

WEIGHTED END-POINT WEAK TYPE (p, p) ESTIMATES FOR g_λ^* -FUNCTION WITH KERNELS OF LOWER REGULARITIES

ZHENGYANG LI AND QINGYING XUE

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Abstract. In 1970, if $1 < p < 2$ and $\lambda = 2/p$, C. Fefferman obtained the end-point weak (p, p) boundedness of g_λ^* -function. In this paper, the authors essentially improved the result given by C. Fefferman, by showing that the weighted end-point weak type (p, p) boundedness of g_λ^* -function still holds with lower regularities assumed on the kernel for $1 < p < 2$ and $\lambda = 2/p$. Moreover, similar results can also be extended to parametric Littlewood-Paley g_λ^* -function with more rough kernels.

1. Introduction

As is well known, Littlewood-Paley g_λ^* -function plays very important roles in the problems associated with multipliers ([11], p. 94 and p. 232) and function spaces (for example, Sobolov spaces (see [11], p. 162). The classical g_λ^* -function of higher dimensions was first introduced by Stein [10] in 1961 as follows,

$$g_\lambda^*(f)(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} |\nabla_{y,t} u(y, t)|^2 t^{1-n} dy dt \right)^{1/2}, \quad \lambda > 1 \quad (1.1)$$

where $u(y, t) = (P_t * f)(y)$ is the Poisson integral of f and $\nabla_{y,t} = (\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}, \frac{\partial}{\partial t})$. In [10], if $\lambda > 2$, Stein gave the weak $(1, 1)$ boundedness and L^p ($p > 1$) bounds of g_λ^* -function. In the same paper, Stein also showed that g_λ^* -function is not of weak type $(1, 1)$ for $1 < \lambda \leq 2$. Since $g_\lambda^*(f)(x)$ is decreasing with respect to λ , thus, a natural question arises, what happens if $1 < \lambda < 2$? In 1970, Fefferman [4] improved Stein's L^p ($p > 1$) result by enlarging the region of the restriction $\lambda > 2$ to $\lambda > \max\{2/p, 1\}$. Moreover, if $1 < p < 2$, $\lambda = 2/p$ becomes the end-point case, though the weak $(1, 1)$ estimate doesn't hold by Stein's results, Fefferman surprisingly succeeded to prove that the following weak (p, p) boundedness still holds for $1 < p < 2$ and $\lambda = 2/p$.

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THEOREM A. ([4]) *Let g_λ^* be defined as in (1.1), then*

- (i) *If $1 < p < 2$ and $\lambda = 2/p$, then g_λ^* is of weak type (p, p) ;*
- (ii) *If $1 < p < \infty$ and $\lambda > \max\{2/p, 1\}$, then g_λ^* is of type (p, p) .*

REMARK 1. Note that in (i), $1 < p < 2$, which means that $1 < \lambda = 2/p < 2$. Moreover, if p tends to one from the right side, then $2/p$ tends to two from the left side. Thus the weak (p, p) boundedness can be looked as a replacement of Stein’s weak $(1, 1)$ estimate for $1 < \lambda < 2$.

In 1974, by using a different method compared with Theorem A and considering the estimates of a kind of maximal function associated with g_λ^* -function, which was first introduced by Fefferman and Stein [5] in 1970, Muckenhoupt and Wheeden obtained the following weighted results of Theorem A.

THEOREM B. ([6]) *Let g_λ^* be defined as in (1.1), then*

- (i) *If $1 < p < 2$ and $\lambda = 2/p$, then g_λ^* is of weighted weak type (p, p) for $w \in A_1$;*
- (ii) *If $1 < p < \infty$ and $\lambda > \max\{2/p, 1\}$, then g_λ^* is of weighted strong type (p, p)*

for $w \in A_{p\lambda/2}$.

Since then, efforts have been made to decrease the smoothness condition of g_λ^* -function. On one hand, the following more generalized Littlewood-Paley g_λ^* -function has been studied:

$$g_{\lambda, \psi}^*(f)(x) = \left(\int \int_{\mathbb{R}^n} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} |\psi_t * f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \quad \lambda > 1, \tag{1.2}$$

where $\psi_t(x) = t^n \psi(x/t)$, ψ is a function on \mathbb{R}^n such that there exist positive constants C_0, C_1, δ and γ satisfying

$$\psi \in L^1(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} \psi(x) dx = 0; \tag{1.3}$$

$$|\psi(x)| \leq C_0(1 + |x|)^{-n-\delta}; \tag{1.4}$$

$$|\psi(x+y) - \psi(x)| \leq C_1|y|^\gamma(1 + |x|)^{-n-\gamma-\delta} \text{ for } 2|y| \leq |x|. \tag{1.5}$$

As a more generalized g_λ^* -function, L^p ($p > 1$) bounds of $g_{\lambda, \psi}^*$ -function is also well known (see for example, [7, pp. 309-318]). In 2009, Xue and Ding [13] established the weak type $(1, 1)$ bounds of $g_{\lambda, \psi}^*$ for $\lambda > 2$. From [10], we know that if $1 < \lambda \leq 2$, then $g_{\lambda, \psi}^*$ is also not of weak type $(1, 1)$. Inspired by the Fefferman’s work [4], in this paper we considered the end-point weighted weak (p, p) boundedness of $g_{\lambda, \psi}^*$ -function defined by (1.1) for $1 < p < 2$ and $\lambda = 2/p$. Our first main result is as follows.

THEOREM 1.1. *Let ψ be a function satisfying (1.3)–(1.5) and $\omega \in A_1$, then for $1 < p < 2$ and $\lambda = 2/p$, there exists a constant $C > 0$ such that*

$$\omega(\{x \in \mathbb{R}^n \mid g_{\lambda, \psi}^*(f)(x) > \alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p. \tag{1.6}$$

REMARK 2. It is easy to see that the Poisson kernel of g_λ^* satisfies (1.3)–(1.5). One may obtain immediately the weighted weak (p, p) boundedness of g_λ^* -function for $1 < p < 2$ and $\lambda = 2/p$, which coincides with (i) of Theorem B.

If we take $\omega = 1$ in Theorem 1.1, we obtain the unweighted weak (p, p) estimate of $g_{\lambda, \psi}^*$ -function, which essentially improves the conclusion (i) in Theorem A.

COROLLARY 1. Let ψ be a function satisfying (1.3)–(1.5). If $1 < p < 2$ and $\lambda = 2/p$, then $g_{\lambda, \psi}^*$ is of weak type (p, p) .

On the other hand, efforts have also been made to study $g_{\lambda, \psi}^*$ with even more rough kernels. For example, one may pay attention to the kind of kernels with compact support (the classical operator $g_{\lambda, \psi}^*$ does not need to satisfy the compact condition, thus they are quite different in this sense). This compact support assumption allows us to consider the corresponding theory along with more rough kernels. Then one may consider the parametrized Littlewood-Paley g_λ^* -function. Let us first give the definition. Suppose that $\Omega \in L^1(S^{n-1})$ is homogeneous of degree zero on \mathbb{R}^n and satisfies

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0, \quad (1.7)$$

where S^{n-1} denote the unit sphere of \mathbb{R}^n ($n \geq 2$) equipped with Lebesgue measure $d\sigma = d\sigma(x')$. Let $\varphi^\rho(x) = \Omega(x)|x|^{-n+\rho} \chi_B(x)$, where $\rho > 0$ and B denotes the unit ball in \mathbb{R}^n . Then the parametrized Littlewood-Paley g_λ^* -function is defined by

$$\mu_\lambda^{*,\rho}(f)(x) = \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} |\varphi_t^\rho * f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2},$$

where $\lambda > 1$ and $\varphi_t^\rho(x) = \frac{1}{t^n} \varphi^\rho(x/t)$.

If $\rho = 1$, in 1990, Torchinsky and Wang [8] obtained the weighted L^2 boundedness of $\mu_\lambda^{*,\rho}$. In 1999, Sakamoto and Yabuta [9] established the L^p boundedness of $\mu_\lambda^{*,\rho}$, their results can be summarized as follows.

THEOREM C. ([9]) If Ω satisfies (1.7) and $\Omega \in Lip_\beta(S^{n-1})$, $0 < \beta \leq 1$, i.e.

$$|\Omega(x') - \Omega(y')| \leq |x' - y'|^\beta, \quad x', y' \in S^{n-1}, \quad (1.8)$$

then

- (a) for $\lambda > 1$, $\rho > 0$ and $2 \leq p < \infty$, $\|\mu_\lambda^{*,\rho}(f)\|_p \leq C_{n,p,\rho,\beta} \|f\|_p$;
- (b) for $\lambda > 2/p$, $0 < \rho \leq n/2$ and $2n/(n+2\rho) < p < 2$, $\|\mu_\lambda^{*,\rho}(f)\|_p \leq C_{n,p,\rho,\beta} \|f\|_p$;
- (c) for $\lambda > 2/p$, $\rho > n/2$ and $1 < p < 2$, $\|\mu_\lambda^{*,\rho}(f)\|_p \leq C_{n,p,\rho,\beta} \|f\|_p$;
- (d) for $0 < \rho \leq n/2$, $1 \leq p \leq 2n/(n+2\rho)$, there exists a function $\Omega \in Lip_\beta(S^{n-1})$ satisfies (1.7), such that $\|\mu_\lambda^{*,\rho}(f)\|_p \leq C_{n,p,\rho,\beta} \|f\|_p$ are not bounded on $L^p(\mathbb{R}^n)$.

In 2002, Ding, Lu and Yabuta [2] improved the conclusion (a) in Theorem B with more rough kernels, $\Omega \in L \log L(S^{n-1})$. In 2005, Ding and Xue [3] established

the weak type (1,1) bounds of $\mu_\lambda^{*,p}$ for $\rho > n/2$ and $\lambda > 2$. In 2007, Xue, Ding and Yabuta [14] studied the weighted weak-type (1,1) bounds of $\mu_\lambda^{*,p}$ for $\lambda > 2$ and the weighted L^p bounds of $\mu_\lambda^{*,p}$. The main point in the above results is that the properties of $\mu_\lambda^{*,p}$ depends heavily on whether it is bigger or smaller than one half of ρ . In this paper, our second main purpose is to show the following weighted weak-type (p,p) boundedness of $\mu_\lambda^{*,p}$ for $\lambda = 2/p$.

THEOREM 1.2. *If $\omega \in A_1$, $\Omega \in Lip_\beta(S^{n-1})$, $0 < \beta \leq 1$, satisfies (1.7), then for $\lambda = 2/p$, $0 < \rho \leq n/2$ and $\max\{2n/(n + \rho), 2n/(n + 1/2\rho)\} < p < 2$, there exists a constant $C > 0$ such that*

$$\omega(\{x \in \mathbb{R}^n \mid \mu_\lambda^{*,p}(f)(x) > \alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L^p_\omega}^p. \tag{1.9}$$

THEOREM 1.3. *Let $\omega \in A_1$. Suppose that $\Omega \in Lip_\beta(S^{n-1})$ ($0 < \beta \leq 1$) and satisfies (1.7). Then, for $\lambda = 2/p$, $\rho > n/2$ and $2n/(n + 1/2\rho) < p < 2$, there exists a constant $C > 0$ such that*

$$\omega(\{x \in \mathbb{R}^n \mid \mu_\lambda^{*,p}(f)(x) > \alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L^p_\omega}^p. \tag{1.10}$$

REMARK 3. One may guess the best region of p in Theorem 1.2-1.3 is $2n/(n + 2\rho) < p < 2$, which is the same as in Theorem B. However, we don't know whether it is true or not. In fact, our estimates in the proof of Theorem 1.2-1.3 have already been very delicate and complex.

This paper is organized as follows. In Section 2, we give the proof of Theorem 1.1. The proof of 1.2 will be given in Section 3. The proof of Theorem 1.3 is quite similar to the proof of Theorem 1.2, we therefore omit the details of it. In the appendix, we give a simple proof of a lemma used in the proof of our theorems.

2. Proof of the weighted weak-type (p,p) bounds of $g_{\lambda,\psi}^*$

In this section, we will prove Theorem 1.1, the weighted weak-type (p,p) bounds of $g_{\lambda,\psi}^*$. Some basic ideas in the proof of Theorem 1.1 are taken from [4]. We begin with a lemma, which is essential to our proofs.

LEMMA 1. *Let $\omega \in A_1$, $f \in L^p_\omega(\mathbb{R}^n)$ and $\alpha > 0$ be given, there is a collection $\{Q_j\}$ of pairwise disjoint Whitney cubes, functions u and b on \mathbb{R}^n with the following properties:*

- (i) $\Omega = \cup_j Q_j$, $\omega(\Omega) \leq \frac{C}{\alpha^p} \|f\|_{L^p_\omega}^p$.
- (ii) $\frac{1}{\omega(Q_j)} \int_{Q_j} |f(y)|^p \omega(y) dy \leq C\alpha^p$, for any one of the cubes $\{Q_j\}$.

- (iii) For any cube Q_j of the collection, let \tilde{Q}_j be a cube with the same center as Q_j , but with twice as large a side. Then no point of \mathbb{R}^n lies in more than N of the cubes \tilde{Q}_j . We say that the \tilde{Q}_j have bounded overlaps.
- (iv) $f = u + b$.
- (v) $|u| \leq C\alpha$ almost everywhere, and $\|u\|_{L_\omega^p} \leq \|f\|_{L_\omega^p}$.
- (vi) $\frac{1}{\omega(Q_j)} \int_{Q_j} |b(y)|^p \omega(y) dy \leq C\alpha^p$ for each cube Q_j from the collection.
- (vii) b is supported in Ω , $b = \sum_j b_j$, where each b_j is supported in a Whitney cube Q_j . $\int_{Q_j} b_j(y) dy = 0$ for each cube Q_j from the collection.

Lemma 1 is the weighted extension of the decomposition lemma in [4]. The proof is just a modification of the classical proof, for completeness, we give the proof of this lemma in the appendix.

Now we will apply the above lemma to prove Theorem 1.1.

Proof of Theorem 1.1. Let $f \in L_\omega^p(\mathbb{R}^n)$ and $\alpha > 0$. We need to show that

$$\omega(\{x \in \mathbb{R}^n \mid g_\lambda^*(f)(x) > \alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p. \tag{2.1}$$

Applying Lemma 1 to f and α , we obtain a decomposition $f = u + b$, satisfying all properties in Lemma 1. Since $g_\lambda^*(f) \leq g_\lambda^*(u) + g_\lambda^*(b)$, we have

$$\omega(\{x \in \mathbb{R}^n \mid g_\lambda^*(f) > (C+1)\alpha\}) \leq \omega(\{x \in \mathbb{R}^n \mid g_\lambda^*(u) > \alpha\}) + \omega(\{x \in \mathbb{R}^n \mid g_\lambda^*(b) > C\alpha\}).$$

To estimate $\omega(\{x \in \mathbb{R}^n \mid g_\lambda^*(u) > \alpha\})$, note that conclusion (v) in Lemma 1 and the L_ω^2 bounds [1] of g_λ^* , we obtain

$$\begin{aligned} \omega(\{x \in \mathbb{R}^n \mid g_\lambda^*(u) > \alpha\}) &\leq \frac{1}{\alpha^2} \int_{\mathbb{R}^n} |g_\lambda^*(u)(x)|^2 \omega(x) dx \leq \frac{C}{\alpha^2} \int_{\mathbb{R}^n} |u(x)|^2 \omega(x) dx \\ &\leq \frac{C}{\alpha^p} \int_{\mathbb{R}^n} |u(x)|^p \omega(x) dx \leq \frac{C}{\alpha^p} \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx. \end{aligned}$$

Thus we have proved

$$\omega(\{x \in \mathbb{R}^n \mid g_\lambda^*(u)(x) > \alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p. \tag{2.2}$$

Hence, to show (2.1), it is sufficient to prove that

$$\omega(\{x \in \mathbb{R}^n \mid g_\lambda^*(b)(x) > C\alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p. \tag{2.3}$$

To show inequality (2.3), we need a basic decomposition of g_λ^* , the method of this decomposition is coming from [4]. Firstly, we introduce some notation. If $x \in \mathbb{R}^n$ and Q_j is Whitney cube from the collection Q_j in Lemma 1, then $x \sim Q_j$ means that x

belongs to a cube Q_j , which touches or coincides with Q_j . In addition, we denote $b_j(x) = b(x) \cdot \chi_{Q_j}(x)$.

By definition,

$$g_\lambda^*(b)(x) = \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} \left| \sum_j \psi_t * b_j(y) \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

Then we have $g_\lambda^*(b)(x) \leq g_\lambda^{(1)}(b)(x) + g_\lambda^{(2)}(b)(x)$, where

$$g_\lambda^{(1)}(b)(x) = \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} \left| \sum_{y \approx Q_j} \psi_t * b_j(y) \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}$$

and

$$g_\lambda^{(2)}(b)(x) = \left(\int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} \left| \sum_{y \sim Q_j} \psi_t * b_j(y) \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

So to prove (2.3), and thus to finish the proof of Theorem 1, we need to show that

$$\omega(\{x \in \mathbb{R}^n \mid g_\lambda^{(1)}(b)(x) > C\alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p \tag{2.4}$$

and

$$\omega(\{x \in \mathbb{R}^n \mid g_\lambda^{(2)}(b)(x) > C\alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p. \tag{2.5}$$

Now we prove (2.4) first.

Proof of (2.4). By the condition (1.5) of ψ , conclusion (vi) in Lemma 1,

$$\begin{aligned} \left| \sum_{y \approx Q_j} \psi_t * b_j(y) \right| &= \left| \sum_{y \approx Q_j} \int_{Q_j} \psi_t(y - t) b(z) dz \right| \leq \sum_{y \approx Q_j} \int_{Q_j} |\psi_t(y - t)| |b(z)| dz \\ &\leq \sum_{y \approx Q_j} \sup_{z \in Q_j} \frac{C_0 t^\delta}{(t + |y - z|)^{n+\delta}} \int_{Q_j} |b(z)| dz \\ &\leq C \sum_{y \approx Q_j} \sup_{z \in Q_j} \frac{t^\delta}{(t + |y - z|)^{n+\delta}} \alpha |Q_j|. \end{aligned}$$

On the other hand, it is clear that

$$\sup_{z \in Q_j} \frac{t^\delta}{(t + |y - z|)^{n+\delta}} |Q_j| \leq C \int_{Q_j} \frac{t^\delta}{(t + |y - z|)^{n+\delta}} dz$$

for any cube Q_j satisfying $y \approx Q_j$, the constant C is independent of t . Therefore,

$$\left| \sum_{y \approx Q_j} \psi_t * b_j(y) \right| \leq C\alpha \sum_{y \approx Q_j} \int_{Q_j} \frac{t^\delta dz}{(t + |y - z|)^{n+\delta}} \leq C\alpha \int_{\mathbb{R}^n} \frac{t^\delta dz}{(t + |y - z|)^{n+\delta}} dz \leq C\alpha.$$

Putting the estimate we just proved into the definition of $g_\lambda^{(1)}$, we obtain

$$|g_\lambda^{(1)}(b)(x)|^2 \leq C\alpha \int \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \sum_{y \approx Q_j} \psi_t * b_j(y) \right| \frac{dy dt}{t^{n+1}} =: C\alpha \mathcal{J}(x).$$

So to prove (2.4), we have to show that $\omega(\{x \in \mathbb{R}^n \mid \mathcal{J}(x) > \alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p$. By the Chebyshev inequality, it suffices to prove that

$$\int_{\mathbb{R}^n} \mathcal{J}(x) \omega(x) dx \leq C\alpha^{1-p} \|f\|_{L_\omega^p}^p. \quad (2.6)$$

By the definition of \mathcal{J} , we have

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{J}(x) \omega(x) dx &= \int_{\mathbb{R}^n} \iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \sum_{y \approx Q_j} \psi_t * b_j(y) \right| \frac{dy dt}{t^{n+1}} \omega(x) dx \\ &\leq \iint_{\mathbb{R}_+^{n+1}} \left(\frac{1}{t^n} \int_{|x-y| \leq t} \omega(x) dx \right) \frac{1}{t} \left| \sum_{y \approx Q_j} \psi_t * b_j(y) \right| dy dt \\ &\quad + \iint_{\mathbb{R}_+^{n+1}} \left(\frac{1}{t^n} \int_{|x-y| > t} \frac{t^{\lambda n - n}}{|x-y|^{\lambda n}} \omega(x) dx \right) \frac{1}{t} \left| \sum_{y \approx Q_j} \psi_t * b_j(y) \right| dy dt \\ &\leq C \iint_{\mathbb{R}_+^{n+1}} \frac{1}{t} \left| \sum_{y \approx Q_j} \psi_t * b_j(y) \right| M\omega(y) dy dt \\ &\leq C \sum_j \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ y \approx Q_j}} \frac{1}{t} |\psi_t * b_j(y)| \omega(y) dy dt. \end{aligned} \quad (2.7)$$

We denote z_i as the center of the cube Q_i , and r_i the sidelength of Q_i , respectively. Recalling the geometry of the Whitney cubes, that is, if Q_i touches Q_j , then $l(Q_i)/4 \leq l(Q_j) \leq 4l(Q_i)$, where $l(Q)$ means the diameter of Q . So we have

$$\{y \in \mathbb{R}^n \mid y \approx Q_j\} \subset \{y \in \mathbb{R}^n, |y-z| > r_j/4\}, \quad \forall z \in Q_j. \quad (2.8)$$

Now we consider $\int_{\mathbb{R}^n} \int_0^\infty \frac{1}{t} |\psi_t * b_j(y)| dy dt$, the j th summand in the right-hand side of (2.7). By using property (2.8) and $\int_{Q_j} b_j(z) dz = 0$, it can be controlled by a constant times

$$\begin{aligned} &\iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ y \approx Q_j}} \frac{1}{t} \left| \int_{Q_j} (\psi_t(y-z) - \psi_t(y-z_k)) b_j(z) dz \right| \omega(y) dy dt \\ &\leq \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ |y-z| \geq \frac{1}{4} r_j}} \frac{1}{t} \int_{Q_j} |\psi_t(y-z) - \psi_t(y-z_k)| b_j(z) dz \omega(y) dy dt \\ &:= \int_{Q_j} J_1(z) b_j(z) dz + \int_{Q_j} J_2(z) b_j(z) dz. \end{aligned} \quad (2.9)$$

where

$$J_1(z) = \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ \frac{1}{4}r_j \leq |y-z| \leq 2|z-z_j|}} \frac{1}{t} |\psi_t(y-z) - \psi_t(y-z_k)| \omega(y) dy dt$$

$$J_2(z) = \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ |y-z| \geq 2|z-z_j|}} \frac{1}{t} |\psi_t(y-z) - \psi_t(y-z_k)| \omega(y) dy dt.$$

Estimate for $J_1(z)$. Recall the condition (1.4) of ψ , i.e. $|\psi(x)| \leq C_0(1 + |x|)^{-n-\delta}$, and note that since $z, z_j \in Q_j$, $\frac{1}{4}r_j < |y-z| \leq 2|z-z_j|$, then $|y-z|$, $|y-z_k|$ and r_j are comparable. Hence, by definition of $J_1(z)$,

$$J_1(z) \leq C \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ \frac{1}{4}r_j \leq |y-z| \leq 2|z-z_j|}} \frac{t^{\delta-1}}{(t + |y-z|)^{n+\delta}} \omega(y) dy dt.$$

We consider two subcases, $t > |y-z|$ and $t \leq |y-z|$, respectively. Note that if $t > |y-z|$, then

$$J_1(z) \leq C \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1}, t > |y-z| \\ \frac{1}{4}r_j \leq |y-z| \leq 2|z-z_j|}} t^{-n-1} \omega(y) dy dt \leq \frac{C}{(r_j)^n} \int_{|y-z| \leq 2\sqrt{n}r_j} \omega(y) dy \leq C\omega(z).$$

If $t \leq |y-z|$, with a similar estimate above, we also have $J_1(z) \leq C\omega(z)$.

Estimate for $J_2(z)$. Recall the condition (1.5) of ψ , i.e.

$$|\psi(x+y) - \psi(x)| \leq C_1 |y|^\gamma (1 + |x|)^{-n-\gamma-\delta}, \quad \text{for } 2|y| \leq |x|.$$

By definition of $J_2(z)$,

$$J_2(z) \leq C_1 \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ |y-z| \geq 2|z-z_j|}} \frac{|z-z_j|^\gamma t^{\delta-1}}{(t + |y-z|)^{n+\gamma+\delta}} \omega(y) dy dt$$

$$= C_1 \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1}, t > |y-z| \\ |y-z| \geq 2|z-z_j|}} \frac{|z-z_j|^\gamma t^{\delta-1} \omega(y)}{(t + |y-z|)^{n+\gamma+\delta}} dy dt$$

$$+ C_1 \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1}, t \leq |y-z| \\ |y-z| \geq 2|z-z_j|}} \frac{|z-z_j|^\gamma t^{\delta-1} \omega(y)}{(t + |y-z|)^{n+\gamma+\delta}} dy dt := C_1 (J_{2,1}(z) + J_{2,2}(z)).$$

Firstly, we consider $J_{2,1}(z)$, note that $t > |y - z| \geq 2|z - z_j|$, we have

$$\begin{aligned} J_{2,1}(z) &= \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ |y-z| \geq 2|z-z_j|, t > |y-z|}} \frac{|z - z_j|^\gamma t^{\delta-1} \omega(y)}{(t + |y - z|)^{n+\gamma+\delta}} dy dt \\ &\leq \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ |y-z| \geq 2|z-z_j|, t > |y-z|}} |z - z_j|^\gamma t^{-n-\gamma-1} \omega(y) dy dt \\ &\leq \int_{|y-z| \geq 2|z-z_j|} \frac{|z - z_j|^\gamma}{|y - z|^{n+\gamma}} \omega(y) dy \\ &\leq CM\omega(z). \end{aligned}$$

As for $J_{2,2}(z)$, we denote $N = \max\{2|z - z_j|, t\}$, then

$$\begin{aligned} J_{2,2}(z) &= \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ |y-z| \geq 2|z-z_j|, t > |y-z|}} \frac{|z - z_j|^\gamma t^{\delta-1}}{(t + |y - z|)^{n+\gamma+\delta}} \omega(y) dy dt \\ &\leq \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ |y-z| \geq N}} \frac{|z - z_j|^\gamma t^{\delta-1}}{|y - z|^{n+\gamma+\delta}} \omega(y) dy dt \end{aligned}$$

We consider two cases, $t > 2|z - z_j|$ and $t \leq 2|z - z_j|$, respectively.

Note that if $t > 2|z - z_j|$, then $N = t$, and

$$\begin{aligned} J_{2,2}(z) &\leq \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ |y-z| \geq t}} \frac{|z - z_j|^\gamma t^{\delta-1}}{|y - z|^{n+\gamma+\delta}} \omega(y) dy dt \leq C \int_{|y-z| \geq 2|z-z_j|} \frac{|z - z_j|^\gamma}{|y - z|^{n+\gamma}} \omega(y) dy \\ &\leq CM\omega(z) \leq C\omega(z). \end{aligned}$$

If $t \leq 2|z - z_j|$, then $N = 2|z - z_j|$, with a similar estimate, we have $J_{2,2}(z) \leq CM\omega(z)$.

Thus, we obtain $J_2(z) \leq CM\omega(z)$. Putting the estimates of $J_1(z)$ and $J_2(z)$ into (2.9), we have the j th summand in the right-hand side of (2.7) satisfying

$$\int_{\mathbb{R}^n} \int_0^\infty \frac{1}{t} |\psi_t * b_j(y)| \omega(y) dy dt \leq C \int_{Q_j} |b_j(z)| \omega(z) dz.$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{J}(x) \omega(x) dx &\leq C \sum_j \int_{Q_j} |b_j(z)| \omega(z) dz \leq C \sum_j \alpha \omega(Q_j) \\ &= C\alpha\omega(\Omega) \leq C\alpha^{1-p} \|f\|_{L_\omega^p}, \end{aligned} \quad (2.10)$$

this completes the proof of (2.6). Since we have reduced inequality (2.4) to inequality (2.6), we have also proved (2.4).

Now we are in the position to prove (2.5), we will use the condition $p = 2/\lambda$.

Proof of (2.5). Recall that inequality (2.5) states that

$$\omega(\{x \in \mathbb{R}^n \mid g_\lambda^{(2)}(b)(x) > C\alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p.$$

Since $\omega(\Omega) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p$, it will be enough to prove that

$$\omega(\{x \in \mathbb{R}^n \setminus \Omega \mid g_\lambda^{(2)}(b)(x) > C\alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p.$$

Let y_j be the center of Q_j , now if $x \in \mathbb{R}^n \setminus \Omega$, then for any point y in Q_j , there exist constants C_2 and C_3 that independent of variables satisfying $C_2|x - y_j| \leq |x - y| \leq C_3|x - y_j|$. Since $\sum_{y \sim Q_j} |\psi_t * b_j(y)|^2$ is an empty sum if $y \notin \Omega$, because of that if $x \notin \Omega$ then $x \sim Q_j$ never holds. therefore, for $x \in \mathbb{R}^n \setminus \Omega$,

$$\begin{aligned} (g_\lambda^{(2)}(b)(x))^2 &= \iint_{\Omega \times (0, \infty)} \left(\frac{t}{t + |x - y|}\right)^{\lambda n} \left| \sum_{y \sim Q_i} \psi_t * b_i(y) \right|^2 \frac{dy dt}{t^{n+1}} \\ &\leq \sum_j \int_{Q_j} \int_0^\infty \left(\frac{t}{t + |x - y|}\right)^{\lambda n} \left| \sum_{y \sim Q_i} \psi_t * b_i(y) \right|^2 \frac{dy dt}{t^{n+1}} \\ &\leq C \sum_j \frac{1}{|x - y_j|^{n\lambda}} \int_{Q_j} \int_0^\infty t^{\lambda n - n - 1} \left| \sum_{y \sim Q_i} \psi_t * b_i(y) \right|^2 dy dt. \end{aligned} \tag{2.11}$$

By definition of $y \sim Q_i$, the relation $y \sim Q_i$ depends only on which cube y is located in, we can assured that

$$\left| \sum_{y \sim Q_j} \psi_t * b_j(y) \right|^2 = \left| \sum_{y_j \sim Q_j} \psi_t * b_j(y) \right|^2, \quad y \in Q_j. \tag{2.12}$$

Denote that $b^j = \sum_{y_j \sim Q_i} b_i$ and $\phi(x) = (1 + |x|^2)^{-\frac{n+\delta}{2}}$. Recall the condition (b) of ψ that $|\psi(x)| \leq C_0(1 + |x|)^{-n-\delta}$, we have

$$\int_{Q_j} \int_0^\infty t^{\lambda n - n - 1} \left| \sum_{y \sim Q_j} \psi_t * b_j(y) \right|^2 dy dt = \int_{Q_j} \int_0^\infty t^{\lambda n - n - 1} |\psi_t * b^j(y)|^2 dy dt \tag{2.13}$$

$$\leq \int_{\mathbb{R}^n} \int_0^\infty t^{\lambda n - n - 1} (|\psi_t| * |b^j|(y))^2 dy dt \tag{2.14}$$

$$\leq C \int_{\mathbb{R}^n} \int_0^\infty t^{\lambda n - n - 1} (\phi_t * |b^j|(y))^2 dy dt \tag{2.15}$$

We claim that $\forall L \in \mathbb{N}, \exists C_L$ s.t.

$$|\hat{\phi}(\xi)| \leq \frac{C_L}{(|\xi| + 1)^L}, \tag{2.16}$$

Now we first prove the claim (2.16). Since ϕ is a smooth function, and its any derivative is integrable, then

$$\|\widehat{\phi}(\xi)\|_{\infty} \leq \|\phi\|_1 \leq C, \quad (2.17)$$

and for $\forall \alpha \in \mathbb{R}^n$, we have

$$|\xi^{\alpha} \widehat{\phi}(\xi)| = |CD^{\alpha} \widehat{\phi}(\xi)| \leq C \|D^{\alpha} \phi(\xi)\|_1 \leq C,$$

so for any given $L \in \mathbb{N}$ and $|\xi| > 0$, we get

$$|\widehat{\phi}(\xi)| \leq \frac{C}{|\xi|^{\alpha}} \leq \frac{C}{|\xi|^L}, \quad (2.18)$$

combine (2.17) and (2.18), we conclude that for any given $L \in \mathbb{N}$, we have

$$|\widehat{\phi}(\xi)| \leq \frac{C_L}{(|\xi| + 1)^L}.$$

Thus we have proved the claim (2.16). By (2.16), we get

$$|\widehat{\phi}_t(\xi)| \leq \frac{C_L}{(|t\xi| + 1)^L}. \quad (2.19)$$

Recalling

$$I_r(f)(x) = 2^{-r} \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n-r}{2})}{\Gamma(\frac{r}{2})} \int_{\mathbb{R}^n} f(x-y) |y|^{-n+r} dy$$

is Riesz potential operator with order r . Since $p = 2/\lambda$, then

$$\frac{1}{p} - \frac{1}{2} = \frac{(\frac{\lambda n - n}{2})}{n}. \quad (2.20)$$

By Plancherel theorem, (2.19), (2.20), we get

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_0^{\infty} t^{\lambda n - n - 1} |\phi_t * |b^j|(y)|^2 dy dt \\ &= \int_{\mathbb{R}^n} \int_0^{\infty} t^{\lambda n - n - 1} |\widehat{\phi}_t \cdot \widehat{|b^j|}(\xi)|^2 d\xi dt \\ &\leq C_L \int_{\mathbb{R}^n} \int_0^{\infty} t^{\lambda n - n - 1} \frac{1}{(|t\xi| + 1)^{2L}} |\widehat{|b^j|}(\xi)|^2 d\xi dt, \end{aligned}$$

Take $t|\xi| = s$, and by the boundedness of the Riesz potential operator (see e.g., [12, p. 11]), we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_0^{\infty} t^{\lambda n - n - 1} |\phi_t * |b^j|(y)|^2 dy dt \\ &\leq C \int_{\mathbb{R}^n} \int_0^{\infty} s^{\lambda n - n - 1} \frac{1}{(s+1)^{2L}} |\xi|^{n-\lambda n} |\widehat{|b^j|}(\xi)|^2 d\xi ds \\ &\leq C \int_{\mathbb{R}^n} (|\xi|^{\frac{n-\lambda n}{2}} |\widehat{|b^j|}(\xi)|)^2 d\xi = C \|I_{(\lambda n - n)/2}(|b^j|)\|_{L^2}^2 \leq C \|b^j\|_{L^p}^2, \end{aligned}$$

combing this with (2.13) and (2.11), we obtain

$$(g_\lambda^{(2)}(b)(x))^2 \leq \sum_j \frac{C}{|x - y_j|^{\lambda n}} \|b^j\|_{L^p}^2.$$

By the estimate above, it yields that

$$\begin{aligned} \omega(\{x \in \mathbb{R}^n \setminus \Omega \mid g_\lambda^{(2)}(b)(x) > C\alpha\}) &\leq \frac{C}{\alpha^2} \int_{\mathbb{R}^n \setminus \Omega} (g_\lambda^{(2)}(b)(x))^2 \omega(x) dx \\ &\leq \frac{C}{\alpha^2} \sum_j \|b^j\|_{L^p}^2 \int_{\mathbb{R}^n \setminus \Omega} \frac{\omega(x)}{|x - y_j|^{\lambda n}} dx. \end{aligned} \tag{2.21}$$

Note that $\omega \in A_1$, then $\omega(\beta Q_j) \leq C\beta^n \omega(Q_j)$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n \setminus \Omega} \frac{\omega(x)}{|x - y_j|^{\lambda n}} dx &\leq \int_{\mathbb{R}^n \setminus Q_j} \frac{\omega(x)}{|x - y_j|^{\lambda n}} dx = \sum_{i=1}^\infty \int_{2^i Q_j / 2^{i-1} Q_j} \frac{\omega(x)}{|x - y_j|^{\lambda n}} dx \\ &\leq \sum_{i=1}^\infty \frac{\omega(2^i Q_j)}{2^i l(Q_j)^{\lambda n}} \leq \frac{\omega(Q_j)}{|Q_j|^\lambda}. \end{aligned} \tag{2.22}$$

Recalling the geometry of the Whitney cubes, that is, if Q_i touches Q_j , then $l(Q_i)/4 \leq l(Q_j) \leq 4l(Q_i)$, and for fixed x , $x \sim Q_j$, holds for at most N Whitney cubes. Putting inequality (2.22) into (2.21), we have

$$\begin{aligned} \omega(\{x \in \mathbb{R}^n \setminus \Omega \mid g_\lambda^{(2)}(b)(z) > C\alpha\}) &\leq \frac{C}{\alpha^2} \sum_j \|b^j\|_{L^p}^2 \frac{\omega(Q_j)}{|Q_j|^\lambda} \\ &\leq \frac{C}{\alpha^2} \sum_j \sum_{y_j \sim Q_i} \left(\frac{1}{|Q_i|} \int_{Q_i} |b_i(y)|^p dy \right)^{\frac{2}{p}} \omega(Q_j) \\ &\leq \frac{C}{\alpha^2} \sum_j \sum_{y_j \sim Q_i} \left(\frac{1}{\omega(Q_j)} \int_{Q_j} |b_j(y)|^p \omega(y) dy \right)^{\frac{2}{p}} \omega(Q_j) \\ &= \frac{C}{\alpha^2} \sum_j \alpha^2 \omega(Q_j) \leq C \|f\|_{L_\omega^p}^p. \end{aligned} \tag{2.23}$$

Hence, we completed the proof of (2.5).

The proof of Theorem 1.1 is thus finished. \square

3. Proof of the weighted weak-type (p, p) bounds of $\mu_\lambda^{*,p}$

In this section, we will prove Theorem 1.2. Since the proof of Theorem 1.3 is similar to Theorem 1.2, we omit the details here.

Proof of Theorem 1.2. Fix $\alpha > 0$ and let f be in L_ω^p , we have to show

$$\omega(\{x \in \mathbb{R}^n \mid \mu_\lambda^{*,p}(f)(x) > \alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p.$$

Following the arguments in the proof of Theorem 1.1, it is sufficient to show that

$$\omega(\{x \in \mathbb{R}^n \mid \mu_\lambda^{*,p}(b)(x) > \alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p. \quad (3.1)$$

To do so, we have only to prove

$$\omega(\{x \in \mathbb{R}^n \mid \mu_\lambda^{1,p}(b)(x) > C\alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p \quad (3.2)$$

and

$$\omega(\{x \in \mathbb{R}^n \mid \mu_\lambda^{2,p}(b)(x) > C\alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p, \quad (3.3)$$

where

$$\mu_\lambda^{1,p}(b)(x) = \left(\int \int_{\mathbb{R}_+^n} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} \left| \sum_{y \sim Q_j} \varphi_t^p * b_j(y) \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}$$

and

$$\mu_\lambda^{2,p}(b)(x) = \left(\int \int_{\mathbb{R}_+^n} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} \left| \sum_{y \sim Q_j} \varphi_t^p * b_j(y) \right|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

Now let us begin with the proof of (3.2).

Proof of (3.2). By definition of φ and conclusion (vi) in Lemma 1, we have

$$\begin{aligned} \left| \sum_{y \sim Q_j} \varphi_t^p * b_j(y) \right| &= \left| \sum_{y \sim Q_j} \int_{\substack{z \in Q_j \\ |y-z| < t}} \frac{|\Omega(y-z)|}{t^\rho |y-z|^{n-\rho}} |b_j(z)| dz \right| \\ &\leq \sum_{y \sim Q_j} \sup_{\substack{z \in Q_j \\ |y-z| < t}} \frac{|\Omega(y-z)|}{t^\rho |y-z|^{n-\rho}} \int_{Q_j} |b_j(z)| dz \\ &\leq C \sum_{y \sim Q_j} \sup_{\substack{z \in Q_j \\ |y-z| < t}} \frac{|\Omega(y-z)|}{t^\rho |y-z|^{n-\rho}} \alpha |Q_j|. \end{aligned}$$

Since $y \sim Q_j$, it is clear that

$$\sup_{\substack{z \in Q_j \\ |y-z| < t}} \frac{|\Omega(y-z)|}{t^\rho |y-z|^{n-\rho}} \leq C \frac{|\Omega(y-z)|}{t^\rho |y-z|^{n-\rho}}.$$

Therefore,

$$\left| \sum_{y \sim Q_j} \varphi_t^p * b_j(y) \right| \leq C\alpha \int_{|y-z| < t} \frac{|\Omega(y-z)|}{t^\rho |y-z|^{n-\rho}} dz = C\alpha.$$

By the estimate above, we obtain

$$(\mu_\lambda^{1,\rho}(b)(x))^2 \leq C\alpha \int \int_{\mathbb{R}_+^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left| \sum_{y \approx Q_j} \varphi_t^\rho * b_j(y) \right| \frac{dydt}{t^{n+1}} =: C\alpha \mathcal{L}(x).$$

To prove (3.2), it suffices to prove that

$$\int_{\mathbb{R}^n} \mathcal{L}(x)\omega(x)dx \leq C\alpha^{1-\rho} \|f\|_{L_\omega^\rho}^\rho.$$

Using the method in proof of estimate (2.7) in Theorem 1.1, we also have

$$\int_{\mathbb{R}^n} \mathcal{L}(x)\omega(x)dx \leq C \sum_j \int_{y \approx Q_j} \int_0^\infty \frac{1}{t} |\varphi_t * b_j(y)| \omega(y) dy dt.$$

Recall z_i is the center of the cube Q_i , r_i is the sidelength of Q_i , and

$$\{y \in \mathbb{R}^n | y \approx Q_j\} \subset \{y \in \mathbb{R}^n | |y-z| > \frac{1}{4}r_j\}, \quad \forall z \in Q_j. \tag{3.4}$$

Since $\int_{Q_j} b_j(z)dz = 0$, then

$$\begin{aligned} & \int_{y \approx Q_j} \int_0^\infty \frac{1}{t} |\varphi_t * b_j(y)| \omega(y) dy dt \\ &= \int_{y \approx Q_j} \int_0^\infty \frac{1}{t^{\rho+1}} \left| \int_{\substack{|y-z|<t \\ z \in Q_j}} \left(\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-z_j)}{|y-z_j|^{n-\rho}} \right) b_j(z) dz \right| \omega(y) dy dt \\ &\leq \int_{Q_j} \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ \frac{1}{4}r_j < |y-z| < t}} \frac{1}{t^{\rho+1}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-z_j)}{|y-z_j|^{n-\rho}} \right| \omega(y) dy dt |b_j(z)| dz. \end{aligned} \tag{3.5}$$

In order to estimate (3.5), recall $z, z_k \in Q_j$, $y \approx Q_j$, by the mean value theorem, we have

$$\begin{aligned} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-z_j)}{|y-z_j|^{n-\rho}} \right| &\leq \left| \frac{\Omega(y-z_j)}{|y-z|^{n-\rho}} - \frac{\Omega(y-z_j)}{|y-z_j|^{n-\rho}} \right| + \left| \frac{\Omega(y-z)}{|y-z_j|^{n-\rho}} - \frac{\Omega(y-z_j)}{|y-z_j|^{n-\rho}} \right| \\ &\leq C \frac{r_j}{|y-z|^{n-\rho+1}} + C \frac{r_j^\beta}{|y-z|^{n-\rho+\beta}}. \end{aligned}$$

Putting the estimate into (3.5), then

$$\begin{aligned} & \int_{y \approx Q_j} \int_0^\infty \frac{1}{t} |\varphi_t * b_j(y)| \omega(y) dy dt \\ &\leq \int_{Q_j} \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ \frac{1}{4}r_j < |y-z| < t}} \frac{1}{t^{\rho+1}} \left(\frac{r_j}{|y-z|^{n-\rho+1}} + \frac{r_j^\beta}{|y-z|^{n-\rho+\beta}} \right) \omega(y) dy dt |b_j(z)| dz \\ &:= \int_{Q_j} (K_1(z) + K_2(z)) |b_j(z)| dz. \end{aligned} \tag{3.6}$$

Now we consider $K_1(z)$, doing the t -integration first,

$$K_1(z) = \iint_{\substack{(y,t) \in \mathbb{R}_+^{n+1} \\ \frac{1}{4}r_j < |y-z| < t}} \frac{1}{t^{\rho+1}} \frac{r_j}{|y-z|^{n-\rho+1}} \omega(y) dy dt \leq CM\omega(z) \leq C\omega(z).$$

Similarly, we obtain $K_2(z) \leq C\omega(z)$, putting these two estimates into (3.6), we have

$$\int_{y \sim Q_j} \int_0^\infty \frac{1}{t} |\varphi_t * b_j(y)| \omega(y) dy dt \leq C \int_{Q_j} |b_j(z)| \omega(z) dz.$$

Similar arguments as in (2.10) yield that

$$\int_{\mathbb{R}^n} \mathcal{L}(x) \omega(x) dx \leq C \sum_j \int_{Q_j} |b_j(z)| \omega(z) dz \leq C\alpha^{1-p} \|f\|_{L_\omega^p}^p.$$

So we have proved (3.2). \square

Proof of (3.3). Similar to (2.5), to prove (3.3), it is sufficient to prove

$$\omega(\{x \in \mathbb{R}^n \setminus \Omega \mid \mu_\lambda^{2,p}(b)(x) > C\alpha\}) \leq \frac{C}{\alpha^p} \|f\|_{L_\omega^p}^p. \quad (3.7)$$

By the method in proof of estimate (2.11), we have

$$(\mu_\lambda^{2,p}(b)(x))^2 \leq C \sum_j \frac{1}{|x-y_j|^{n\lambda}} \int_{Q_j} \int_0^\infty t^{\lambda n-n-1} \left| \sum_{y \sim Q_j} \varphi_t * b_i(y) \right|^2 dy dt. \quad (3.8)$$

Recall the argument of (2.12), we have

$$\left| \sum_{y \sim Q_j} \psi_t * b_j(y) \right|^2 = \left| \sum_{y_j \sim Q_j} \psi_t * b_j(y) \right|^2, \quad y \in Q_j. \quad (3.9)$$

Note that $b^j = \sum_{y_j \sim Q_j} b_i$, by Plancherel theorem, then we have

$$\begin{aligned} & \int_{Q_j} \int_0^\infty t^{\lambda n-n-1} \left| \sum_{y \sim Q_j} \varphi_t * b_j(y) \right|^2 dy dt \\ & \leq \int_{\mathbb{R}^n} \int_0^\infty t^{\lambda n-n-1} |\varphi_t * b^j(y)|^2 dy dt \\ & = \int_0^\infty t^{\lambda n-n-1} \int_{\mathbb{R}^n} |\hat{b}_j(\xi)|^2 \left| \frac{1}{t^\rho} \int_{|y|<t} e^{-i\xi \cdot y} \frac{\Omega(y)}{|y|^{n-\rho}} \right|^2 d\xi dt \\ & =: \int_0^\infty t^{\lambda n-n-1} \int_{\mathbb{R}^n} |\hat{b}_j(\xi)|^2 |k_t^\rho(\xi)|^2 d\xi dt. \end{aligned}$$

Denote $\gamma = \frac{1}{2} \min(1, 1/\rho)$, $|k_t^\rho(\xi)|$ satisfying following properties [9, p. 110–112]:

If $t \leq 1/|\xi|$, then $|k_t^p(\xi)| \leq C|\xi|t$. If $t \geq 1/|\xi|$, then $|k_t^p(\xi)| \leq C|\xi|^{-\gamma p}t^{-\gamma p} + |\xi|^{|\gamma/2}t^{\gamma/2}$. By these estimates, we have

$$\begin{aligned}
 & \int_{Q_j} \int_0^\infty t^{\lambda n-n-1} \left| \sum_{y \sim Q_j} \varphi_t * b_j(y) \right|^2 dy dt \\
 & \leq C \int_0^{1/|\xi|} t^{\lambda n-n-1} \int_{\mathbb{R}} |\hat{b}(\xi)|^2 |\xi^2| t^2 d\xi dt \\
 & \quad + C \int_{1/|\xi|}^\infty t^{\lambda n-n-1} \int_{\mathbb{R}} |\hat{b}(\xi)|^2 (|\xi|^{-\gamma p} t^{-\gamma p} + |\xi|^{|\gamma/2} t^{\gamma/2})^2 d\xi dt \\
 & \leq C \int_{\mathbb{R}^n} |b(\hat{\xi})|^2 |\xi|^2 \int_0^{1/|\xi|} t^{\lambda n-n-1} dt d\xi \\
 & \quad + C \int_{\mathbb{R}^n} |b(\hat{\xi})|^2 |\xi|^{-2\gamma p} \int_{1/|\xi|}^\infty t^{\lambda n-n-1-2\gamma p} dt d\xi \\
 & \quad + C \int_{\mathbb{R}^n} |b(\hat{\xi})|^2 |\xi|^{-\gamma} \int_{1/|\xi|}^\infty t^{\lambda n-n-1-\gamma} dt d\xi \\
 & \leq C \int_{\mathbb{R}^n} (|\xi|^{|\frac{n-\lambda}{2}|} |\hat{b}^j(\xi)|)^2 d\xi = C \|I_{(\lambda n-n)/2}(f)\|_{L^2}^2 \leq C \|b^j\|_{L^p}^2,
 \end{aligned} \tag{3.10}$$

since $\lambda = 2/p$, $0 < \rho \leq n/2$ and $\max\{2n/(n+\rho), 2n/(n+1/2\rho)\} < p < 2$, we can assure that the right side of the second inequality is integrabel. Thus, we obtain

$$(\mu_{\lambda}^{2,\rho}(b)(x))^2 \leq C \sum_j \frac{1}{|x-y_j|^{\lambda n}} \|b^j\|_{L^p}^2,$$

repeat the same estimate of (2.21), (2.22) and (2.23) we complete the proof of (3.3). Since we have reduced the proof of inequality (3.1) to inequality (3.2) and inequality (3.3), we have also proved (3.1).

The proof of Theorem 1.2 is finished. \square

4. Appendix

Proof of Lemma 1. First, noting that if we take the open set $\Omega = \{x \in \mathbb{R}^n | f^*(x) > \alpha^p\}$, where

$$f^*(x) = \sup_{x \in Q} \frac{1}{\omega(Q)} |f(y)|^p \omega(y) dy,$$

then the proofs of conclusion (i), (ii) and (iii) in Lemma 1 is quite similar to the decomposition Lemma in [4], we therefore omit the details of them. Now, we define

$$b_j = \left(f - \frac{1}{|Q_j|} \int_{Q_j} f dx \right) \chi_{Q_j},$$

$b = \sum_j b_j$ and $u = f - b$. Hence, it's clear that conclusion (iv) and (v) are true.

Next, we need to obtain the estimates of u . By definition,

$$u(x) = \begin{cases} \frac{1}{|\mathcal{Q}_j|} \int_{\mathcal{Q}_j} f(y) dy & \text{if } x \in \mathcal{Q}_j, \\ f(x) & \text{if } x \notin \mathcal{Q}_j. \end{cases}$$

Hence, on the set $\mathbb{R}^n \setminus \Omega$, $u = f$. On each \mathcal{Q}_j , for $\omega \in A_1$, the Hölder inequality gives that

$$\begin{aligned} \int_{\mathcal{Q}_j} |u(x)|^p \omega(x) dx &\leq \int_{\mathcal{Q}_j} \left| \frac{1}{|\mathcal{Q}_j|} \int_{\mathcal{Q}_j} |f(y)| dy \right|^p \omega(x) dx \\ &= \left(|\mathcal{Q}_j|^{\frac{1}{p}-1} \int_{\mathcal{Q}_j} \left(\frac{\omega(\mathcal{Q}_j)}{|\mathcal{Q}_j|} \right)^{\frac{1}{p}} |f(y)| dy \right)^p \\ &\leq C \left(|\mathcal{Q}_j|^{\frac{1}{p}-1} \int_{\mathcal{Q}_j} \omega(y)^{\frac{1}{p}} |f(y)| dy \right)^p \\ &\leq C \int_{\mathcal{Q}_j} |f(x)|^p \omega(x) dx. \end{aligned}$$

Thus, we have proved $\|u\|_{L_{\omega}^p} \leq \|f\|_{L_{\omega}^p}$. Now we need to show that $|u| \leq C\alpha$ holds almost everywhere. Note that $\omega \in A_1$, then $\omega \in A_p$, by conclusion (ii) and Hölder inequality

$$\begin{aligned} \frac{1}{|\mathcal{Q}_j|} \int_{\mathcal{Q}_j} |f(x)| dx &= \frac{1}{|\mathcal{Q}_j|} \int_{\mathcal{Q}_j} |f(x)| \omega^{\frac{1}{p}}(x) \omega^{-\frac{1}{p}}(x) dx \\ &\leq \left(\frac{1}{|\mathcal{Q}_j|} \int_{\mathcal{Q}_j} |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} \left(\frac{1}{|\mathcal{Q}_j|} \int_{\mathcal{Q}_j} \omega^{\frac{1}{p-1}}(x) dx \right)^{\frac{p-1}{p}} \\ &\leq C \left(\frac{1}{|\mathcal{Q}_j|} \int_{\mathcal{Q}_j} |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} \left(\frac{1}{|\mathcal{Q}_j|} \int_{\mathcal{Q}_j} \omega(x) dx \right)^{-\frac{1}{p}} \\ &= C \left(\frac{1}{\omega(\mathcal{Q}_j)} \int_{\mathcal{Q}_j} |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} \leq C\alpha^p. \end{aligned} \tag{4.1}$$

Thus, if $x \in \mathcal{Q}_j$, we have $|u(x)| \leq C\alpha$. If $x \in \mathbb{R}^n \setminus \Omega$, by estimate (4.1) and Lebesgue differentiable theorem

$$u = f(x) = \lim_{\substack{x \in \mathcal{Q}_j \\ |\mathcal{Q}_j| \rightarrow 0}} \frac{1}{|\mathcal{Q}_j|} \int_{\mathcal{Q}_j} |f(t)| dt \leq \lim_{\substack{x \in \mathcal{Q}_j \\ |\mathcal{Q}_j| \rightarrow 0}} C \left(\frac{1}{\omega(\mathcal{Q}_j)} \int_{\mathcal{Q}_j} |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} \leq C\alpha^p.$$

Therefore, we proved the conclusion (v).

Finally, it is obviously that conclusion (vi) follows from $\|u\|_{L_{\omega}^p} \leq \|f\|_{L_{\omega}^p}$. This finished the proof of this Lemma. \square

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Zhengyang Li

School of Mathematics and Computing Sciences
 Hunan University of Science and Technology
 Xiangtan 411201, People's Republic of China
 e-mail: zhengyli@mail.bnu.edu.cn

Corresponding author: Qingying Xue
 School of Mathematical Sciences, Beijing Normal University
 Laboratory of Mathematics and Complex Systems, Ministry of Education
 Beijing 100875, People's Republic of China
 e-mail: qyxue@bnu.edu.cn