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ESSENTIAL NORM OF WEIGHTED COMPOSITION OPERATORS ON ZYGMUND–TYPE SPACES WITH NORMAL WEIGHT

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Abstract. In this paper, we investigate the boundedness, compactness and essential norm of weighted composition operators between Zygmund-type spaces with normal weight.

1. Introduction

A positive continuous function μ on $[0,1)$ is called normal, if there exist positive numbers a and b, $0<a<b$, and $\delta\in[0,1)$ such that (see [22])

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^a}=0;$$
$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^b}=\infty.$$

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Let $H^\infty(\mathbb{D})$ denote the bounded analytic function space on $\mathbb{D}$.

Suppose ω is normal on $[0,1)$. An $f\in H(\mathbb{D})$ is said to belong to the Bloch-type space, denoted by $\mathscr{B}_\omega$, if (see [9], for example)

$$\|f\|_{\mathscr{B}_\omega}=|f(0)|+\sup_{z\in\mathbb{D}}\omega(|z|)|f'(z)|<\infty.$$

It is easy to check that $\mathscr{B}_\omega$ is a Banach space with the norm $\|\cdot\|_{\mathscr{B}_\omega}$. When $0<\alpha<\infty$ and $\omega(t)=(1-t^2)^\alpha$, we get the α-Bloch space (often also called the Bloch-type space), denoted by $\mathscr{B}^\alpha$. In particular, when $\omega(t)=1-t^2$, we get the Bloch space, denoted by $\mathscr{B}$. See [35] for more information of the Bloch space.

Let μ be normal on $[0,1)$. The Zygmund-type space, denoted by $\mathscr{Z}_\mu$, is the space of all $f\in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{Z}_\mu}=|f(0)|+|f'(0)|+\sup_{z\in\mathbb{D}}\mu(|z|)|f''(z)|<\infty.$$

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It is also easy to see that $\mathscr{Z}_\mu$ is a Banach space with the norm $\|\cdot\|_{\mathscr{Z}_\mu}$. When $\mu(t) = 1-t^2$, we get the Zygmund space (the terminology seems was introduced in [11]). For more information on the Zygmund space, see, for example, [11, 13, 14, 23, 26]. When $\mu(t) = (1-t^2)^\alpha$, we get the Zygmund-type space $\mathscr{Z}_\alpha$. For the corresponding space in the unit ball setting, see, for example, [23, 26]. For some generalizations of Bloch-type and Zygmund-type spaces, see, for example, the spaces introduced and studied by Stević in [24, 25, 27, 28, 29].

Throughout the paper, $S(\mathbb{D})$ denotes the set of all analytic self-maps of $\mathbb{D}$. Associated with $\varphi \in S(\mathbb{D})$ is the composition operator C_φ, which is defined by

$$C_\varphi f = f \circ \varphi, \quad f \in H(\mathbb{D}).$$

Let $u \in H(\mathbb{D})$. The weighted composition operator, denoted by uC_φ, is defined on $H(\mathbb{D})$ as follows.

$$(uC_\varphi f)(z) = u(z) f(\varphi(z)), \ \ f \in H(\mathbb{D}).$$

We refer to the books [4, 35] for the theory of composition operators and weighted composition operators.

The boundness, compactness and essential norm of composition operators and some related operators on Bloch-type spaces and Zygmund-type spaces with $\omega(t) = (1-t^2)^\alpha$ were studied, for example, in [1, 2, 3, 4, 5, 7, 10, 11, 12, 13, 15, 17, 18, 19, 20, 21, 30, 31, 32, 34, 36, 37]. See [5, 6, 8, 9, 16, 33] for some related results on Bloch-type spaces $\mathscr{B}_\mu$ and Zygmund-type spaces $\mathscr{Z}_\mu$.

Recently, Ye and Hu in [32] have characterized the boundedness and compactness of weighted composition operators on the Zygmund space $\mathscr{Z}$. In [7], Esmaeili and Lindström extended the results in [32] to the case of Zygmund-type spaces with $\mu(t) = (1-t^2)^\alpha$. Moreover, they gave some estimates of the essential norm of weighted composition operators.

Motivated by [7, 32], in this paper we obtain some sufficient and necessary conditions for the boundedness and compactness of the operator $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$. Moreover, we give some estimates of the essential norm of weighted composition operators $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$.

Recall that the essential norm of $uC_\varphi, \mathscr{Z}_\mu \to \mathscr{Z}_\omega$, denoted by $\|uC_\varphi\|_{e,\mathscr{Z}_\mu \to \mathscr{Z}_\omega}$, is defined by

$$\|uC_\varphi\|_{e,\mathscr{Z}_\mu \to \mathscr{Z}_\omega} = \inf\left\{\|uC_\varphi - K\|_{\mathscr{Z}_\mu \to \mathscr{Z}_\omega} : K \text{ is a compact operator of } \mathscr{Z}_\mu \to \mathscr{Z}_\omega\right\}.$$

Constants are denoted by C, they are positive and may differ from one occurrence to the next. We say that $A \lesssim B$ if there exists a constant C such that $A \leqslant CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Auxiliary results

In this section, we give some auxiliary results which will be used in proving the main results of this paper. They are incorporated in the lemmas which follow.

LEMMA 1. [8] *Suppose* $\mu(t)$ *is normal on* $[0,1)$, *then there exists* $\mu_* \in H(\mathbb{D})$, *such that*

(I) For any $t \in [0,1)$, $\mu_*(t) \in R^+$, $\mu_*(t)$ *is increasing on* $[0,1)$;

(II) $\inf_{t\in[0,1)} \mu(t)\mu_*(t) > 0$; $\sup_{z\in\mathbb{D}} \mu(|z|)|\mu_*(z)| < \infty$.

In the rest of the paper, we will always use μ_* to denote the analytic function related to μ in Lemma 1.

LEMMA 2. *Suppose* μ *is normal on* $[0,1)$. *Then the following statements hold.*

(I) There exists a $\delta \in (0,1)$, *such that* μ *is decreasing on* $[\delta,1)$, $\lim_{t\to 1} \mu(t) = 0$.

(II) For all fixed $\alpha > 1, \beta \in (0,1)$, *when* $t \in (0,1), s \in (\beta,1)$,

$$\mu(t) \approx \mu(t^\alpha) \approx \frac{1}{\mu_*(t)}, \quad \int_0^{s^\alpha} \frac{1}{\mu(t)}dt \approx \int_0^s \frac{1}{\mu(t)}dt, \quad \int_0^{s^\alpha} \frac{s^\alpha - t}{\mu(t)}dt \approx \int_0^s \frac{s-t}{\mu(t)}dt.$$

(III) For any $z \in \mathbb{D}, |\int_0^z \mu_*(\eta)d\eta| \lesssim \int_0^{|z|} \mu_*(t)dt$. *If* $|\eta| \leqslant |z|$, $\mu(|z|)|\mu_*(\eta)| < C$.

Proof. Suppose $\beta \in (0,1)$ and $\alpha > 1$. We only prove that

$$\int_0^{s^\alpha} \frac{s^\alpha - t}{\mu(t)}dt \approx \int_0^s \frac{s-t}{\mu(t)}dt$$

when $s > \beta$. The proofs of the other statements can be found, for example, in [6].

For any $t \in (\frac{\beta}{2}, s)$, there is an $\eta \in (t,s) \subset (\frac{\beta}{2}, 1)$ such that

$$\frac{s^\alpha - t^\alpha}{s-t} = \alpha\eta^{\alpha-1},$$

so $s^\alpha - t^\alpha \approx s-t$. Therefore

$$\int_{\frac{\beta^\alpha}{2^\alpha}}^{s^\alpha} \frac{s^\alpha - t}{\mu(t)}dt = \int_{\frac{\beta}{2}}^{s} \frac{s^\alpha - t^\alpha}{\mu(t^\alpha)}\alpha t^{\alpha-1}dt \approx \int_{\frac{\beta}{2}}^{s} \frac{s-t}{\mu(t)}dt.$$

When $s > \beta$, since

$$0 < \int_{\frac{\beta}{2}}^{\beta} \frac{\beta - t}{\mu(t)}dt < \int_{\frac{\beta}{2}}^{\beta} \frac{s-t}{\mu(t)}dt < \int_0^{\beta} \frac{s-t}{\mu(t)}dt < \int_0^{\beta} \frac{1}{\mu(t)}dt < +\infty,$$

we have

$$\int_{\frac{\beta}{2}}^{\beta} \frac{s-t}{\mu(t)}dt \approx \int_0^{\beta} \frac{s-t}{\mu(t)}dt \approx 1.$$

Therefore

$$\int_0^{s^\alpha} \frac{s^\alpha - t}{\mu(t)}dt \geqslant \int_{\frac{\beta\alpha}{2^\alpha}}^{s^\alpha} \frac{s^\alpha - t}{\mu(t)}dt \approx \int_{\frac{\beta}{2}}^{\beta} \frac{s-t}{\mu(t)}dt + \int_{\beta}^{s} \frac{s-t}{\mu(t)}dt \approx \int_0^{s} \frac{s-t}{\mu(t)}dt.$$

It is obvious that

$$\int_0^{s^\alpha} \frac{s^\alpha - t}{\mu(t)}dt \leqslant \int_0^{s} \frac{s-t}{\mu(t)}dt.$$

So, we get

$$\int_0^{s^\alpha} \frac{s^\alpha - t}{\mu(t)}dt \approx \int_0^{s} \frac{s-t}{\mu(t)}dt,$$

as desired. The proof is compete. □

The following estimates can be found in [26, 33].

LEMMA 3. *Suppose μ is normal on $[0,1)$. Then for every $z \in \mathbb{D}$ and $f \in \mathscr{Z}_\mu$, we have*

$$|f'(z)| \leqslant \left(1 + \int_0^{|z|} \frac{1}{\mu(t)}dt\right)\|f\|_{\mathscr{Z}_\mu}, \text{ and } \quad |f(z)| \leqslant \left(1 + \int_0^{|z|} \frac{|z|-t}{\mu(t)}dt\right)\|f\|_{\mathscr{Z}_\mu}.$$

LEMMA 4. [33] *Suppose that μ is normal on $[0,1)$ such that $\int_0^1 \frac{1}{\mu(t)}dt < \infty$. If $\{f_n\}$ is bounded in $\mathscr{B}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$, then*

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f_n(z)| = 0.$$

LEMMA 5. [5, 26] *Suppose that μ is normal on $[0,1)$ such that $\lim\limits_{|z|\to 1} \int_0^{|z|} \frac{|z|-t}{\mu(t)}dt < \infty$. If $\{f_n\}$ is bounded in $\mathscr{Z}_\mu$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$, then*

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f_n(z)| = 0.$$

To study the compactness, we need the following lemma, which can be get by Lemma 2.10 in [30].

LEMMA 6. *Suppose that ω and μ are normal on $[0,1)$. If $T : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded, then T is compact if and only if whenever $\{f_n\}$ is bounded in $\mathscr{Z}_\mu$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, $\lim\limits_{n\to\infty} \|Tf_n\|_{\mathscr{Z}_\omega} = 0$.*

3. Main results and proofs

In this section, we will use the following symbols. Suppose that $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and μ is normal on $[0,1)$, we define

$$M_0(z) = u''(z), M_1(z) = 2u'(z)\varphi'(z) + u(z)\varphi''(z), M_2(z) = u(z)(\varphi'(z))^2,$$

$$G_\mu(z) = 1 + \int_0^{|z|} \frac{1}{\mu(t)} dt, \quad H_\mu(z) = 1 + \int_0^{|z|} \frac{|z|-t}{\mu(t)} dt, \quad z \in \mathbb{D}.$$

Then for any $f \in H(\mathbb{D})$, we have

$$(uC_\varphi f)''(z) = M_0(z) f(\varphi(z)) + M_1(z) f'(\varphi(z)) + M_2(z) f''(\varphi(z)).$$

THEOREM 1. *Suppose that $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$. Then $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \omega(|z|) \left(|M_0(z)| H_\mu(\varphi(z)) + |M_1(z)|\, G_\mu(\varphi(z)) + \frac{|M_2(z)|}{\mu(|\varphi(z)|)} \right) < \infty. \tag{1}$$

Proof. Assume that (1) holds. By Lemma 3, we get

$$|(uC_\varphi f)(0)| \leqslant |u(0)| H_\mu(\varphi(0)) \|f\|_{\mathscr{Z}_\mu} \lesssim \|f\|_{\mathscr{Z}_\mu},$$

$$|(uC_\varphi f)'(0)| \leqslant \big(|u'(0)| H_\mu(\varphi(0)) + |u(0)\varphi'(0)| G_\mu(\varphi(0)) \big) \|f\|_{\mathscr{Z}_\mu} \lesssim \|f\|_{\mathscr{Z}_\mu}$$

and

$$|(uC_\varphi f)''(z)| \leqslant \left(|M_0(z)| H_\mu(\varphi(z)) + |M_1(z)|\, G_\mu(\varphi(z)) + \frac{|M_2(z)|}{\mu(|\varphi(z)|)} \right) \|f\|_{\mathscr{Z}_\mu}.$$

The above three inequalities and the assumed conditions imply that $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded.

Conversely, suppose $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded. Since $1, z, z^2 \in \mathscr{Z}_\mu$, we get

$$\sup_{z\in\mathbb{D}} \omega(|z|)|M_0(z)| < \infty, \quad \sup_{z\in\mathbb{D}} \omega(|z|)|M_1(z)| < \infty, \quad \sup_{z\in\mathbb{D}} \omega(|z|)|M_2(z)| < \infty. \tag{2}$$

For any $\xi \in \mathbb{D}$ with $|\varphi(\xi)| > \frac{1}{2}$, let $a = \overline{\varphi(\xi)}$. Now, we define

$$h_a(z) = \int_0^{az} \int_0^{\eta} \mu_*(t)\, dt\, d\eta,$$

$$g_a(z) = h_{a^2}(z^2) - 2h_{a^3|a|^{-2}}(z^3) + h_{a^4|a|^{-4}}(z^4)$$

and

$$f_a(z) = 6h_{a^2}(z^2) - 8h_{a^3|a|^{-2}}(z^3) + 3h_{a^4|a|^{-4}}(z^4).$$

By a calculation, we have

(a) $\|h_a\|_{\mathscr{Z}_\mu} \leqslant C$, $\|g_a\|_{\mathscr{Z}_\mu} \leqslant C$, $\|f_a\|_{\mathscr{Z}_\mu} \leqslant C$.

(b) $h_a(\overline{a}) = \int_0^{|a|^2}(|a|^2 - t)\mu_*(t)dt$, $h'_a(\overline{a}) = a\int_0^{|a|^2}\mu_*(t)dt$, $|h''_a(\overline{a})| = a^2\mu_*(|a|^2)$.

(c) $g_a(\overline{a}) = g'_a(\overline{a}) = 0$, $g''_a(\overline{a}) = 2a^2|a|^4\mu_*(|a|^4) + 2a^2\int_0^{|a|^4}\mu_*(t)dt$.

(d) $f_a(\overline{a}) = \int_0^{|a|^4}(|a|^4 - t)\mu_*(t)dt$, $f'_a(\overline{a}) = f''_a(\overline{a}) = 0$.

By (c), (d), Lemmas 1 and 2, we get

$$\frac{\omega(|\xi|)|M_2(\xi)|}{\mu(|\varphi(\xi)|)} \lesssim \omega(|\xi|)\left|(uC_\varphi g_a)''(\xi)\right| \lesssim \|uC_\varphi\|\|g_a\|_{\mathscr{Z}_\mu}$$

and

$$\omega(|\xi|)|M_0(\xi)|\int_0^{|\varphi(\xi)|}\frac{|\varphi(\xi)|-t}{\mu(t)}dt \approx \omega(|\xi|)|(uC_\varphi f_a)''(\xi)| \leqslant \|uC_\varphi\|\|f_a\|_{\mathscr{Z}_\mu}$$

when $|\varphi(\xi)| > \frac{1}{2}$. From the last two inequalities and (2), we have

$$\sup_{z\in\mathbb{D}}\frac{\omega(z)|M_2(z)|}{\mu(|\varphi(z)|)} < \infty, \quad \sup_{z\in\mathbb{D}}\omega(|z|)|M_0(z)|H_\mu(\varphi(z)) < \infty. \tag{3}$$

By Lemma 2, we have

$$\begin{aligned}
&\omega(|\xi|)|M_1(\xi)|\int_0^{|\varphi(\xi)|}\frac{1}{\mu(t)}dt \approx \omega(|\xi|)|M_1(\xi)||h'_a(\varphi(\xi))| \\
&\leqslant \|uC_\varphi h_a\|_{\mathscr{Z}_\omega} + \omega(|\xi|)\left(|M_0(\xi)h_a(\varphi(\xi))| + \left|M_2(\xi)h''_a(\varphi(\xi))\right|\right) \\
&\lesssim \|uC_\varphi\|\|h_a\|_{\mathscr{Z}_\mu} + \omega(|\xi|)|M_0(\xi)|H_\mu(\varphi(\xi)) + \frac{\omega(|\xi|)|M_2(\xi)|}{\mu(|\varphi(\xi)|)}.
\end{aligned}$$

By (2), (3) and the boundness of uC_φ, we get

$$\sup_{z\in\mathbb{D}}\omega(|z|)\,|M_1(z)|\,G_\mu(\varphi(z)) < \infty.$$

The proof is complete. □

THEOREM 2. *Suppose that $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, ω and μ are normal on $[0,1)$. If $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded, then the following statements hold.*

(I) When $\sup_{z\in\mathbb{D}} G_\mu(z) < \infty$ and $\sup_{z\in\mathbb{D}} H_\mu(z) < \infty$,

$$\|uC_\varphi\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \approx \limsup_{|\varphi(z)|\to 1}\frac{\omega(|z|)|M_2(z)|}{\mu(|\varphi(z)|)}.$$

(II) When $\sup_{z\in\mathbb{D}} G_\mu(z) = \infty$ *and* $\sup_{z\in\mathbb{D}} H_\mu(z) < \infty$,

$$\|uC_\varphi\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \approx \limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|M_2(z)|}{\mu(|\varphi(z)|)} + \limsup_{|\varphi(z)|\to 1} \omega(|z|)\,|M_1(z)|\,G_\mu(\varphi(z)).$$

(III) When $\sup_{z\in\mathbb{D}} G_\mu(z) = \infty$ *and* $\sup_{z\in\mathbb{D}} H_\mu(z) = \infty$,

$$\begin{aligned}\|uC_\varphi\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \approx & \limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|M_2(z)|}{\mu(|\varphi(z)|)} + \limsup_{|\varphi(z)|\to 1} \omega(|z|)\,|M_1(z)|\,G_\mu(\varphi(z)) \\ & + \limsup_{|\varphi(z)|\to 1} \omega(|z|)|M_0(z)|H_\mu(\varphi(z)).\end{aligned}$$

Proof. Since uC_φ is bounded, (1) and (2) hold. For any fixed $\rho_n = 1 - \frac{1}{n+1}$, by (2) and Lemma 6, it easily follows that $uC_{\rho_n\varphi} : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact. Fix $s \in (0,1)$,

$$\begin{aligned}
\|uC_\varphi\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} &\leqslant \|uC_\varphi - uC_{\rho_n\varphi}\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega} = \sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1} \|(uC_\varphi - uC_{\rho_n\varphi})f\|_{\mathscr{Z}_\omega} \\
&\leqslant \sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1} \big(|u(0)k_{n,f}(\varphi(0))| + |u'(0)k_{n,f}(\varphi(0))| + |u(0)\varphi'(0)k'_{n,f}(\varphi(0))|\big) \\
&\quad + \sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1} \sup_{|\varphi(z)|\leqslant s} \omega(|z|)|M_0(z)||k_{n,f}(\varphi(z))| \\
&\quad + \sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1} \sup_{s<|\varphi(z)|<1} \omega(|z|)|M_0(z)||k_{n,f}(\varphi(z))| \\
&\quad + \sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1} \sup_{|\varphi(z)|\leqslant s} \omega(|z|)|M_1(z)||k'_{n,f}(\varphi(z))| \\
&\quad + \sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1} \sup_{s<|\varphi(z)|<1} \omega(|z|)|M_1(z)||k'_{n,f}(\varphi(z))| \\
&\quad + \sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1} \sup_{|\varphi(z)|\leqslant s} \omega(|z|)|M_2(z)||k''_{n,f}(\varphi(z))| \\
&\quad + \sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1} \sup_{s<|\varphi(z)|<1} \omega(|z|)|M_2(z)||k''_{n,f}(\varphi(z))|,
\end{aligned} \tag{4}$$

where $k_{n,f}(z) = f(z) - f(\rho_n z)$. When $\|f\|_{\mathscr{Z}_\mu} \leqslant 1$, we see that $\|k_{n,f}\|_{\mathscr{Z}_\mu} \leqslant 2$. By Lemma 3,

$$|k_{n,f}(z)| \leqslant 2H_\mu(z), \quad |k'_{n,f}(z)| \leqslant 2G_\mu(z), \quad |k''_{n,f}(z)| \leqslant \frac{2}{\mu(|z|)}. \tag{5}$$

By Lemma 3 and

$$\int_{\rho_n|z|}^{|z|}\int_0^\eta \frac{1}{\mu(t)}dtd\eta = \int_0^{\rho_n|z|} \frac{|z|-\rho_n|z|}{\mu(t)}dt + \int_{\rho_n|z|}^{|z|} \frac{|z|-t}{\mu(t)}dt = H_\mu(z) - H_\mu(\rho_n z),$$

we have

$$|k_{n,f}(z)| = \left|\int_{\rho_n z}^{z}\int_0^\eta f''(t)dtd\eta + \int_{\rho_n z}^{z} f'(0)d\eta\right| \leqslant H_\mu(z) - H_\mu(\rho_n z) + 1 - \rho_n \tag{6}$$

and

$$|k'_{n,f}(z)| = \left| \int_{\rho_n z}^{z} f''(t)dt + (1-\rho_n)\left(\int_0^{\rho_n z} f''(t)dt + f'(0) \right) \right|$$
$$\leqslant \int_{\rho_n|z|}^{|z|} \frac{1}{\mu(t)} dt + (1-\rho_n)G_\mu(\rho_n z). \tag{7}$$

When $|z| \leqslant s$, by Cauchy's estimate, we have

$$|f'''(z)| \leqslant \frac{2}{1-|z|} \max_{|\xi - z| \leqslant \frac{1-|z|}{2}} |f''(\xi)| \leqslant \frac{2}{1-s} \max_{|\xi| = \frac{1+s}{2}} |f''(\xi)| \leqslant \frac{2}{(1-s)\mu(\frac{1+s}{2})},$$

which implies that

$$|k''_{n,f}(z)| = \left| \int_{\rho_n z}^{z} f'''(\eta)d\eta + (1-\rho_n^2)f''(\rho_n z) \right| \leqslant \frac{2(1-\rho_n)}{(1-s)\mu(\frac{1+s}{2})} + \frac{1-\rho_n^2}{\mu(|\rho_n z|)}. \tag{8}$$

By (2), (6), (7) and (8), we get

$$\lim_{s\to 1} \lim_{n\to\infty} \sup_{\|f\|_{\mathscr{Z}_\mu} \leqslant 1} \sup_{|\varphi(z)| \leqslant s} \omega(|z|)|M_0(z)||k_{n,f}(\varphi(z))| = 0, \tag{9}$$

$$\lim_{s\to 1} \lim_{n\to\infty} \sup_{\|f\|_{\mathscr{Z}_\mu} \leqslant 1} \sup_{|\varphi(z)| \leqslant s} \omega(|z|)|M_1(z)||k'_{n,f}(\varphi(z))| = 0, \tag{10}$$

$$\lim_{s\to 1} \lim_{n\to\infty} \sup_{\|f\|_{\mathscr{Z}_\mu} \leqslant 1} \sup_{|\varphi(z)| \leqslant s} \omega(|z|)|M_2(z)||k''_{n,f}(\varphi(z))| = 0 \tag{11}$$

and

$$\lim_{n\to\infty} \sup_{\|f\|_{\mathscr{Z}_\mu} \leqslant 1} \left(|u(0)k_{n,f}(\varphi(0))| + |u'(0)k_{n,f}(\varphi(0))| + |u(0)\varphi'(0)k'_{n,f}(\varphi(0))| \right) = 0. \tag{12}$$

(I). Suppose $\sup_{z\in\mathbb{D}} G_\mu(z) < \infty$ and $\sup_{z\in\mathbb{D}} H_\mu(z) < \infty$. By (2), (6), (7) and (8), we obtain

$$\lim_{s\to 1} \lim_{n\to\infty} \sup_{\|f\|_{\mathscr{Z}_\mu} \leqslant 1} \sup_{s<|\varphi(z)|<1} \omega(|z|)|M_0(z)||k_{n,f}(\varphi(z))| = 0,$$

$$\lim_{s\to 1} \lim_{n\to\infty} \sup_{\|f\|_{\mathscr{Z}_\mu} \leqslant 1} \sup_{s<|\varphi(z)|<1} \omega(|z|)|M_1(z)||k'_{n,f}(\varphi(z))| = 0$$

and

$$\lim_{s\to 1} \lim_{n\to\infty} \sup_{\|f\|_{\mathscr{Z}_\mu} \leqslant 1} \sup_{s<|\varphi(z)|<1} \omega(|z|)|M_2(z)||k''_{n,f}(\varphi(z))| \lesssim \limsup_{|\varphi(z)|\to 1} \frac{\omega(z)|M_2(z)|}{\mu(|\varphi(z)|)},$$

Hence, by (4) and (9)–(12), we get

$$\|uC_\varphi\|_{e,\mathscr{Z}_\mu \to \mathscr{Z}_\omega} \leqslant \lim_{n\to\infty} \|uC_\varphi - uC_{\rho_n\varphi}\|_{\mathscr{Z}_\mu \to \mathscr{Z}_\omega} \lesssim \limsup_{|\varphi(z)|\to 1} \frac{\omega(z)|M_2(z)|}{\mu(|\varphi(z)|)}.$$

Next, we prove that $\|uC_\varphi\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \gtrsim \limsup_{|\varphi(z)|\to 1} \frac{\omega(z)|M_2(z)|}{\mu(|\varphi(z)|)}$. Assume $\{\xi_n\}_{n=1}^\infty \subset \mathbb{D}$ such that $\lim_{n\to\infty}|\varphi(\xi_n)| = 1$. Let $a_n = \overline{\varphi(\xi_n)}$ and

$$p_n(z) = \mu(|a_n|)\int_0^{a_n z}\int_0^\eta \mu_*^2(t)\,dt\,d\eta.$$

Then $\{p_n\}$ is bounded in $\mathscr{Z}_\mu$ and $p_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. If $K : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact, by Lemmas 4-6, we have

$$\lim_{n\to\infty}\sup_{z\in\mathbb{D}}|p_n'(z)| = 0, \lim_{n\to\infty}\sup_{z\in\mathbb{D}}|p_n(z)| = 0, \lim_{n\to\infty}\|Kp_n\|_{\mathscr{Z}_\omega} = 0. \tag{13}$$

Thus

$$\begin{aligned}\|uC_\varphi - K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega} &\gtrsim \|(uC_\varphi - K)p_n\|_{\mathscr{Z}_\omega} \geqslant \|uC_\varphi p_n\|_{\mathscr{Z}_\omega} - \|Kp_n\|_{\mathscr{Z}_\omega}\\ &\geqslant \omega(|\xi_n|)|(uC_\varphi p_n)''(\xi_n)| - \|Kp_n\|_{\mathscr{Z}_\omega}\\ &\geqslant \omega(|\xi_n|)|M_2(\xi_n)p''(\varphi(\xi_n))| - \omega(|\xi_n|)|M_0(\xi_n)p_n(\varphi(\xi_n))|\\ &\quad -\omega(\xi_n)|M_1(\xi_n)p_n'(\varphi(\xi_n))| - \|Kp_n\|_{\mathscr{Z}_\omega}.\end{aligned} \tag{14}$$

Let $n\to\infty$. By Lemmas 1 and 2, (2) and (13), we get

$$\|uC_\varphi - K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \gtrsim \limsup_{n\to\infty}\frac{\omega(|\xi_n|)|M_2(\xi_n)|}{\mu(\varphi(\xi_n))},$$

which implies

$$\|uC_\varphi\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \gtrsim \limsup_{|\varphi(z)|\to 1}\frac{\omega(|z|)|M_2(z)|}{\mu(\varphi(z))},$$

as desired.

(II). Suppose $\sup_{z\in\mathbb{D}} G_\mu(z) = \infty$ and $\sup_{z\in\mathbb{D}} H_\mu(z) < \infty$. By (2), (6) and (7), we get

$$\lim_{s\to 1}\lim_{n\to\infty}\sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1}\sup_{s<|\varphi(z)|<1}\omega(|z|)|M_0(z)||k_{n,f}(\varphi(z))| = 0,$$

$$\lim_{s\to 1}\lim_{n\to\infty}\sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1}\sup_{s<|\varphi(z)|<1}\omega(|z|)|M_1(z)||k_{n,f}'(\varphi(z))| \lesssim \limsup_{|\varphi(z)|\to 1}\omega(|z|)|M_1(z)|G_\mu(\varphi(z)),$$

and

$$\lim_{s\to 1}\lim_{n\to\infty}\sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1}\sup_{s<|\varphi(z)|<1}\omega(|z|)|M_2(z)||k_{n,f}''(\varphi(z))| \lesssim \limsup_{|\varphi(z)|\to 1}\frac{\omega(|z|)|M_2(z)|}{\mu(|\varphi(z)|)}.$$

Thus, by (4) and (9)–(12), we get

$$\begin{aligned}\|uC_\varphi\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} &\leqslant \lim_{n\to\infty}\|uC_\varphi - uC_{\rho_n\varphi}\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega}\\ &\lesssim \limsup_{|\varphi(z)|\to 1}\frac{\omega(|z|)|M_2(z)|}{\mu(|\varphi(z)|)} + \limsup_{|\varphi(z)|\to 1}\omega(|z|)|M_1(z)|G_\mu(\varphi(z)).\end{aligned}$$

Next, we prove

$$\|uC_\varphi\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \gtrsim \limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|M_2(z)|}{\mu(|\varphi(z)|)} + \limsup_{|\varphi(z)|\to 1} \omega(|z|)|M_1(z)|G_\mu(\varphi(z)).$$

Let $\{\xi_n\}_{n=1}^\infty \subset \mathbb{D}$ such that $\lim_{n\to\infty} |\varphi(\xi_n)| = 1$. Let $a_n = \overline{\varphi(\xi_n)}$ and

$$r_n(z) = \frac{\int_0^{a_n z}\left(\int_0^{\eta^2} \mu_*(t)dt\right)^2 d\eta}{\int_0^{|a_n|} \mu_*(t)dt} - \frac{\int_0^{a_n z}\left(\int_0^{\frac{\eta^3}{|a_n|^2}} \mu_*(t)dt\right)^2 d\eta}{\int_0^{|a_n|} \mu_*(t)dt}.$$

Then $\{r_n\}_{n=1}^\infty$ is bounded in $\mathscr{Z}_\mu$ and $\{r_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. If $K : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact, by a calculation and Lemmas 5 and 6, we get

$$\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |r_n(z)| = 0, \quad \lim_{n\to\infty} \|Kr_n\|_{\mathscr{Z}_\omega} = 0, \quad r_n'(\overline{a_n}) = 0, \quad |r_n''(\overline{a_n})| \approx \frac{1}{\mu(|a_n|)}. \tag{15}$$

Similarly to (14), we have

$$\begin{aligned} &\|uC_\varphi - K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \\ &\gtrsim \omega(|\xi_n|)|M_2(\xi_n)r_n''(\varphi(\xi_n))| - \omega(|\xi_n|)|M_0(\xi_n)r_n(\varphi(\xi_n))| - \|Kr_n\|_{\mathscr{Z}_\omega}. \end{aligned}$$

Let $n\to\infty$. By (2) and (15), we have

$$\|uC_\varphi - K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \gtrsim \limsup_{n\to\infty} \frac{\omega(|\xi_n|)|M_2(\xi_n)|}{\mu(|\varphi(\xi_n)|)}.$$

In the same way, taking

$$q_n(z) = \frac{3\int_0^{a_n z}\left(\int_0^{\eta^2} \mu_*(t)dt\right)^2 d\eta}{\int_0^{|a_n|} \mu_*(t)dt} - \frac{2\int_0^{a_n z}\left(\int_0^{\frac{\eta^3}{|a_n|^2}} \mu_*(t)dt\right)^2 d\eta}{\int_0^{|a_n|} \mu_*(t)dt},$$

we have

$$\|uC_\varphi - K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \gtrsim \limsup_{n\to\infty} \omega(|\xi_n|)|M_1(\xi_n)|G_\mu(\varphi(\xi_n)).$$

From the arbitrary of K and $\{\xi\}_{n=1}^\infty$, when $\sup_{z\in\mathbb{D}} G_\mu(z) = \infty$ and $\sup_{z\in\mathbb{D}} H_\mu(z) < \infty$, we obtain the desired result.

(III). Suppose $\sup_{z\in\mathbb{D}} G_\mu(z) = \infty$ and $\sup_{z\in\mathbb{D}} H_\mu(z) = \infty$. By (6), we have

$$\lim_{s\to 1}\lim_{n\to\infty} \sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1} \sup_{s<|\varphi(z)|<1} \omega(|z|)|M_0(z)||k_{n,f}(\varphi(z))| \lesssim \limsup_{|\varphi(z)|\to 1} \omega(|z|)|M_0(z)|H_\mu(\varphi(z)),$$

$$\lim_{s\to 1}\lim_{n\to\infty} \sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1} \sup_{s<|\varphi(z)|<1} \omega(|z|)|M_1(z)||k_{n,f}'(\varphi(z))| \lesssim \limsup_{|\varphi(z)|\to 1} \omega(|z|)|M_1(z)|G_\mu(\varphi(z))$$

and

$$\lim_{s\to 1}\lim_{n\to\infty}\sup_{\|f\|_{\mathscr{Z}_\mu}\leqslant 1}\sup_{s<|\varphi(z)|<1}\omega(|z|)|M_2(z)||k''_{n,f}(\varphi(z))|\lesssim \limsup_{|\varphi(z)|\to 1}\frac{\omega(|z|)|M_2(z)|}{\mu(|\varphi(z)|)}.$$

By (4) and (9)–(12), we get

$$\|uC_\varphi\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega}\lesssim \limsup_{|\varphi(z)|\to 1}\omega(|z|)|M_0(z)|H_\mu(\varphi(z))+\limsup_{|\varphi(z)|\to 1}\omega(|z|)|M_1(z)|G_\mu(\varphi(z)) + \limsup_{|\varphi(z)|\to 1}\frac{\omega(|z|)|M_2(z)|}{\mu(|\varphi(z)|)}.$$

Next we give the lower estimate of $\|uC_\varphi\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega}$. Assume $\{\xi_n\}_{n=1}^\infty\subset\mathbb{D}$ such that $\lim_{n\to\infty}|\varphi(\xi_n)|=1$. Let $a_n=\overline{\varphi(\xi_n)}$. Set

$$\tau_n(z)=\frac{10h^2_{a_n^3}(z^3)-15h^2_{a_n^4|a_n|^{-2}}(z^4)+6h^2_{a_n^5|a_n|^{-4}}(z^5)}{h_{a_n}(\overline{a_n})},$$

where

$$h_a(z)=\int_0^{az}\int_0^\eta \mu_*(t)dtd\eta.$$

In [5], we have proved that

$$\frac{\mu(|z|)\left(\int_0^{|a_nz|}\mu_*(t)dt\right)^2}{\int_0^{|a_n|}\int_0^s\mu_*(s)dsdt}\lesssim 1.$$

Thus $\{\tau_n\}_{n=1}^\infty$ is bounded in $\mathscr{Z}_\mu$ and converges to 0 uniformly on compact subset of $\mathbb{D}$. By Lemma 6 and a calculation,

$$\lim_{n\to\infty}\|K\tau_n\|_{\mathscr{Z}_\omega}=0,\quad \tau_n'(\overline{a_n})=\tau''(\overline{a_n})=0,\quad |\tau_n(\overline{a_n})|\approx\int_0^{|a_n|}\int_0^\eta\mu_*(t)dtd\eta. \quad (16)$$

Similar to (14), we have

$$\|uC_\varphi-K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega}\gtrsim \omega(|\xi_n|)|M_0(\xi_n)\tau_n(\varphi(\xi_n))|-\|K\tau_n\|_{\mathscr{Z}_\omega}.$$

Let $n\to\infty$. By (16), we have

$$\|uC_\varphi-K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega}\gtrsim \limsup_{n\to\infty}\omega(|\xi_n|)|M_0(\xi_n)|H_\mu(\varphi(\xi_n)). \quad (17)$$

Since the test functions $\{r_n\}_{n=1}^\infty$ are bounded in $\mathscr{Z}_\mu$ and converges to 0 uniformly on compact subset of $\mathbb{D}$, we have

$$\lim_{n\to\infty}\|Kr_n\|_{\mathscr{Z}_\omega}=0,\quad r_n'(\overline{a_n})=0,\quad |r_n''(\overline{a_n})|\approx\frac{1}{\mu(|a_n|)}.$$

By Lemma 3,

$$\begin{aligned}&\|uC_\varphi - K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega}\\ \gtrsim\ & \omega(|\xi_n|)|M_2(\xi_n)r_n''(\varphi(\xi_n))| - \omega(|\xi_n|)|M_0(\xi_n)r_n(\varphi(\xi_n))| - \|Kr_n\|_{\mathscr{Z}_\omega}\\ \gtrsim\ & \frac{\omega(|\xi_n|)|M_2(\xi_n)|}{\mu(|\varphi(\xi_n)|)} - \omega(|\xi_n|)|M_0(\xi_n)|H_\mu(\varphi(\xi_n)) - \|Kr_n\|_{\mathscr{Z}_\omega}.\end{aligned}$$

So

$$\|uC_\varphi - K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega} + \omega(|\xi_n|)|M_0(\xi_n)|H_\mu(\varphi(\xi_n)) \gtrsim \frac{\omega(|\xi_n|)|M_2(\xi_n)|}{\mu(|\varphi(\xi_n)|)} - \|Kr_n\|_{\mathscr{Z}_\omega}.$$

Let $n\to\infty$. We get

$$\|uC_\varphi - K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \gtrsim \limsup_{n\to\infty} \frac{\omega(|\xi_n|)|M_2(\xi_n)|}{\mu(|\varphi(\xi_n)|)}. \tag{18}$$

In the same way, using test functions $\{q_n\}_{n=1}^\infty$, we have

$$\|uC_\varphi - K\|_{\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \gtrsim \limsup_{n\to\infty} \omega(|\xi_n|)|M_1(\xi_n)|G_\mu(\varphi(\xi_n)). \tag{19}$$

From the arbitrary of K and $\{\xi_n\}_{n=1}^\infty$, when $\sup_{z\in\mathbb{D}} G_\mu(z) = \infty$ and $\sup_{z\in\mathbb{D}} H_\mu(z) = \infty$, by (17)–(19), we have

$$\begin{aligned}\|uC_\varphi\|_{e,\mathscr{Z}_\mu\to\mathscr{Z}_\omega} \gtrsim\ & \limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|M_2(z)|}{\mu(\varphi(z))} + \limsup_{|\varphi(z)|\to 1} \omega(|z|)|M_1(z)|G_\mu(\varphi(z))\\ &+ \limsup_{|\varphi(z)|\to 1} \omega(|z|)|M_0(z)|H_\mu(\varphi(z)),\end{aligned}$$

as desired. The proof is complete. □

COROLLARY 1. *Suppose that $u\in H(\mathbb{D})$, $\varphi\in S(\mathbb{D})$, ω and μ are normal on $[0,1)$. If $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is bounded, then following statements hold.*

(I) When $\sup_{z\in\mathbb{D}} G_\mu(z) < \infty$ and $\sup_{z\in\mathbb{D}} H_\mu(z) < \infty$, $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact if and only if

$$\limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|M_2(z)|}{\mu(|\varphi(z)|)} = 0.$$

(II) When $\sup_{z\in\mathbb{D}} G_\mu(z) = \infty$ and $\sup_{z\in\mathbb{D}} H_\mu(z) < \infty$, $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact if and only if

$$\limsup_{|\varphi(z)|\to 1} \frac{\omega(|z|)|M_2(z)|}{\mu(|\varphi(z)|)} + \lim_{|\varphi(z)|\to 1} \omega(|z|)\,|M_1(z)|\,G_\mu(\varphi(z)) = 0.$$

(III) When $\sup_{z\in\mathbb{D}} G_\mu(z) = \infty$ and $\sup_{z\in\mathbb{D}} H_\mu(z) = \infty$, $uC_\varphi : \mathscr{Z}_\mu \to \mathscr{Z}_\omega$ is compact if and only if

$$\limsup_{|\varphi(z)|\to 1} \left(\frac{\omega(|z|)|M_2(z)|}{\mu(|\varphi(z)|)} + \omega(|z|)\,|M_1(z)|\,G_\mu(\varphi(z)) + \omega(|z|)|M_0(z)|H_\mu(\varphi(z)) \right) = 0.$$

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