'^2\mathscr{K}(r)\mathscr{K}'(r)^2\left[\mathscr{K}(r)-\mathscr{E}(r)\right]}{x'^2\mathscr{K}(x)^3\left[\mathscr{K}(x)-\mathscr{E}(x)\right]} > 0, \tag{3.38}$$

$$\lim_{K\to\infty} F_2(K) = -\infty, \tag{3.39}$$

since

$$\begin{aligned}
&\left[x'^2\mathscr{K}(x)^3\left[\mathscr{K}(x)-\mathscr{E}(x)\right]\right]\left[\left(\frac{\mathscr{K}'(x)}{\mathscr{K}(x)}\right)^2 - p\frac{r'^2\mathscr{K}(r)\mathscr{K}'(r)^2\left[\mathscr{K}(r)-\mathscr{E}(r)\right]}{x'^2\mathscr{K}(x)^3\left[\mathscr{K}(x)-\mathscr{E}(x)\right]}\right]\\
=&x'^2\mathscr{K}(x)\mathscr{K}'(x)^2\left[\mathscr{K}(x)-\mathscr{E}(x)\right] - pr'^2\mathscr{K}(r)\mathscr{K}'(r)^2\left[\mathscr{K}(r)-\mathscr{E}(r)\right]\\
>&x'^2\mathscr{K}(x)\mathscr{K}'(x)\mathscr{K}'(r)\left[\mathscr{K}(r)-\mathscr{E}(r)\right] - pr'^2\mathscr{K}(r)\mathscr{K}'(r)^2\left[\mathscr{K}(r)-\mathscr{E}(r)\right]\\
=&\frac{\pi}{2}\left[\mathscr{M}(r^p) - p\mathscr{M}(r)\right]\mathscr{K}'(r)\left[\mathscr{K}(r)-\mathscr{E}(r)\right] > 0.
\end{aligned}$$

It follows from (3.37)–(3.39) and the monotonicity of $F_2(K)$ that there exists $K_2 \in (1,\infty)$ such that $F_2(K) > 0$ for $K \in (0,K_2)$, and $F_2(K) < 0$ for $K \in (K_2,\infty)$. By (3.32) we clearly see that $F_1(K)$ is strictly increasing on $(0,K_2)$, and strictly decreasing on (K_2,∞). Thus from (3.27)–(3.31) and Lemma 3.4 one has

$$\partial F/\partial K \geqslant 0 \text{ for all } K \in (0,1] \Longleftrightarrow A(r) \leqslant \inf_{K\in(0,1]} F_1(K) = p\mathscr{U}(r) - \mathscr{U}(r^p),$$

$$\partial F/\partial K \leqslant 0 \text{ for all } K \in (0,1] \Longleftrightarrow A(r) \geqslant \sup_{K\in(0,1]} F_1(K) = p\mathscr{M}(r) - \mathscr{M}(r^p),$$

$$\partial F/\partial K \geqslant 0 \text{ for all } K \in [1,\infty) \Longleftrightarrow A(r) \geqslant \inf_{K\in[1,\infty)} F_1(K) = p\mathscr{M}(r) - \mathscr{M}(r^p).$$

The assertion of part (1) for $0 < p \leqslant 1$ is clear. □

Proof of Theorem 1.5. Theorem 1.5 follows from Theorem 3.7 together with the similar proof of Theorem 1.4. □

4. Remarks

REMARK 4.1. The following well-known upper and lower bounds for $\mathscr{M}(r)$ were given in [30, 31]:

$$I_1(r) = [(1-\log 4)r + \log 4]r'^2 \frac{\operatorname{arth} r}{r} < \mathscr{M}(r) < J_1(r) = r'^2 \frac{\operatorname{arth} r}{r} \log 4 \tag{4.1}$$

$$\mathscr{M}(r) < J_2(r) = r'^{3/2} \log 4 \tag{4.2}$$

for all $r \in (0,1)$, where arth is the inverse of the hyperbolic tangent function. Consequently, for each $z \in \mathbb{B}$ and $f \in QC_K(\mathbb{B})$,

$$|f(z)| \leqslant \min\{4^{(1-|z|^2)^{3/4}(1-1/K)}|z|^{1/K}, 4^{(1-|z|^2)\frac{\operatorname{arth}|z|}{|z|}(1-1/K)}|z|^{1/K}\}.$$

Computational and numerical experiments show that the upper bound in (1.6) is tighter than the upper bounds in (4.1) and (4.2), and the lower bound in (1.6) is also better than that in (4.1) for $0 < r \leqslant 0.963$ (see Figures 1–2). Therefore, Corollary 1.3 is an improvement of the quasiconformal Schwarz lemma and the estimation for the solutions of Ramanujan's modular equations.

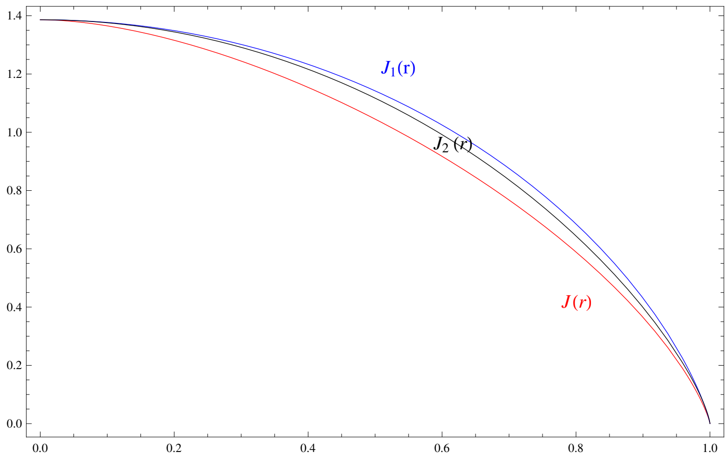

Figure 1: *The graph of* $J(r)$, $J_1(r)$ *and* $J_2(r)$ *on* $(0,1)$

REMARK 4.2. In 1999, Qiu and Vuorinen [33] found a infinite series for $\mathscr{U}(r) = \mu(r) + \log r$ as follows:

$$\mathscr{U}(r) = \sum_{n=0}^{\infty} \frac{1}{2^n} \log(1 + r_n)$$

for all $r \in (0,1)$, where r_n is defined as in Theorem 1.1.

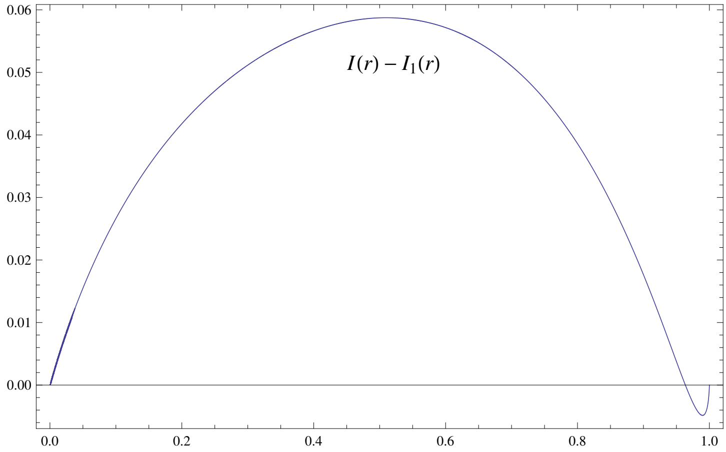

Figure 2: *The graph of the difference of* $I(r)-I_1(r)$ *on* $(0,1)$

Simple computations lead to

$$I^*(r) < \mathscr{U}(r) < J^*(r),$$

where $I^*(r)=\log(1+r')+[\log(1+r_1)+\log(1+r_2)]/2$, and $J^*(r)=\log(1+r')+[\log(1+r_1)+\log 2]/2$.

Let $I(r)$ and $J(r)$ be defined as in Theorem 1.2. Then the explicit forms of inequalities (1.7)–(1.15) involving $\varphi_K(r)$ in Theorems 1.4 and 1.5 can be derived. Here we omit the details.

REMARK 4.3. As described in Section 3, Anderson, Vamanamurthy and Vuorinen [6] proved that

$$\varphi_K(r)\varphi_K(t) \leqslant 4^{1-1/K}\varphi_K(rt),\ \varphi_K(r)^p < 4^{(p-1)(1-1/K)}\varphi_K(r^p)$$

for $K,p\in(1,\infty)$ and $r,t\in(0,1)$.

From Lemma 3.1 (6) and Lemma 3.3 we clearly see that inequalities (1.7) and (1.13) with $a(r,t)=\mathscr{M}(r)+\mathscr{M}(t)-\mathscr{M}(rt)$ and $A(r)=p\mathscr{M}(r)-\mathscr{M}(r^p)$ in Theorems 1.4 and 1.5 refine the above inequalities.

REMARK 4.4. One of the referees communicated to us that some upper and lower bounds of $\varphi_K(r)$ were obtained by the descending and ascending Landen transformations in [7, Corollary 5.44], and these results might be better than those in Corollary 1.3. Actually, Vuorinen et al. in [7, Corollary 5.44], using the identity $\varphi_K(r)=\varphi_{2^p}(\varphi_K(\varphi_{2^{-p}}(r)))$ and inequalities

$$r^{1/K} \leqslant \varphi_K(r) \leqslant 4^{1-1/K}r^{1/K},\quad K>1, \tag{4.3}$$

$$4^{1-K}r^K \leqslant \varphi_{1/K}(r) \leqslant r^K,\quad K>1, \tag{4.4}$$

provided refinements of (4.3) and (4.4). Note that our Corollary 1.3 improves the right-hand inequalities of (4.3) and (4.4). If we employ $\varphi_K(r) = \varphi_{2^p}(\varphi_K(\varphi_{2^{-p}}(r)))$ and inequalities in Corollary 1.3, then the analog of Corollary 5.44 in [7] can be also derived in the same way, which also refine their results.

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